

Recall the random walk in  $\mathbb{Z}^d$ .

We will compute whether or not it is recurrent.

In particular, we will check if  $\sum_{n=0}^{\infty} p_{00}^{(n)} = \infty$  or not.

Notation: unit vectors in  $\mathbb{Z}^d$ :  $\vec{e}_j = (0, \dots, 0, \overset{j\text{th position}}{1}, 0, \dots, 0)$   
 (e.g.,  $d=3$ :  $\vec{e}_1 = (1, 0, 0)$ ,  $\vec{e}_2 = (0, 1, 0)$ ,  $\vec{e}_3 = (0, 0, 1)$ ).

So, the walk takes  $i^{\text{th}}$  step  $\vec{X}_i$  with probabilities:

$$P(\vec{X}_i = \vec{e}_j) = P(X_i = -\vec{e}_j) = \frac{1}{2d}.$$

Let  $\vec{S}_0 = 0$  initial position

$\vec{S}_n = \text{position after } n \text{ steps}$   
 $= \vec{X}_1 + \vec{X}_2 + \dots + \vec{X}_n \quad (n \geq 1)$ .

Then,  $\vec{S}_{n+1} = \vec{S}_n + \vec{X}_{n+1}$ , so

$$P(\vec{S}_{n+1} = \vec{X} + \vec{e}_j \mid \vec{S}_n = \vec{X}) = P(\vec{X}_{n+1} = \vec{e}_j) = \frac{1}{2d},$$

and

$$P(\vec{S}_{n+1} = \vec{X} - \vec{e}_j \mid \vec{S}_n = \vec{X}) = P(\vec{X}_{n+1} = -\vec{e}_j) = \frac{1}{2d}.$$

This defines the transition probabilities for the symmetric R.W. on  $\mathbb{Z}^d$ .

For  $\vec{k} \in \mathbb{R}^d$ ,  $k = (k_1, \dots, k_d)$  we can define  $\Phi_n(\vec{k})$  to be the characteristic function of  $\vec{S}_n$  evaluated at  $\vec{k}$ :

$$\Phi_n(\vec{k}) = \mathbb{E}[e^{i\vec{k} \cdot \vec{S}_n}],$$

where  $\vec{k} \cdot \vec{S}_n$  is the standard dot product in  $\mathbb{R}^d$ , i.e.,

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n.$$

$$\text{So, } \vec{k} \cdot \vec{S}_n = \vec{k} \cdot \left( \sum_{i=1}^n \vec{X}_i \right) = \sum_{i=1}^n \vec{k} \cdot \vec{X}_i.$$

Theorem: We will show that

$$(1). \sum_{n=0}^{\infty} P_{\vec{S}_n}^{(n)} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{1 - \Phi_1(\vec{k})} d\vec{k}$$

$$(2). \text{ Furthermore, this equals } \begin{cases} +\infty & d=1, 2 \\ -\infty & d \geq 3. \end{cases}$$

Note that

$$\Phi_n(\vec{k}) = \mathbb{E}[e^{i\vec{k} \cdot \vec{S}_n}] = \mathbb{E}[e^{i\vec{k} \cdot \vec{X}_1} e^{i\vec{k} \cdot \vec{X}_2} \dots e^{i\vec{k} \cdot \vec{X}_n}]$$

$$\begin{aligned} & \text{since } \vec{X}_1, \dots, \vec{X}_n \text{ are independent} \\ & = \mathbb{E}[e^{i\vec{k} \cdot \vec{X}_1}] \dots \mathbb{E}[e^{i\vec{k} \cdot \vec{X}_n}] \\ & = (\mathbb{E}[e^{i\vec{k} \cdot \vec{X}_1}])^n \quad \text{since } \vec{X}_1, \dots, \vec{X}_n \text{ have the same distribution.} \end{aligned}$$

Since  $\vec{X}_i = \vec{S}_i$ , we have

$$\boxed{\Phi_n(\vec{k}) = (\mathbb{E}[e^{i\vec{k} \cdot \vec{S}_1}])^n = (\Phi_1(\vec{k}))^n.}$$

Now, we show that

$$\Phi_1(\vec{k}) = \frac{1}{d} \sum_{j=1}^d \cos k_j.$$

Proof:

$$\begin{aligned}\Phi_1(\vec{k}) &= \mathbb{E}[e^{i\vec{k} \cdot \vec{X}_1}] = \sum_{j=1}^d p(\vec{X}_1 = e_j) e^{ik_j} + \sum_{j=1}^d p(\vec{X}_1 = -e_j) e^{-ik_j} \\ &= \sum_{j=1}^d \frac{1}{2d} (\cos k_j + i \sin k_j + \cos k_j - i \sin k_j) \\ &= \frac{1}{2d} \sum_{j=1}^d 2 \cos k_j = \frac{1}{d} \sum_{j=1}^d \cos k_j.\end{aligned}$$

Thus,

$$\boxed{\Phi_n(\vec{k}) = \left( \frac{1}{d} \sum_{j=1}^d \cos k_j \right)^n}, \quad \forall n \geq 0. \quad (n=0 \text{ is also true}).$$

The characteristic function is useful for the study of recurrence as we can connect it with

$$P(\vec{S}_n = \vec{X}) = P_{\vec{S} \vec{X}}^{(n)} :$$

$$\Phi_n(\vec{k}) = \mathbb{E}[e^{i\vec{k} \cdot \vec{S}_n}] = \sum_{\vec{x} \in \mathbb{Z}^d} e^{i\vec{k} \cdot \vec{x}} \underbrace{P(\vec{S}_n = \vec{x})}_{= P_{\vec{S} \vec{X}}^{(n)}}.$$

Using Fourier inversion, we will show:

$$\boxed{P_{\vec{S} \vec{X}}^{(n)} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \Phi_n(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} d\vec{k}}$$

Proof: We first introduce Fourier series.

Let  $f(\vec{x}) = \sum_{\vec{x} \in \mathbb{Z}^d} e^{i\vec{k}\vec{x}} c_{\vec{x}}$   $c_{\vec{x}} \in \mathbb{C}$  is the  $\vec{x}$ -th Fourier coefficient.

Fourier inversion formula:

$$c_{\vec{x}} = \int_{[-\pi, \pi]^d} \frac{f(\vec{k}) e^{-i\vec{k}\vec{x}}}{(2\pi)^d} d\vec{k}$$

Therefore, since

$$\Phi_n(\vec{x}) = \sum_{\vec{x} \in \mathbb{Z}^d} e^{i\vec{k}\vec{x}} \underbrace{p_{\vec{0}\vec{x}}^{(n)}}_{\text{Fourier coefficient}}$$

We have that  $p_{\vec{0}\vec{x}}^{(n)} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \Phi_n(\vec{k}) e^{-i\vec{k}\vec{x}} d\vec{k}$ .

In particular, setting  $\vec{x} = \vec{0}$ , we get:

$$\begin{aligned} p_{\vec{0}\vec{0}}^{(n)} &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \Phi_n(\vec{k}) d\vec{k} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} (\Phi_1(\vec{k}))^n d\vec{k} \\ &\Rightarrow \sum_{n=0}^{\infty} p_{\vec{0}\vec{0}}^{(n)} = \frac{1}{(2\pi)^d} \sum_{n=0}^{\infty} \int_{[-\pi, \pi]^d} (\Phi_1(\vec{k}))^n d\vec{k} \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \sum_{n=0}^{\infty} (\Phi_1(\vec{k}))^n d\vec{k} \end{aligned}$$

$$\boxed{\sum_{n=0}^{\infty} p_{\vec{0}\vec{0}}^{(n)} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{1 - \Phi_1(\vec{k})} d\vec{k}} \quad (1)$$

Finally,  
We will show this is  $\begin{cases} +\infty & d=1, 2 \\ -\infty & d \geq 3. \end{cases}$

Proof: Recall that  $\overline{\Phi}_k(\vec{r}) = \frac{1}{d} \sum_{j=1}^d \cos k_j$

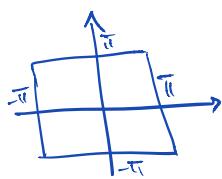
$$\Rightarrow \frac{1}{1 - \overline{\Phi}_k(\vec{r})} = \frac{1}{1 - \frac{1}{d} \sum_{j=1}^d \cos k_j} = \frac{d}{d - \sum_{j=1}^d \cos k_j} = \frac{d}{\sum_{j=1}^d (1 - \cos k_j)}$$

$\Rightarrow$  (1) defines an improper integral at  $\vec{r} = \vec{0}$ .

Close to 0,  $1 - \cos t \sim \frac{1}{2}t^2$  (Taylor's formula:  
 $\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \dots$ )

$\Rightarrow$  (1) is finite if and only if

$$I = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{\sum_{j=1}^d k_j^2} d\vec{k} \quad \text{is finite.}$$

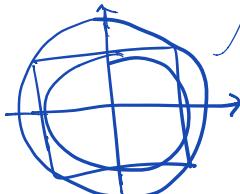
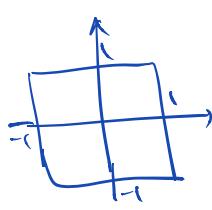


$$M \int_{\|\vec{k}\| \leq 1} \dots \leq \int_{[-1, 1]^d} \dots \leq M' \int_{\|\vec{k}\| \leq 1} \dots$$

Thus,

$$I \propto \int_{\|\vec{k}\| \leq 1} \frac{1}{\sum_{j=1}^d k_j^2} d\vec{k} = \int_{|r| \leq 1} \frac{1}{r^2} d\vec{r},$$

where  $r = \|\vec{k}\| = \sqrt{k_1^2 + \dots + k_d^2}$ .



Now, change coordinates:

$(k_1, \dots, k_d) \rightarrow (r, \theta_1, \dots, \theta_{d-1})$   
hyperspherical coordinates.

(e.g.,  $d=2$ :  $dx dy = r dr d\theta$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$d=3$ :

$$x_1 = r \cos \theta_1$$

$$x_2 = r \sin \theta_1 \cos \theta_2$$

$$x_3 = r \sin \theta_1 \sin \theta_2$$

$$dx_1 dx_2 dx_3 = r^2 dr \sin \theta_1 d\theta_1 d\theta_2$$

)

In general,

$$dx_1 \dots dx_d = r^{d-1} dr f(\theta_1, \dots, \theta_{d-1}) d\theta_1 \dots d\theta_{d-1}$$

and, thus,  $I$  is finite if and only if

$$\int_{|r| \leq 1} \frac{r^{d-1} dr}{r^d} = \int_{|r| \leq 1} \frac{1}{r^{3-d}} dr \text{ is finite.}$$

$$\int_{|r| \leq 1} \frac{1}{r^{3-d}} dr = \begin{cases} \infty & \text{if } 3-d \geq 1 \text{ i.e. } d \leq 2 \\ -\infty & \text{if } 3-d < 1 \text{ i.e. } d > 2 \end{cases}$$

Thus, The value is  $\begin{cases} \text{recurrent if } d=1, 2 \\ \text{transient if } d \geq 3. \end{cases}$