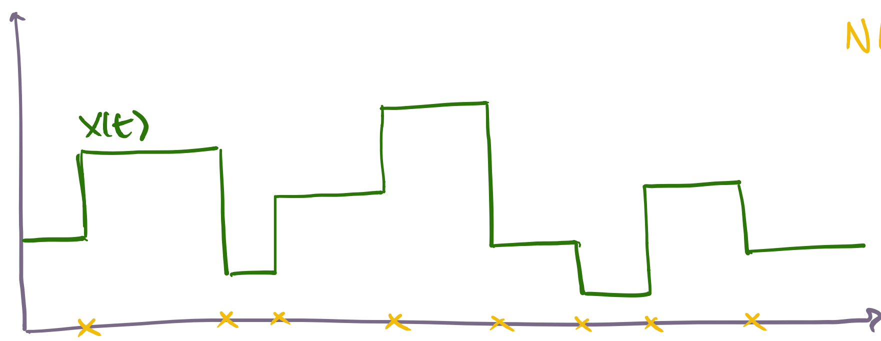
Uniformization (Section 6.7)

- Let $\{X(t), t \geq 0\}$ be a CTMC in which the mean time spent in a state is the same for all states, i.e., $\nu_i = \nu$ for all i.

Let $N(t) = \#$ of state transitions by time t.

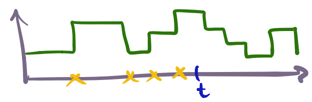


$N(t) = \#$ steps by time t.

Since time between 2 state transitions is $\mathrm{Exp}(\nu)$, then $N(t)$ is a Poisson process of rate ν!

Transition probabilities $P_{ij}(t)$ for $X(t)$, we can condition on $N(t)$:

$$
\begin{aligned}
P_{ij}(t) &= P(X(t)=j \mid X(0)=i) \\
&= \sum_{n=0}^{\infty} P(X(t)=j, N(t)=n \mid X(0)=i) \\
&= \sum_{n=0}^{\infty} P(X(t)=j \mid X(0)=i, N(t)=n) P(\overbrace{N(t)=n \mid X(0)=i}^{\text{Poisson}(t\nu)}) \\
&= \sum_{n=0}^{\infty} P(X(t)=j \mid X(0)=i, N(t)=n)\, e^{-\nu t} \frac{(\nu t)^n}{n!}
\end{aligned}
$$

What is $P(X(t)=j \mid X(0)=i, N(t)=n)$? 

Given that there were n transitions by time t, since the distribution of time spent in each state is the same,

$$P(X(t)=j \mid X(0)=i, N(t)=n) = P_{ij}^n, \text{ where}$$

P^n is the n-step transition probability of the embedded chain.

Therefore, $$\boxed{P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^n \, e^{-\nu t} \frac{(\nu t)^n}{n!}.}$$

This is often useful for computing approximations of $P_{ij}(t)$, by taking a partial sum.

• How can we turn any CTMC into one with $\nu_i = \nu, \forall i$?

Consider a CTMC $X(t)$ for which the ν_i are bounded. Let ν be s.t.

$$\nu_i \leq \nu \quad \forall i.$$

In state i the process leaves at rate ν_i. This is equivalent to supposing the transitions occur at rate ν, (i.e. more often) but
- only the fraction $\frac{\nu_i}{\nu}$ of the transitions are real
- the remaining $1-\frac{\nu_i}{\nu}$ "transition" to state i itself.

Thus, $X(t)$ - spends $Exp(\nu)$ in state i, and

- transitions to j with probability:

$$P^*_{ij} = \begin{cases} 1 - \frac{\nu_i}{\nu}, & \text{if } i=j \\ \frac{\nu_i}{\nu} P_{ij}, & \text{if } i \neq j. \end{cases}$$

Thus, the transition probabilities can be computed as

$$\boxed{P_{ij}(t) = \sum_{n=0}^{\infty} P^{*n}_{ij} \, e^{-\nu t} \frac{(\nu t)^n}{n!},}$$

where P^{*n} are the n-stage transitions of the discrete-time MC with transition probabilities P^*_{ij}.

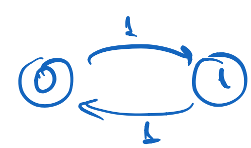
This technique is called uniformization.

Example: Recall the 2-state CTMC (6.11 textbook)

state 0: working, state 1: broken

time to breakdown $\sim Exp(\overset{=\nu_0}{\lambda})$

time to repair $\sim Exp(\overset{=\nu_1}{\mu})$.



$P_{01} = 1$

$P_{10} = 1$

Q: Find $P_{ij}(t)$.

Solution: Let $\nu = \lambda + \mu$

Uniformized version: $P^*_{00} = 1 - \frac{\nu_0}{\nu} = \frac{\mu}{\lambda+\mu}$; $P^*_{01} = \frac{\lambda}{\lambda+\mu}$

$P^*_{10} = \frac{\mu}{\lambda+\mu}$; $P^*_{11} = \frac{\lambda}{\lambda+\mu}$

$$P^* = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \mu & \lambda \\ 1 & \mu & \lambda \end{array} \cdot \frac{1}{\lambda+\mu} \qquad (P^*)^2 = \frac{1}{(\lambda+\mu)^2}\begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix}\begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix}$$

$$= \frac{1}{(\lambda+\mu)^2}\begin{pmatrix} \mu(\lambda+\mu) & \lambda(\lambda+\mu) \\ \mu(\lambda+\mu) & \lambda(\lambda+\mu) \end{pmatrix}$$

$$\Rightarrow P^{*n} = P^*$$

$$= \frac{1}{\lambda+\mu}\begin{pmatrix} \mu & \lambda \\ \mu & \lambda \end{pmatrix} = P^*$$

$$\Rightarrow P_{ij}(t) = \sum_{n=0}^{\infty} \underbrace{P^{*n}_{ij}}_{= P^*_{ij} \text{ if } n \geq 1} e^{-(\lambda+\mu)t} \frac{(t(\lambda+\mu))^n}{n!} = e^{-(\lambda+\mu)t} + \sum_{n=1}^{\infty} P^*_{ij} e^{-(\lambda+\mu)t} \frac{(t(\lambda+\mu))^n}{n!}$$

$$= e^{-(\lambda+\mu)t} + P^*_{ij}\, e^{-(\lambda+\mu)t} \underbrace{\sum_{n=1}^{\infty} \frac{(t(\lambda+\mu))^n}{n!}}_{e^{(\lambda+\mu)t} - 1}$$

$$= e^{-(\lambda+\mu)t} + P^*_{ij}\left(1 - e^{-(\lambda+\mu)t}\right)$$

$$\Rightarrow P^*_{00}(t) = e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right)$$

$$= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$\vdots$

We recover the same formulas as we found in lecture 29 on March 20, when we used the forward and backward Kolmogorov equations!