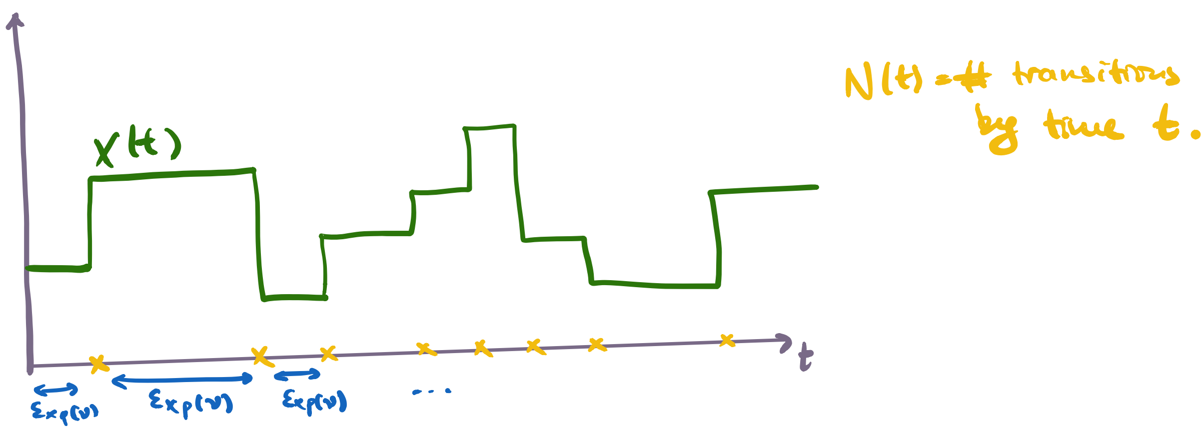
Uniformization (Section 6.7 in Ross)

- Let $\{X(t), t \geq 0\}$ be a CTMC for which $\nu_i = \nu$ for all states i. (i.e., $T_i \sim Exp(\nu)$ $\forall i$)

Goal: give a formula for $P_{ij}(t)$.

Let $N(t) = \#$ state transitions by time t.



Since time between 2 state transitions of $X(t)$ is $Exp(\nu)$, then $N(t)$ is a Poisson process of rate ν!

Transition probabilities $P_{ij}(t)$ for $X(t)$ satisfy:

$$P_{ij}(t) = P(X(t)=j \mid X(0)=i)$$

split according to value of $N(t)$

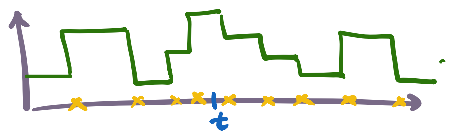
$$= \sum_{n=0}^{\infty} P(X(t)=j, N(t)=n \mid X(0)=i)$$

condition on $N(t)$

$$= \sum_{n=0}^{\infty} P(X(t)=j \mid X(0)=i, N(t)=n) \overbrace{P(N(t)=n \mid X(0)=i)}^{\sim \text{Poisson}(t\nu)}$$

$$= \sum_{n=0}^{\infty} P(X(t)=j \mid X(0)=i, N(t)=n)\, e^{-t\nu} \frac{(\nu t)^n}{n!}$$

What is $P(X(t)=j \mid X(0)=i, N(t)=n)$?



Since the distribution of time spent in each state k is the same for all k, given there were n transitions:

$$P(X(t)=j \mid X(0)=i, N(t)=n) = P^n_{ij},$$

where P^n is the n-step transition probability matrix of the embedded chain.

Therefore,
$$\boxed{P_{ij}(t) = \sum_{n=0}^{\infty} P^n_{ij} \, e^{-\nu t} \frac{(\nu t)^n}{n!}}.$$

This is often useful for approximating $P_{ij}(t)$ by taking a partial sum.

- How can we turn any CTMC into one with $\nu_i = \nu$?

Consider a CTMC $X(t)$ for which the ν_i are bounded, i.e., $\exists \nu$ s.t. $\nu_i \leq \nu \; \forall i$.

When in state i $X(t)$ leaves at rate ν_i. This is equivalent to supposing the transitions occur at rate ν (i.e. more often), but

- only the fraction $\frac{\nu_i}{\nu}$ of these transitions are real (i.e. they leave state i)
- the remaining $1-\frac{\nu_i}{\nu}$ "transition" to state i itself

Thus, $X(t)$

- spends $\text{Exp}(\nu)$ in state i, and
- transitions to state j with probability:

$$P^*_{ij} = \begin{cases} 1 - \dfrac{\nu_i}{\nu}, & \text{if } i=j \\[2ex] \dfrac{\nu_i}{\nu} \cdot P_{ij}, & \text{if } i \neq j \end{cases}$$

Thus, the transition probabilities can be combuted as

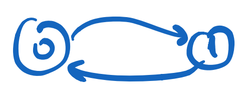
$$\boxed{P_{ij}(t) = \sum_{n=0}^{\infty} P^{*n}_{ij} \, e^{-\nu t} \frac{(\nu t)^n}{n!}},$$

where P^{*n} is the n-step transition matrix of a discrete-time MC with transition matrix P^*.

This technique is called uniformization.

Example (6.11 textbook): Recall the 2-state CTMC

state 0: working, state 1: broken



time to breakdown $\sim \text{Exp}(\lambda)$ $\quad \nu_0 = \lambda$ $\quad P_{01} = 1$

time to repair $\sim \text{Exp}(\mu)$ $\quad \nu_1 = \mu$ $\quad P_{10} = 1$

Q: Find $P_{ij}(t)$.

Solution: Let $\nu = \lambda + \mu \Rightarrow \nu \geq \lambda, \mu$.

Uniformized transition probabilities: $\quad P^*_{00} = 1 - \dfrac{\nu_0}{\nu} = \dfrac{\mu}{\lambda+\mu}$; $P^*_{01} = \dfrac{\lambda}{\lambda+\mu}$

$$P^*_{10} = \frac{\nu_1}{\nu} = \frac{\mu}{\lambda+\mu} \; ; \quad P^*_{11} = \frac{\lambda}{\lambda+\mu}.$$

$$P^* = \frac{1}{\lambda+\mu}\begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix}. \quad (P^*)^2 = \frac{1}{(\lambda+\mu)^2}\begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix}\begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix}$$

$$= \frac{1}{(\lambda+\mu)^2}\begin{pmatrix} \mu(\lambda+\mu) & \lambda(\lambda+\mu) \\ \mu(\lambda+\mu) & \lambda(\lambda+\mu) \end{pmatrix} = \frac{1}{\lambda+\mu}\begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix}$$

$$= P^*.$$

$$\Rightarrow \quad P^{*n} = P^* \quad \forall n \geq 1.$$

$$\Rightarrow P_{ij}(t) = \sum_{n=0}^{\infty} \overbrace{P^{*n}_{ij}}^{=P^*_{ij} \text{ if } n \geq 1} e^{-(\lambda+\mu)t} \frac{((\lambda+\mu)t)^n}{n!}$$

$$= e^{-(\lambda+\mu)t} + \sum_{n=1}^{\infty} P^*_{ij} e^{-(\lambda+\mu)t} \frac{((\lambda+\mu)t)^n}{n!}$$

$$= e^{-(\lambda+\mu)t} + P^*_{ij} e^{-(\lambda+\mu)t} \underbrace{\sum_{n=1}^{\infty} \frac{((\lambda+\mu)t)^n}{n!}}_{e^{(\lambda+\mu)t} - 1}$$

$$= e^{-(\lambda+\mu)t} + P^*_{ij} e^{-(\lambda+\mu)t}\left(e^{(\lambda+\mu)t} - 1\right)$$

$$\boxed{P_{ij}(t) = e^{-(\lambda+\mu)t} + P^*_{ij}\left(1 - e^{-(\lambda+\mu)t}\right).}$$

$$P^* = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ 1 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{array}$$

Therefore, $P_{00}(t) = e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right)$

$$= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}.$$

$\vdots$

We recover the same formulas as we found in lecture 29 from March 20, when we used the Kolmogorov backward and forward equations!

* Please fill in the course evaluations!