Recall the Kolmogorov Backward equations:

$$\boxed{P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t).}$$

looking "backward" from $k \to j$

Theorem (Kolmogorov Forward equations):

$$\boxed{P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t) q_{kj} - \nu_j P_{ij}(t).}$$

looking "forward" from $i \to k$

Proof: Same idea as the backward equations, except we now use the Chapman-Kolmogorov eq's as:

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \neq j} P_{ik}(t) P_{kj}(h) - P_{ij}(t).$$

Limiting probabilities

We're interested in the long-term behaviour of the CTMC, and, in particular, in finding if it exists:

$$P_j = \lim_{t \to \infty} P_{ij}(t) \quad \text{, independent of } i.$$

Remark: If $\lim_{t \to \infty} P_{ij}(t) = P_j$ exists, then, $\lim_{t \to \infty} P_{ij}'(t) = 0$

Therefore, using the Kolmogorov backward equations:

$$0 = \lim_{t\to\infty} P_{ij}'(t) = \lim_{t\to\infty} \sum_{k\neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

$$= \sum_{k\neq i} q_{ik} \lim_{t\to\infty} P_{kj}(t) - \nu_i \lim_{t\to\infty} P_{ij}(t)$$

$$= \sum_{k\neq i} q_{ik} P_j - \nu_i P_j$$

$$\Leftrightarrow \quad \nu_i P_j = \underbrace{\left(\sum_{k\neq i} q_{ik}\right)}_{\sum_{k\neq i} \nu_i P_{ik} = \nu_i \underbrace{\sum_{k\neq i} P_{ik}}_{=1} = \nu_i} P_j$$

$$\Leftrightarrow \quad \nu_i P_j = \nu_i P_j$$

This is always true, so the backward equations do not give any additional info.

Using the Kolmogorov forward equations:

$$0 = \lim_{t\to\infty} P_{ij}'(t) = \lim_{t\to\infty} \sum_{k\neq j} P_{ik}(t) q_{kj} - \nu_j P_{ij}(t)$$

$$= \sum_{k \neq j} q_{kj} \lim_{t \to \infty} P_{ik}(t) - \nu_j \lim_{t \to \infty} P_{ij}(t)$$

$$= \sum_{k \neq j} q_{kj} P_k - \nu_j P_j$$

Thus,

$$\begin{cases} \nu_j P_j = \displaystyle\sum_{k \neq j} q_{kj} P_k \\ \displaystyle\sum_j P_j = 1 \end{cases} \qquad (*)$$

Interpretation of $(*)$:

LHS: $\nu_j P_j \rightarrow$ overall rate at which the CTMC leaves state j in the long run.

- P_j: prob of being in state j in the long run
- ν_j: rate of leaving state j

$T_j \sim \text{Exp}(\nu_j)$, $\mathbb{E}[T_j] = \frac{1}{\nu_j}$ hrs

"ν_j times per hr we leave state j"

RHS: $\sum_{k \neq j} q_{kj} P_k \rightarrow$ overall rate at which CTMC enters j!

$q_{kj} = \nu_k P_{kj}$ rate at which CTMC enters j from k.

$\Rightarrow (*)$ means that at equilibrium, the rate at which CTMC leaves j equals rate at which it enters j.

Therefore (*) is a "balance equation".

(*) Can be rewritten as $\begin{cases} \pi Q = 0 \\ \sum \pi_i = 1, \end{cases}$

where $\pi = (P_0, P_1, P_2, \cdots)$ is the limiting prob. distribution

and $$Q = \begin{pmatrix} -\nu_0 & q_{01} & q_{02} & \cdots \\ q_{10} & -\nu_1 & q_{12} & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix}$$

$$\pi Q = (P_0, P_1, P_2, \cdots) \begin{pmatrix} -\nu_0 & q_{01} & q_{02} \cdots \\ q_{10} & -\nu_1 & \cdots \\ q_{20} & \vdots & \ddots \end{pmatrix}$$

$$= \Big(\cdots \underbrace{\sum_k P_k Q_{kj}}_{j\text{th entry}} \cdots \Big).$$

Q is the intensity matrix of the CTMC.

In fact, we can rewrite the Kolmogorov backward and forward equations using the matrix Q!

Kolmogorov backward equations:

$$p'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

$$P(t) := \begin{bmatrix} P_{00}(t) & P_{01}(t) & \cdots \\ P_{10}(t) & \ddots & \end{bmatrix}$$

$$= \sum_k Q_{ik} P_{kj}(t) = \left(Q \cdot P(t)\right)_{ij}$$

$$\Rightarrow \boxed{P'_{ij}(t) = \left(Q \cdot P(t)\right)_{ij}}$$

Kolmogorov forward equations:

$$P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t) q_{kj} - P_{ij}(t) \nu_j$$

$$= \sum_k P_{ik}(t) Q_{kj} = \left(P(t) \cdot Q\right)_{ij}.$$

$$\Rightarrow \boxed{P'_{ij}(t) = \left(P(t) \cdot Q\right)_{ij}}$$

Definition: The distribution π is stationary if

$$P(X(t) = y \mid X(0) \sim \pi) = \pi_y.$$

Here $P(X(t) = y \mid X(0) \sim \pi) = \sum_x P(X(0) = x \mid X(0) \sim \pi) \cdot P(X(t) = y \mid X(0) = x)$

$$= \sum_x \pi_x P_{xy}(t).$$

Thus, π is stationary $\iff$ $\boxed{\sum_x \pi_x P_{xy}(t) = \pi_y \quad \forall t \text{ and } \forall y.}$

<u>Theorem</u>: The distribution π is stationary if and only if $\pi Q = 0$.