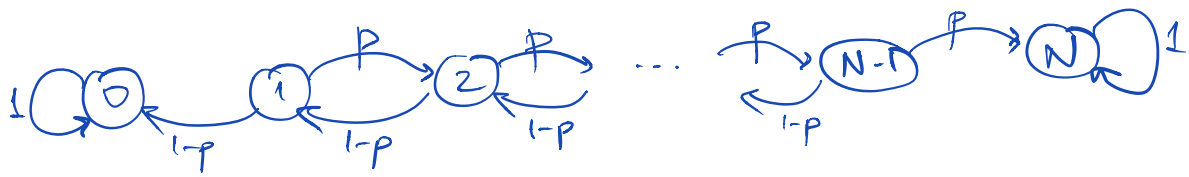
Last time: Gambler's ruin problem



Q: What is $P(\text{lose} \mid X_0 = n) =: \bar{p}(n)$.

We found a recurrence relation:

$$\boxed{\begin{aligned} &\bar{p}(n) = \bar{p}(n+1)\cdot p + \bar{p}(n-1)\cdot(1-p) \\ &\left.\begin{aligned} \bar{p}(0) &= 1 \\ \bar{p}(N) &= 0 \end{aligned}\right\} \text{boundary / initial conditions} \end{aligned}}$$

Characteristic equation:

$$x^2 - \frac{1}{p}x + \frac{(1-p)}{p} = 0$$

solutions $\quad y_1 = 1, \quad y_2 = \frac{1-p}{p}$

Case 1: $y_1 \neq y_2$, i.e. $p \neq \frac{1}{2}$.

Then, $\bar{p}(n) = \alpha y_1^n + \beta y_2^n = \alpha + \beta \cdot \left(\frac{1-p}{p}\right)^n$

Substitute initial conditions:

$$\begin{cases} \bar{p}(0) = 1 \\ \bar{p}(N) = 0 \end{cases} \iff \begin{cases} \alpha + \beta = 1 \\ \alpha + \beta\left(\frac{1-p}{p}\right)^N = 0 \end{cases} \iff \begin{cases} \beta = 1-\alpha \\ \alpha + (1-\alpha)\left(\frac{1-p}{p}\right)^N = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = \dfrac{\left(\frac{1-p}{p}\right)^N}{\left(\frac{1-p}{p}\right)^N - 1} \\ \beta = 1-\alpha = \dfrac{-1}{\left(\frac{1-p}{p}\right)^N - 1} \end{cases}$$

$$\Rightarrow \quad \bar{p}(n) = \frac{\left(\frac{1-p}{p}\right)^N}{\left(\frac{1-p}{p}\right)^N - 1} - \frac{1}{\left(\frac{1-p}{p}\right)^N - 1}\left(\frac{1-p}{p}\right)^n = \boxed{\frac{\left(\frac{1-p}{p}\right)^N - \left(\frac{1-p}{p}\right)^n}{\left(\frac{1-p}{p}\right)^N - 1}}$$

<u>Case 2</u>: $y_1 = y_2$, i.e., $p = \frac{1}{2}$.

Then, $\bar{p}(n) = \alpha y_1^n + \beta n y_1^n = \alpha + \beta n$.

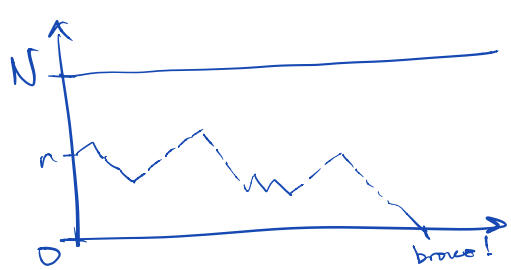
Substitute initial conditions:

$$\begin{cases} \bar{p}(0) = 1 \\ \bar{p}(N) = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 1 \\ \alpha + N\beta = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 1 \\ \beta = -\frac{1}{N} \end{cases}$$

Thus, $\bar{p}(n) = \alpha + \beta n = \boxed{1 - \frac{n}{N}}$.

<u>Remark</u>: We can also interpret the Gambler's ruin problem as a 1-D random walk with absorbing boundaries s.t. we want to find

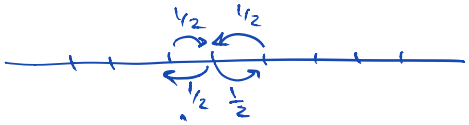
P (hit 0 before we hit N)



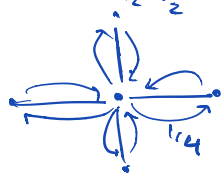
(B). Random walk in $\mathbb{Z}^d$

We consider the uniform r.w. in $\mathbb{Z}^d$

$d=1$:



$d=2$:



$d=3$:

Question: Is the process recurrent or transient?

We study $\sum_{n=0}^{\infty} p_{00}^{(n)}$.

$d=1$: Starting from 0, we can only come back in an even # of steps.

$p_{00}^{(2n)}$ = Prob to move n times backward and n times forward

$$= \underbrace{\left(\frac{1}{2}\right)\times\left(\frac{1}{2}\right)\times\cdots\times\left(\frac{1}{2}\right)}_{2n}\times\underbrace{\binom{2n}{n}}_{\text{Total \# of trajectories choosing } n \text{ times when the trajectory goes forward.}} = \frac{\binom{2n}{n}}{2^{2n}}$$

$$\Rightarrow \sum_{n=0}^{+\infty} p_{00}^{(n)} = \sum_{n=0}^{+\infty}\frac{\binom{2n}{n}}{2^{2n}} = \sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2\,2^{2n}}$$

Stirling formula: $n! \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$

$$\Rightarrow \frac{(2n)!}{(n!)^2\,2^{2n}} \approx \cancel{\left(\frac{2n}{e}\right)^{2n}}\sqrt{2\pi\, 2n}\,\cancel{\left(\frac{n}{e}\right)^{-2n}}\,\frac{1}{2\pi n}\cdot\cancel{\frac{1}{2^{2n}}}$$

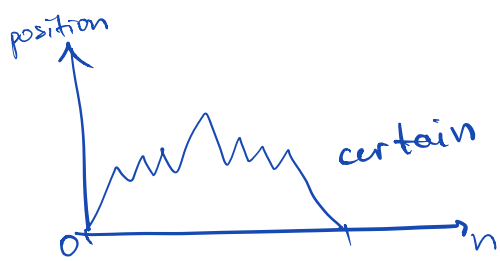
$$\approx \frac{1}{\sqrt{\pi n}} = \frac{1}{(\pi n)^{1/2}} < 1$$

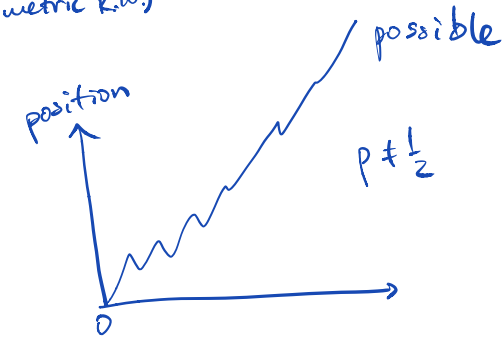
$$\Rightarrow \sum_{n=0}^{+\infty} P_{00}^{(n)} = +\infty$$

$\Rightarrow$ The $\boxed{\text{R.W. is recurrent}}$ when $p = \frac{1}{2}$.

<u>Remark</u>: When $d=1$ and $p \neq \frac{1}{2}$, (assymetric R.W.) we can show that R.W. is transient!

$p = \frac{1}{2}$





- For $d \geq 2$, we can use a similar approach, but the problem becomes more and more complex as d increases, so we are going to use a different approach: characteristic functions.

<u>Notation</u>: unit vectors in $\mathbb{Z}^d$: $\vec{e_j} = (0, \dots, 0, \overset{j^{th}\text{ position}}{1}, 0, \dots, 0)$

(e.g., $d=3$: $\vec{e_1} = (1,0,0)$, $\vec{e_2} = (0,1,0)$, $\vec{e_3} = (0,0,1)$).

So, the walks takes i^{th} step $\vec{X_i}$ with probabilities:

$$P(\vec{X_i} = \vec{e_j}) = P(X_i = -\vec{e_j}) = \frac{1}{2d}.$$

Let $\vec{S_0} = 0$ initial position

$\vec{S_n}$ = position after n steps

$= \vec{X_1} + \vec{X_2} + \dots + \vec{X_n}$ $(n \geq 1)$.

Then, $\vec{S}_{n+1} = \vec{S}_n + \vec{X}_{n+1}$, so

$$P(\vec{S}_{n+1} = \vec{x} + \vec{e}_j \mid \vec{S}_n = \vec{x}) = P(\vec{X}_{n+1} = \vec{e}_j) = \frac{1}{2d},$$

and

$$P(\vec{S}_{n+1} = \vec{x} - \vec{e}_j \mid \vec{S}_n = \vec{x}) = P(\vec{X}_{n+1} = -\vec{e}_j) = \frac{1}{2d}.$$

This defines the transition probabilities for the symmetric R.W. on $\mathbb{Z}^d$.

For $\vec{k} \in \mathbb{R}^d$, $k = (k_1, \dots, k_d)$ we can define $\Phi_n(\vec{k})$ to be the characteristic function of $\vec{S}_n$ evaluated at $\vec{k}$:

$$\Phi_n(\vec{k}) = \mathbb{E}\left[e^{i\vec{k}\cdot\vec{S}_n} \right],$$

Recall Euler's formula:
$e^{i\theta} = \cos\theta + i\sin\theta.$

where $\vec{k}\cdot\vec{S}_n$ is the standard dot product in $\mathbb{R}^d$, i.e.,

$\vec{x}\cdot\vec{y} = x_1 y_1 + \dots + x_n y_n$.