

Announcements: • HW1 due now

- HW2 is online and is due next Friday in class.
- Piazza can be accessed from Canvas

Last time: • period of a state i : largest integer d s.t. $p_{ii}^{(n)} = 0$

whenever n is not divisible by d . Remark: if $p_{ii}^{(n)} = 0$ for all $n \geq 1$, then the period is undefined.

If $d=1$, aperiodic.

• Recurrence and transience: $f_i = P(X_n = i \text{ for some } n \geq 1 | X_0 = i)$

state i is $\begin{cases} \text{recurrent} & \text{if } f_i = 1 \\ \text{transient} & \text{if } f_i < 1. \end{cases}$

Defines the ability of a M.C. to reenter a state.

• $N_i = \#\{n \geq 0 : X_n = i\}$ r.v. on $\mathbb{N} \cup \{\infty\}$.

Proposition: (i) If i is recurrent, then $P(N_i = \infty | X_0 = i) = 1$

(ii) if i is transient & $X_0 = i$, then $N_i \sim \text{Geom}(1-f_i)$,
(i.e., $P(N_i = m) = f_i^{m-1} (1-f_i)$.)

In particular, $\mathbb{E}[N_i] = \frac{1}{1-f_i}$.

Proof: (i) If $f_i = 1 \Rightarrow$ the process returns to i , so

$$\hookrightarrow P(N_i \geq 1 | X_0 = i) = 1, \text{ and}$$

it essentially starts fresh. But that means it returns to i again with probability 1 $\Rightarrow P(N_i \geq 2 | X_0 = i) = 1$, etc.

$$\Rightarrow \forall m, \quad P(N_i \geq m | X_0 = i) = 1 \Rightarrow P(N_i = +\infty | X_0 = i) = 0$$

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$$\Rightarrow P(N_i = +\infty | X_0 = i) = 1.$$

(ii). $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) = f_i < 1.$

$P(N_i = 1) = 1 - f_i$ (i.e. probability it doesn't come back to state i).

$$P(N_i = 2) = f_i(1 - f_i)$$

\downarrow
returns once, but doesn't return again.

$$P(N_i = 3) = f_i^2(1 - f_i)$$

\downarrow
returns twice, but doesn't return again.
 $\# \text{ trials until success.}$

$$\therefore P(N_i = m) = f_i^{m-1} (1 - f_i).$$

$$\Rightarrow \mathbb{E}[N_i | X_0 = i] = \frac{1}{1 - f_i} \Rightarrow P(N_i = +\infty | X_0 = i) = 0.$$

Consequences: (i) If i is recurrent, $\mathbb{E}[N_i | X_0 = i] = +\infty$
(ii) if the state space is finite, then some state must be recurrent.

Proof: If not, all states are transient, so there is no state visited after a certain time. Contradiction!

$$\sum_{i \in S} N_i = +\infty \Rightarrow \sum_{i \in S} \mathbb{E}[N_i] = +\infty$$

$\Rightarrow \mathbb{E}[N_i] = +\infty \text{ for some } i.$

We now characterize recurrence and transience using n-step tail probabilities:

Proposition: i is $\begin{cases} \text{recurrent} & \text{if } \sum_{n=0}^{+\infty} p_{ii}^{(n)} = \infty \\ \text{transient} & \text{if } \sum_{n=0}^{+\infty} p_{ii}^{(n)} < \infty. \end{cases}$

$$\text{Proof: } \mathbb{E}[N_i | X_0 = i] = \sum_{n=0}^{+\infty} \mathbb{E}(1_{(X_n=i)} | X_0 = i) = \sum_{n=0}^{+\infty} P(X_n = i | X_0 = i)$$

$= \sum_{n=0}^{+\infty} p_{ii}^{(n)}. \blacksquare$

Corollary: If i is recurrent, and $i \leftrightarrow j$, then j is recurrent.

Proof: $i \leftrightarrow j \Leftrightarrow \exists k, m: p_{ij}^{(k)} > 0 \text{ and } p_{ji}^{(m)} > 0$

$$\text{and } \forall n \quad p_{ij}^{(n+k)} = \sum_{s,t} p_{js}^{(m)} p_{st}^{(n)} p_{ti}^{(k)}$$

$$\geq p_{ji}^{(m)} p_{ii}^{(n)} p_{ij}^{(k)}.$$

$$\Rightarrow \sum_{N=0}^{+\infty} p_{ij}^{(N)} \geq \sum_{n=0}^{+\infty} p_{ij}^{(m+n+k)} \geq \sum_{n=0}^{+\infty} p_{ji}^{(m)} p_{ii}^{(n)} p_{ij}^{(k)}$$

$$= p_{ji}^{(m)} \underbrace{\left(\sum_{n=0}^{+\infty} p_{ii}^{(n)} \right)}_{=+\infty \text{ since } i \text{ is recurrent.}} p_{ij}^{(k)} = +\infty$$

$$\Rightarrow \sum_{N=0}^{+\infty} p_{ij}^{(N)} = +\infty. \Rightarrow j \text{ is recurrent.}$$