Last time we saw that

If $i \longleftrightarrow j$ and i is recurrent, then j is recurrent as well.

To finish the proof, we need the following

Proposition.

We can characterize recurrence and transience using n-step tail probabilities:

<u>Proposition</u>: i is $\begin{cases} \text{recurrent} & \text{if } \sum_{n=0}^{+\infty} p_{ii}^{(n)} = \infty \\ \text{transient} & \text{if } \sum_{n=0}^{+\infty} p_{ii}^{(n)} < \infty. \end{cases}$

<u>Proof</u>: $\mathbb{E}[N_i | X_0 = i] = \sum_{n=0}^{\infty} \mathbb{E}(\mathbb{1}_{(X_n = i)} | X_0 = i) = \sum_{n=0}^{\infty} P(X_n = i | X_0 = i)$
$= \sum_{n=0}^{\infty} p_{ii}^{(n)}.$ ■

<u>Consequences</u>: (i) Recurrence & Transience are class properties (all states in a communicating class are either all recurrent or all transient)

(ii) We can relax the hypothesis of the corollary as "i recurrent & j accessible from i"

(If i is not accessible from j, $f_i \le 1 - p_{ij}^{(m)} < 1$ for some m).

<u>Examples: Gambler's ruin problem & Random Walk</u>

(Ross, Example 4.19, Sec. 4.5.1)

We use the concepts introduced so far to study fundamental examples of stochastic processes.

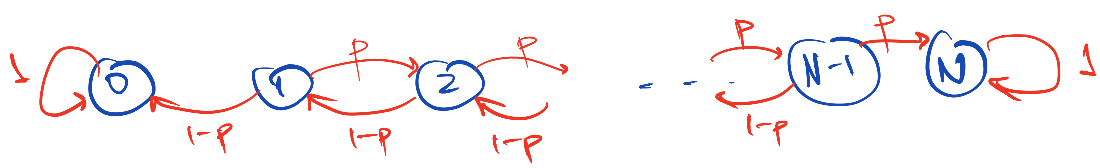
Ⓐ. <u>Gambler's ruin</u>: Smith has \$$n$, and plays a game with probability of winning $0<p<1$;

winning gives him \$1

losing takes away \$1.

Smith plays until he gets broke or reaches a goal of having \$$N$, $N \geq n$.

Smith's wealth follows a M.C.



<u>Transition matrix</u>:

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & \cdots & N-1 & N \\ \hline 0 & 1 & 0 & 0 & \cdots & & 0 \\ 1 & 1-p & 0 & p & 0 & \cdots & 0 \\ & 0 & 1-p & 0 & p & \cdots & 0 \\ & & & & \ddots & & \\ N-1 & 0 & \cdots & 0 & 1-p & 0 & p \\ N & 0 & \cdots & 0 & 0 & 0 & 1 \end{array}$$

period 2

3 communicating classes $\{0\}, \{N\}, \{1, \dots, N-1\}$.

aperiodic recurrent

transient for example, $f_{N-1} < 1-p < 1$

<u>Consequence</u>: At the end of the game, either Smith wins or he goes broke, and probability that game finishes $=1$.

Question: What is $P(\underbrace{\text{Smith goes broke}}_{R} \mid X_0 = n)$?

Let $p(n) = P(R \mid X_0 = n)$.

By definition, $p(0) = 1$ and $p(N) = 0$

For, $1 \le n \le N-1$,

$$p(n) = P(R \mid X_1 = n+1) \cdot P(X_1 = n+1 \mid X_0 = n)$$
$$+ P(R \mid X_1 = n-1) \cdot P(X_1 = n-1 \mid X_0 = n)$$
$$= p(n+1) \cdot p + p(n-1) \cdot (1-p)$$

This is a second order linear recurrence relation.

To solve it, we consider the characteristic equation

$p X^2 - X + (1-p) = 0$ and solve it:

$pX(X-1) - (1-p)(X-1) = 0$ $\qquad X_1 = 1$

$(X-1)(pX - (1-p)) = 0$ $\qquad X_2 = \frac{1-p}{p} = a$

Case 1: $X_1 \neq X_2$. Then $p(n) = \alpha X_1^n + \beta X_2^n$, where α and β are determined by

$$\begin{cases} p(0) = 1 \\ p(N) = 0 \end{cases} \Rightarrow \begin{cases} p(0) = \alpha + \beta = 1 \\ p(N) = \alpha \cdot 1^N + \beta \left(\frac{1-p}{p}\right)^N = 0 \end{cases}$$

$$\Rightarrow \beta = 1 - \alpha \quad \text{and} \quad \alpha + (1-\alpha)\left(\frac{1-p}{p}\right)^N = 0$$

$$\Rightarrow \alpha = \frac{\left(\frac{1-p}{p}\right)^N}{1 - \left(\frac{1-p}{p}\right)^N}$$

and $p(n) = \dfrac{a^N - a^n}{a^N - 1} = \left(1 - \dfrac{a^n - 1}{a^N - 1}\right).$

<u>Case 2</u> : $p = \frac{1}{2}$, $\Rightarrow$ $x_1 = x_2 = 1$, and

$$p(n) = \alpha x_1^n + \beta n x_1^n = \alpha + \beta n$$

$$p(0) = 1 = \alpha$$

$$p(N) = 0 = \alpha + N\beta$$

$$\Rightarrow \begin{cases} \alpha = 1 \\ \beta = -\frac{1}{N} \end{cases}$$

and $\boxed{p(n) = 1 - \frac{n}{N}}$.