Assignment 8: Due Friday, March 27th at 8pm

Problems to be handed in

Problem 1

A frog is in a pond with 5 water lilies numbered from 1 to 5. With exponential rate 1, the frog leaves its current water lily, chooses a new one uniformly among the four others and jumps to it. We assume the frog starts from lily 1. Let $X(t)$ be the number of the lily where the frog is at time t.

1. Admitting that $X(t)$ is a continuous-time Markov chain, give its parameters (i.e. the v_i and p_{ij} of the course). No proof is required.

2. Let $p_{1j}(t) = \mathbb{P}(X(t) = j | X(0) = 1)$. Explain why $p_{12}(t) = p_{13}(t) = p_{14}(t) = p_{15}(t)$ (no computations required).

3. Write the backward Chapman–Kolmogorov equation, and prove that

$$p'_{11}(t) = \frac{1}{4} - \frac{5}{4} p_{11}(t).$$

4. Solve this equation to compute $p_{11}(t)$.

Problem 2

A one-cell organism can be in two possible states called A and B. An organism in state A switches to state B with exponential rate α. An organism in state B divides into two organism in state A with exponential rate β.

Define an appropriate continuous-time Markov chain for a population of such organisms, and give the parameters of the model. No proof is required.

Hint: The state space is formed of pairs of integers such as $\{2, 1\}$. For example, the probability to pass from 2 organisms of type A and 1 of type B to 4 of type A and none of type B can be denoted by $p_{\{2,1\},\{4,0\}}$.

Problem 3

We consider a group of 4 students among which a rumour is spreading. At time 0, only one student is aware of the rumour. Whenever two students meet, if one is aware of the rumour and not the other, the second one becomes aware of it. We assume that for any pair of students, the times at which they meet forms a Poisson process with rate 1. We also assume that, for the 6 possible pairs of students, we get 6 independent Poisson processes.

1. Let $A(t)$ be the number of students aware of the rumour at time t. We admit this is a continuous-time Markov chain. Give its parameters. No proof is required.

2. Let T be the first time at which all students become aware of the rumour. Compute $\mathbb{E}[T]$.

Recommended Problems

These provide additional practice but are not to be handed in. Textbook Chapter 6 (12th ed.): Exercises 3, 4, 9.