

Definition: The distribution π is stationary

$$P(X(t)=y | X(0) \sim \pi) = \pi_y$$

$$\sum_x \pi_x P_{xy}(t)$$

Theorem: The distribution is stationary if and only if $\pi Q = 0$.

Recall $Q_{ij} = \begin{cases} q_{ij} = \nu_i p_{ij} & \text{if } i \neq j \\ -\nu_i & \text{if } i=j. \end{cases}$

Proof: \Rightarrow : Suppose that π is stationary. Then,

$$\sum_x \pi_x P_{xy}(t) = \pi_y \quad \forall y, \forall t.$$

$$\frac{d}{dt} \left(\sum_x \pi_x P_{xy}(t) \right) = 0 \quad \forall y, \forall t.$$

$$\sum_x \pi_x P'_{xy}(t) = 0$$

Using the Kolmogorov forward equations
 $P'_{xy}(t) = (P(t) \cdot Q)_{xy}$

$$\sum_x \pi_x (P(t) \cdot Q)_{xy} = 0$$

$$\sum_x \pi_x \sum_k P_{xk}(t) Q_{ky} = 0$$

$$\sum_k Q_{ky} \underbrace{\sum_x \pi_x P_{xk}(t)}_{= \pi_k \text{ since } \pi \text{ is stationary}} = 0$$

$$\sum_k Q_{ky} \pi_k = 0 \quad \Leftrightarrow \quad \pi Q = 0.$$

by

≤: Now, suppose that $\pi Q = 0$.

Then, from Kolmogorov backward equations, we have:

$$\begin{aligned} \frac{d}{dt} P(X(t)=y | X(0) \sim \pi) &= \frac{d}{dt} \left(\sum_x \pi_x P_{xy}(t) \right) \\ &= \sum_x \pi_x P'_{xy}(t) = \sum_x \pi_x (Q \cdot P(t))_{xy} \\ &= \sum_x \pi_x \sum_k Q_{xk} P_{ky}(t) = \sum_k P_{ky}(t) \sum_x \pi_x Q_{xk} \end{aligned}$$

$$= \sum_y p_{xy}(t) (\pi Q)_y = 0$$

$$\Rightarrow \frac{d}{dt} P(X(t)=y | X(0) \sim \pi) = 0 \quad \forall t, \forall y.$$

$$\Rightarrow P(X(t)=y | X(0) \sim \pi) = c_y \quad \forall t, \forall y.$$

In particular, when $t=0$,

$$P(X(t)=y | X(0) \sim \pi) = \pi_y$$

$$\Rightarrow c_y = \pi_y \quad \forall y$$

$$\Rightarrow P(X(t)=y | X(0) \sim \pi) = \pi_y,$$

i.e., π is a stationary distribution.

- Remarks :
- Finding the stationary distribution, if it exists, simplifies if the chain is time-reversible.
 - Stationary distr. is key to establishing limiting behaviour.

Definition : (a) For a CTMC $\{X(t), t \geq 0\}$, two states $x, y \in S$ communicate ($x \leftrightarrow y$) if

$P_{xy}(s) > 0$ and $P_{yx}(t) > 0$ for some $s, t \geq 0$.

(b) As communication is an equivalence relation, we say that $X(t)$ is **irreducible** if it only has one communicating class.

Remark: One can show that $x \leftrightarrow y$, then $P_{xy}(t) > 0 \quad \forall t > 0$.

Theorem: Let $\{X(t), t \geq 0\}$ be an irreducible CTMC. Then, either

(1). There is no stationary distribution, and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \underbrace{\mathbf{1}_{\{X(s)=y\}}}_{\text{Proportion of time spent in } y} ds = 1 \text{ almost surely for all } y \in S.$$

or

(2). There is a unique stationary distribution,

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{X(s)=y\}} ds = \pi_y \text{ almost surely for all } y \in S.$$

Proof: Omitted.

Remark: Like for discrete-time Markov Chains we can define analogues of transience, null

and positive recurrence. Then, the alternative (1) and (2) happen in the following cases.

Proposition: Alternative (1) happens when all states are transient or all states are null-recurrent.

Alternative (2): all states are positive recurrent, with mean recurrence time is $-\frac{1}{Q_{yy}\pi_y} = \frac{1}{\gamma_y\pi_y}$.