

Definition: The distribution π is stationary if

$$P(X(t)=y | X(0) \sim \pi) = \pi_y.$$

Here $P(X(t)=y | X(0) \sim \pi) = \sum_x P(X(0)=x | X(0) \sim \pi) P(X(t)=y | X(0)=x)$
 $= \sum_x \pi_x P_{xy}(t).$

Theorem: The distribution π is stationary if and only if $\pi Q = 0$.

Proof: \Rightarrow : Suppose that π is stationary. Then,

$$P(X(t)=y | X(0) \sim \pi) = \left[\sum_x \pi_x P_{xy}(t) = \pi_y \right] \forall y.$$

Taking derivatives w.r.t. t , we get

$$\sum_x \pi_x P'_{xy}(t) = 0 \quad \forall t \text{ and } \forall y.$$

Using the Kolmogorov forward equations, we get

$$\sum_x \pi_x (P(t) \cdot Q)_{xy} = 0$$

$$\Leftrightarrow \sum_x \pi_x \sum_k P(t) Q_{ky} = 0$$

$$\Leftrightarrow \sum_k Q_{ky} \underbrace{\sum_x \pi_x P_{xk}(t)}_{= \pi_k \text{ since } \pi \text{ is stationary!}} = 0$$

$$\Leftrightarrow \sum_k Q_{ky} \pi_k = 0$$

$$\Leftrightarrow \pi Q = 0$$

\Leftarrow : Now, suppose that $\pi Q = 0$

Then, from the Kolmogorov backward equations, we have

$$\begin{aligned}
 \frac{d}{dt} P(X(t)=y | X(0) \sim \pi) &= \frac{d}{dt} \left(\sum_x \pi_x P_{xy}(t) \right) \\
 &= \sum_x \pi_x P_{xy}(t) = \sum_x \pi_x (Q \cdot P(t))_{xy} \\
 &= \sum_x \pi_x \sum_k Q_{xk} P_{ky}(t) \\
 &= \sum_k P_{ky}(t) \sum_x \pi_x Q_{xk} = \sum_k P_{ky}(t) (\pi Q)_{yk} = 0. \\
 \Rightarrow \frac{d}{dt} \left(P(X(t)=y | X(0) \sim \pi) \right) &= 0 \\
 \Rightarrow P(X(t)=y | X(0) \sim \pi) &= c_y \quad \forall t \\
 \text{In particular, when } t=0, \quad P(X(t)=y | X(0) \sim \pi) &= \pi_y \\
 \Rightarrow P(X(t)=y | X(0) \sim \pi) &= \pi_y \quad \forall t \\
 \Rightarrow \pi \text{ is stationary.}
 \end{aligned}$$

Remarks: • As for discrete-time Markov Chains, we'll see that finding the stationary distribution, if it exists, simplifies if the chain is time-reversible.

- The stationary distribution is key to establishing limiting behaviour. Before stating it, we need an analogue of irreducibility.

Definition: (a) For a CTMC $\{X(t), t \geq 0\}$, two states $x, y \in S$ are said to communicate ($x \leftrightarrow y$) if $P_{xy}(s) > 0$ and $P_{yx}(t) > 0$ for some $s, t \geq 0$.
(b) As communication is an equivalence relation, we say that $X(t)$ is irreducible if it has only 1 communicating class.

Remark: One can show that if $x \leftrightarrow y$, then $P_{xy}(t) > 0 \forall t > 0$.

Theorem: Let $\{X(t), t \geq 0\}$ be an irreducible CTMC. Then, either

(1). There is no stationary distribution, and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{1}_{\{X(s)=y\}} ds = 1 \text{ almost surely for all } y \in S$$

Average time spent in state y in the long run.

or

(2) There is a unique stationary distribution, and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{1}_{\{X(s)=y\}} ds = \pi_y \text{ almost surely for all } y \in S.$$

Proof: Omitted.

Remark: Like for discrete-time Markov chains, we can define analogues of transience, null and positive recurrence. Then, the alternatives (1) and (2) from the above theorem happen in the following cases:

Proposition: Alternative (1) in the theorem happens when all states are transient or all states are null-recurrent.

Alternative (2) in the theorem happens when all states are positive recurrent, with mean recurrence time

$$= -\frac{1}{Q_{yy}\pi_y} = \frac{1}{\gamma_y\pi_y}.$$