Time reversibility

Let $X(t)$ be an irreducible CTMC with stationary distribution π. Assume that $X(0) \sim \pi$, so that for all $t > 0$, $X(t) \sim \pi$ as well.

The reversed process: Fix a large t, and consider the reverse chain $\overline{X}(s) := X(t-s)$, for $0 \leq s \leq t$.

Claim: conditional on $\overline{X}(0) = i$, the time $\overline{X}$ spends in state i is $\mathcal{E}xp(\nu_i)$, just like X.

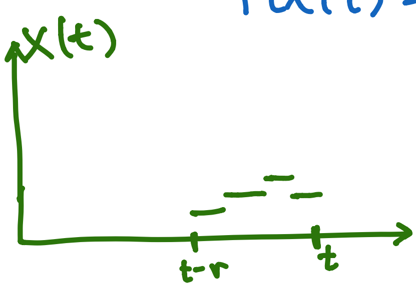
Proof:

$$P(\overline{T}_i \geq r) = P\left(\overline{X}(s) = i \text{ for } s \in [0, r] \,\middle|\, \overline{X}(0) = i\right)$$

$$= P\left(X(t-s) = i \text{ for } s \in [0, r] \,\middle|\, X(t) = i\right)$$

$$= \frac{P(X(t-s) = i \text{ for } s \in [0, r])}{P(X(t) = i)}$$

$$= \frac{P(X(t-s) = i \text{ for } s \in [0, r] \mid X(t-r) = i)\, P(X(t-r) = i)}{P(X(t) = i)}$$



$$= \frac{P(T_i \geq r) \cdot \cancel{\pi_i}}{\cancel{\pi_i}}$$

$$= e^{-\nu_i r} \Rightarrow \underset{\substack{\downarrow \\ \text{time that } \overline{X} \text{ spends in state } i.}}{\overline{T}_i} \sim \mathcal{E}xp(\nu_i)$$

Note that the embedded chain of $\bar{X}$ is the time-reversal of the embedded chain of X.

$\Rightarrow$ By our results for discrete-time MC's, the embedded chain of $\bar{X}$ has transition probabilities

$$\bar{P}_{ij} = \frac{\sigma_j P_{ji}}{\sigma_i}$$, where σ is the stationary distribution of the embedded chain of X.

Recall that last week we showed that

$$\pi_i = \frac{\sigma_i}{\nu_i} \cdot \frac{1}{\underbrace{Z}_{\text{normalizing constant}}}.$$

$$\Rightarrow \sigma_i = \pi_i \nu_i Z$$

$\Rightarrow$ The embedded chain of $\bar{X}$ has transitions:

$$\bar{p}_{ij} = \frac{\pi_j \nu_j \cancel{Z} p_{ji}}{\pi_i \nu_i \cancel{Z}} = \frac{\pi_j \nu_j p_{ji}}{\pi_i \nu_i}.$$

Therefor, $\bar{X}$ is the same as X if and only if

$$\bar{p}_{ij} = p_{ij}, \text{ i.e., } \frac{\pi_j \nu_j p_{ji}}{\pi_i \nu_i} = p_{ij}$$

$$\Leftrightarrow \pi_j \underbrace{\nu_j p_{ji}}_{q_{ji}} = \pi_i \underbrace{\nu_i p_{ij}}_{q_{ij}}$$

$$\Leftrightarrow \pi_j q_{ji} = \pi_i q_{ij}.$$

$\boxed{\text{Recall: } \pi Q = \pi.}$

Thus, we get:

<u>Definition</u>: The CTMC X is time-reversible if it has a stationary distribution π and $\pi_i q_{ij} = \pi_j q_{ji}$ $\forall i, j$.

Like in the discrete case, LHS $= \pi_i q_{ij} = \underbrace{\pi_i}_{\text{prob. of being in } i} \underbrace{\nu_i}_{\text{freq. of leaving } i} \underbrace{p_{ij}}_{\text{prob. of jumping to } j}$ can be interpreted as the frequency of jumping from i to j and RHS $= \pi_j q_{ji} = \pi_j \nu_j p_{ji}$ as the frequency of jumping from j to i.

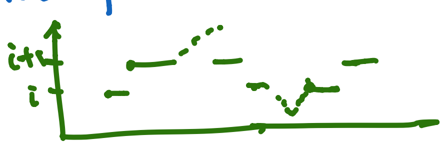
<u>Example</u>: birth and death processes.

<u>Proposition</u>: Any birth and death process with a stationary distribution is time-reversible.

<u>Proof idea</u>: Between 2 consecutive transitions from i to $i+1$

$$i \longrightarrow i+1 \longrightarrow \cdots i+1 \longrightarrow i \cdots \longrightarrow i \longrightarrow i+1$$

the process has to go from $i+1$ to i exactly once.



$\Rightarrow$ In the long run # jumps from i to $i+1$ will equal # jumps from $i+1$ to i.

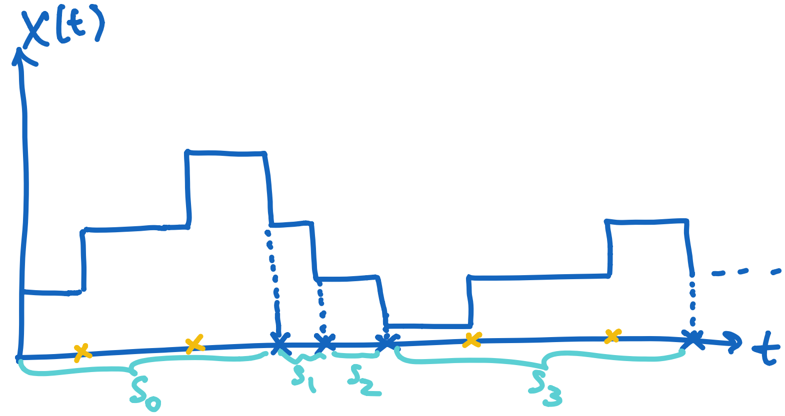
Moreover, for $|i-j| > 1$, both sides of $\pi_i q_{ij} = \pi_j q_{ji}$ are 0.

<u>The M/M/s queue</u>: s servers; new customers arrive at a rate-λ Poisson process. When n customers are present, they leave at rate

$$\mu_n = \begin{cases} n\mu & \text{if } n \le s \\ s\mu & \text{if } n > s \quad \text{all servers are busy} \end{cases}$$

<u>Corollary</u>: For the M/M/s queue, if $\lambda < s\mu$, the departures of the clients form a Poisson process of rate λ.

<u>Proof</u>:



First, we show a stationary distribution exists.

$$\pi_i = \underbrace{\boxed{\frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}}}_{r_i} \pi_0 .$$

$$\mu_1 \cdots \mu_i = \mu_1 \cdots \mu_s \mu_{s+1} \cdots \mu_i = \mu^s \cdot s!\, \mu^{i-s} s^{i-s} = \mu^i s!\, s^{i-s}$$

In the case of M/M/s queue:

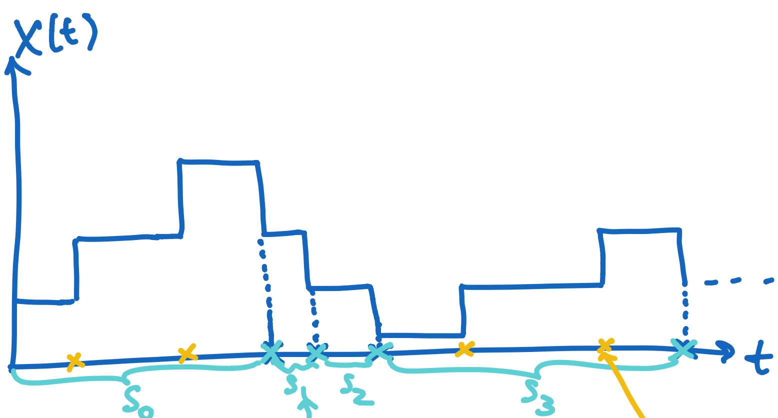
$$r_i = \begin{cases} \dfrac{\lambda^i}{i!\,\mu^i}, & i \le s \\[2ex] \dfrac{\lambda^i s^s}{s!\, s^{i-s} \mu^i s^s}, & i \ge s \end{cases}$$

$$r_i = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} & , i \le s \\ \left(\frac{\lambda}{s\mu}\right)^i \cdot \frac{s^s}{s!} & , i \ge s \end{cases}$$

The stationary distribution π exists if $\sum_{i=1}^{\infty} r_i < \infty$.

$$\sum_{i=1}^{\infty} r_i = \underbrace{\sum_{i=1}^{s} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}}_{\text{const}} + \underbrace{\left(\sum_{i=s+1}^{\infty} \left(\frac{\lambda}{s\mu}\right)^i\right)}_{= \left(\frac{\lambda}{s\mu}\right)^{s+1} \sum_{i=0}^{\infty} \left(\frac{\lambda}{s\mu}\right)^i < \infty \iff \lambda < s\mu.} \frac{s^s}{s!}$$

Thus, stationary distribution π exists $\iff \lambda < s\mu$.



By the Proposition, $X(t)$ is time-reversible. Therefore, the reverse process $\overline{X}(t)$ has the same parameters ν_i and p_{ij} as $X(t)$.

Recall that the times $X(t)$ increases by 1 form a Poisson (λ) process.

$\Rightarrow$ The times $\overline{X}(t)$ increases by 1 form a Poisson(λ) process.

But those are the same as the times $X(t)$ decreases by 1. $\Rightarrow$ The client departures also for a Poisson (λ) process.