

Last time:

Corollary: If $i \leftrightarrow j$ and i is recurrent, then j is recurrent as well.

To finish the proof, we need the following

Proposition: A state i is $\begin{cases} \text{recurrent if } \sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty \\ \text{transient if } \sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty \end{cases}$

Proof: i is recurrent $\Leftrightarrow N_i = +\infty$ ~~wp.s.~~ $X_0 = i$ wp.s.
 $\mathbb{E}[N_i | X_0 = i] = +\infty$

$$+\infty = \mathbb{E}[N_i | X_0 = i] = \mathbb{E}\left[\underbrace{\sum_{n=0}^{\infty} \mathbb{1}_{(X_n = i)}}_{= N_i} \mid X_0 = i\right]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\left[\underbrace{\mathbb{1}_{(X_n = i)}}_{\sim \text{Bernoulli}(p_{ii}^{(n)})} \mid X_0 = i\right]$$

$$= \sum_{n=0}^{\infty} p_{ii}^{(n)} = +\infty$$

If i is transient, $\mathbb{E}[N_i | X_0 = i] < \infty$

$$\text{i.e., } \sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$$

Consequences:

(i) Recurrence and transience are class properties (all states in a communicating class are either all recurrent or all transient).

(ii) We can relax the hypothesis of the corollary to

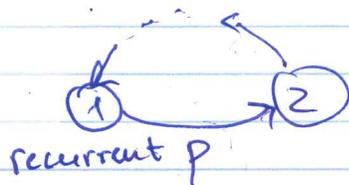
"i is recurrent & j is accessible from i"

(*) implies that j is recurrent"

↳ implies that i is accessible from j

i.e., $i \leftrightarrow j$

Proof: Exercise.



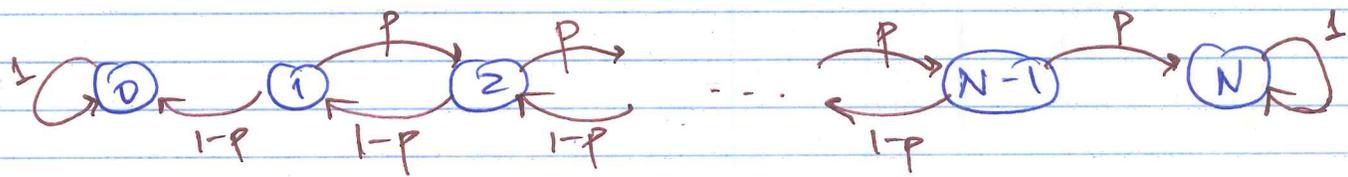
If 1 not acc. from 2, $f_1 \leq 1-p < 1$.

Examples: Gambler's ruin problem & d-dim Random Walk

(A). Gambler's ruin. Smith has \$n, and plays a game many times prob of winning each time is p $0 < p < 1$
→ winning gives him \$1
→ losing takes away \$1

Smith plays until he goes broke or reaches a goal of \$N.

Smith's wealth follows a M.C.



Transition matrix:

	0	1	2	...	N-1	N
0	1	0	0	...	0	0
1	1-p	0	p	...	0	0
2	0	1-p	0	p	...	0
...						
N-1					1-p	0
N	0	...	0	0	0	1

This a second-order recurrence relation

Aside: X_0, X_1, X_2 0, 1, 1, 2, 3, 5

$$\underline{X_{n+1} = \alpha X_n + \beta X_{n-1}}$$

$$X_0 = a$$

$$X_1 = b$$

~~Example~~ Problem: Give a formula for X_n

Example: Fibonacci 0, 1, 1, 2, 3, 5, ...

$$X_{n+1} = X_n + X_{n-1}$$

$$X_0 = 0$$

$$X_1 = 1$$

Solution: characteristic equation

$$x^2 - \alpha x - \beta = 0$$

solve ~~the~~ get ~~the~~ y_1, y_2

formula:

$$\boxed{\text{if } y_1 \neq y_2} \quad \boxed{X_n = s y_1^n + t y_2^n} \quad \text{for some } s, t \in \mathbb{R}$$

$$\text{Plug in } \begin{cases} X_0 = a \rightsquigarrow s + t = a \\ X_1 = b \rightsquigarrow s y_1 + t y_2 = b \end{cases} \text{ find } s, t.$$

$$\boxed{\text{if } y_1 = y_2} \quad \boxed{X_n = s \cdot y_1^n + t \cdot n \cdot y_1^n}$$

