Time reversibility

Let's assume that $X(t)$ is irreducible and has a stationary distribution π. Assume also that $X(0) \sim \pi$, so that for all t, $X(t) \sim \pi$.

The reversed process. Now fix a large t, and considere the reverse chain $\bar{X}(s) := X(t-s)$ for $0 \le s \le t$. We first claim that conditional on $\bar{X}(0)$, the time $\bar{X}$ spends in state i is $Exp(\nu_i)$, just like X. Indeed,

$$P(\bar{X}(s) = i \text{ for } s \in [0,r] \mid \bar{X}(0) = i)$$

$$= P(X(t-s) = i \text{ for } s \in [0,r] \mid X(t) = i)$$

$$= P(X(s) = i \text{ for } s \in [t-r,t] \mid X(t) = i)$$

$$= \frac{P(X(s) = i \text{ for } s \in [t-r,t])}{P(X(t) = i)}$$

$$= \frac{P(X(s) = i \text{ for } s \in [t-r,t] \mid X(t-r) = i)\, P(X(t-r) = i)}{P(X(t) = i)}$$

$$= \frac{P(T_i \ge r)\, \pi_i}{\pi_i} = e^{-\nu_i s}.$$

Therefore, the time $\bar{X}$ spends in state i is also $Exp(\nu_i)$.

Moreover, note that the embedded chain of $\bar{X}$ is the time-reversal of the embedded chain of X. Thus, by the results for discret-time MC, the embedded chain of $\bar{X}$ has transitions $q_{ij} = \frac{\sigma_j P_{ji}}{\sigma_i}$, where σ is the stationary distribution of the embedded chain for X, which we showed in the last lecture satisfies $\pi_i = \frac{\sigma_i}{\nu_i} \cdot \frac{1}{Z}$ ← normalizing constant.

$\Rightarrow$ The embedded chain of $\bar{X}$ has transitions

$$q_{ij} = \frac{\pi_j \nu_j Z\, p_{ji}}{\pi_i \nu_i Z} = \frac{\pi_j \nu_j\, p_{ji}}{\pi_i \nu_i}.$$

Therefore, $\bar{X}$ is the same as X if $p_{ij} = q_{ij}$, i.e.,

$$p_{ij} = \frac{\pi_j \nu_j\, p_{ji}}{\pi_i \nu_i} \iff \pi_i \nu_i p_{ij} = \pi_j \nu_j p_{ji}$$

$$\iff \pi_i q_{ij} = \pi_j q_{ji}.$$

<u>Definition:</u> The CTMC is time-reversible if it has a stationary distribution π and $\pi_i q_{ij} = \pi_j q_{ji}$ for all states i and j.

Like in the discrete case, the left-hand side $\pi_i q_{ij} = \pi_i \nu_i p_{ij}$ can be interpreted as the frequency at which the chain jumps from i to j, and the RHS is $\pi_j q_{ji} = \pi_j \nu_j p_{ji}$ the frequency at which it jumps from j to i.

<u>Example: Birth and death processes:</u>

<u>Proposition:</u> Any birth and death process with a stationary distribution is time-reversible.

<u>Proof idea:</u> Between two transitions $i \to i+1 \underbrace{\cdots}_{i+1 \to i} i \to i+1$, the process has to go from $i+1 \to i$ exactly once. Hence, in the long run, it will jump as many times from i to $i+1$ as from $i+1$ to i. Moreover, for $|i-j| > 1$, both sides will be 0.

The M/M/s queue: s servers, now customers arrive at a rate-λ Poisson process. When n customers are present, they leave at rate

$$\mu_n = \begin{cases} n\mu & \text{if } n \leq s \\ s\mu & \text{if } n > s. \text{ all servers are busy} \end{cases}$$

Corollary: For the M/M/s queue, if $\lambda < s\mu$, the departures of the clients form a Poisson process with rate λ.

Proof:

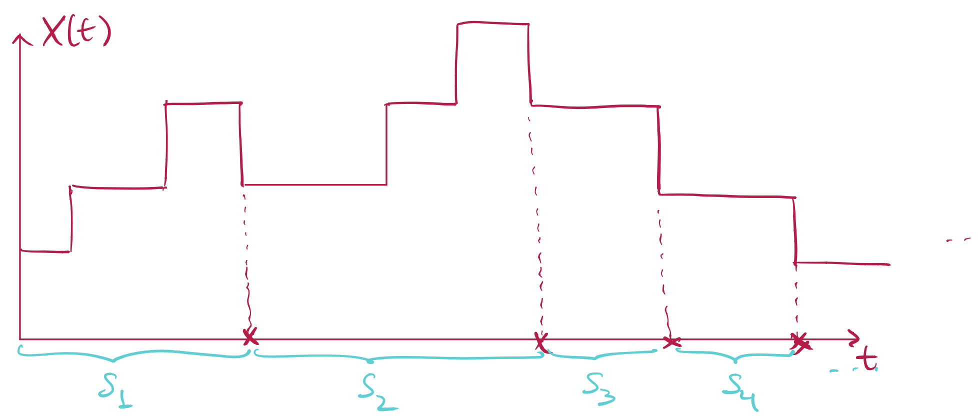
Stationary distribution: $\pi_i = \underbrace{\boxed{\dfrac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}}}_{r_i} \pi_0$

$$r_i = \begin{cases} \dfrac{\lambda^i}{\mu^i \cdot i!} & , i \leq s \\ \dfrac{\lambda^i}{\mu^i s! \, s^{i-s}} & , i \geq s \end{cases}$$

$$\dfrac{\lambda^i}{\mu^i s! \, s^{i-s}} = \left(\frac{\lambda}{s\mu}\right)^i \frac{s^s}{s!}$$

$$\sum r_i = \sum_{i=1}^{s} \left(\frac{\lambda}{\mu}\right)^i \cdot \frac{1}{i!} + \underbrace{\left[\sum_{i=s+1}^{\infty} \left(\frac{\lambda}{s\mu}\right)^i\right]}_{\text{converges} \iff \lambda < s\mu} \frac{s^s}{s!}$$

Let $X(t)$ be the number of customers present at time t.



$Y(t)$ = # of people who have left by time t.

By the proposition, $X(t)$ is time-reversible. Therefore, the reverse process $\overline{X}(s)$ has the same rates ν_i and transition probabilities p_{ij}.

By definition of $X(t)$,

the times X increases by 1 form a Poisson process of rate λ.

Therefore, the times when $\overline{X}(s)$ increases by 1 also form a Poisson(λ) process. But those are exactly the times when $X(t)$ decreases by 1, i.e. a client leaves.