

Recall the Kolmogorov Backward equations:

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t).$$

looking "backward" from $k \rightarrow j$

We can similarly derive the Kolmogorov Forward equations:

Theorem: $P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t) q_{kj} - \nu_j P_{ij}(t)$

looking "forward" from $i \rightarrow k$.

Proof: Same idea as the backward equations, except we now write the Chapman-Kolmogorov equations as:

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \neq j} P_{ik}(t) P_{kj}(h) - P_{ij}(t).$$

Remark: The Kolmogorov equations define a set of linear ODE's that in theory are solved by e^{Qt} (matrix exponential). This can only be done explicitly in a few cases, but the Kolmogorov equations are also useful in studying limiting behaviours and stationary distributions.

Limiting probabilities

Like in the discrete case, we're interested in the long-term behaviour of the CTMC, and, in particular, finding if it exists:

$$P_j = \lim_{t \rightarrow \infty} P_{ij}(t) \quad , \text{ independent of } i.$$

Remark: If $\lim_{t \rightarrow \infty} P_{ij}(t)$ exists, then $\lim_{t \rightarrow \infty} P'_{ij}(t) = 0$.

Therefore, using the Kolmogorov backward equation:

$$0 = \lim_{t \rightarrow \infty} P'_{ij}(t) = \lim_{t \rightarrow \infty} \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

$$= \sum_{k \neq i} q_{ik} p_j - v_i p_j$$

$$\Leftrightarrow v_i p_j = \left(\sum_{k \neq i} q_{ik} \right) p_j$$

$$= \left(\sum_{k \neq i} v_i p_{kj} \right) p_j = v_i p_j$$

This is always true, so the backward equations do not give any additional information.

Using the Kolmogorov forward equations:

$$0 = \lim_{t \rightarrow \infty} P'_{ij}(t) = \lim_{t \rightarrow \infty} \sum_{k \neq j} p_{ik}(t) q_{kj} - v_j p_{ij}(t)$$

$$= \sum_{k \neq j} p_k q_{kj} - v_j p_j$$

Thus,

$$\boxed{\begin{cases} v_j p_j = \sum_{k \neq j} p_k q_{kj} \\ \sum_j p_j = 1 \end{cases}} \quad (*)$$

and also

Interpretation of (*): prob of being in state j in the long run

LHS is $v_j p_j$ → overall rate at which the CTMC leaves state j in the long run!

↑ rate of leaving state j

RHS is $\sum_{k \neq j} q_{kj} p_k$ → overall rate at which CTMC enters j !

→ $v_k p_{kj}$ rate at which CTMC enters j from k

$\Rightarrow (*)$ means that at equilibrium, the rate at which CTMC leaves state j equals rate at which it enters state j .

Therefore, $(*)$ is a "balance equation"

This is not yet "detailed balance" (see later), which requires balance for every pair of states: $P_i q_{ij} = P_j q_{ji}$.

$(*)$ can also be written as $\begin{cases} \pi Q = 0 \\ \sum \pi_i = 1 \end{cases}$,

where $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ is the limiting prob. distribution and $Q = \begin{pmatrix} -\nu_1 & q_{21} & q_{31} & \cdots \\ q_{12} & -\nu_2 & q_{32} & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$ is the intensity matrix of the CTMC.

In fact we can rewrite the Kolmogorov forward and backward equations using the matrix Q !

Kolmogorov backward equation

$$\begin{aligned} P'_{ij}(t) &= \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t) & P(t) := \begin{bmatrix} P_{00}(t) & P_{01}(t) & \dots \\ P_{10}(t) & P_{11}(t) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= \sum_k Q_{ik} P_{kj}(t) = (Q \cdot P(t))_{ij} \end{aligned}$$

$$\Rightarrow P'_{ij}(t) = (Q P(t))_{ij}$$

Kolmogorov forward equation

$$\begin{aligned} P'_{ij}(t) &= \sum_{k \neq j} P_{ik}(t) q_{kj} - P_{ij} \nu_j \\ &= \sum_k P_{ik}(t) Q_{kj} = (P(t) \cdot Q)_{ij}. \end{aligned}$$

$$\Rightarrow P'_{ij}(t) = (P(t) \cdot Q)_{ij}$$

We now study stationary distributions.

Definition: The distribution π is stationary if

$$P(X(t)=y | X(0) \sim \pi) = \pi_y.$$

$$\begin{aligned} \text{Here } P(X(t)=y | X(0) \sim \pi) &= \sum_x P(X(0)=x | X(0) \sim \pi) P(X(t)=y | X(0)=x) \\ &= \sum_x \pi_x P_{xy}(t). \end{aligned}$$

Theorem: The distribution π is stationary if and only if $\pi Q = 0$.