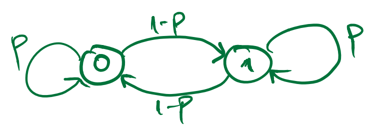
<u>Last time</u>:
- transition matrix: $\mu_{n+1} = \mu_n \cdot \underset{\sim}{P}$.
- stationary distributions: $\mu = \mu \cdot \underset{\sim}{P}$

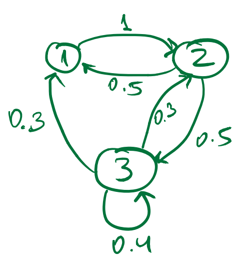
Graph representation of a Markov chain:

We can represent a Markov chain (and its transition matrix) as a directed graph, where each node represents a state, and s.t. we draw an arrow from x to y if $P_{xy} > 0$ with weight P_{xy}. This graph is called a transition diagram.

<u>Example</u> (Ross 4.2): $P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \end{array}$, $S = \{0, 1\}$.



<u>Example</u>: Suppose we have a M.C. on $S = \{1, 2, 3\}$ with transition matrix $P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{pmatrix} \end{array}$. The transition diagram is



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§4.2. Chapman-Kolmogorov equations

$\mu \cdot \underset{\sim}{P}^n$ is the distribution of X_n if μ is the distr. of X_0.

We can generalize this result.

Definition: For $n \in \mathbb{N}$, we define the n-step transition probability:

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i).$$

This defines the n-step transition matrix $(\underset{\sim}{P}^{(n)})_{ij} = p_{ij}^{(n)}$.

Remark: Because we are in the case of a homogeneous Markov Chain,

$$P(X_{n+k} = j \mid X_k = i) = p_{ij}^{(n)}, \quad \forall k.$$

Question: How can we compute the n-step transition probabilities $p_{ij}^{(n)}$?

- $n=0$:

$$P(X_k = j \mid X_k = i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\Rightarrow \underset{\sim}{P}^{(0)} = Id_S.$$

- $n=1$:

$$P(X_{k+1} = j \mid X_k = i) = p_{ij}$$

$$\Rightarrow \underset{\sim}{P}^{(1)} = \underset{\sim}{P}.$$

- In general,

$$p_{ij}^{(n+m)} = P(X_{n+m} = j \mid X_0 = i)$$

$$= \sum_k P(X_{n+m} = j, X_n = k \mid X_0 = i)$$

$$= \sum_k P(X_{n+m} = j \mid X_n = k, \underbrace{\cancel{X_0 = i}}_{M.C.}) P(X_n = k \mid X_0 = i)$$

$$= \sum_k P(X_{n+m} = j \mid X_n = k) P(X_n = k \mid X_0 = i)$$

$$= \sum_k p_{kj}^{(m)} p_{ik}^{(n)}$$

Theorem (Chapman - Kolmogorov equations):

$$P_{ij}^{(n+m)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)}$$

Corollary (Matrix formulation):

- $\underset{\sim}{P}^{(n+m)} = \underset{\sim}{P}^{(n)} \cdot \underset{\sim}{P}^{(m)}$
- $\underset{\sim}{P}^{(n)} = \underset{\sim}{P}^{n}$.

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§4.3. Classification of states

Definition: (1). We say that state j is accessible from i if $P_{ij}^{(n)} > 0$ for some $n \geq 0$.

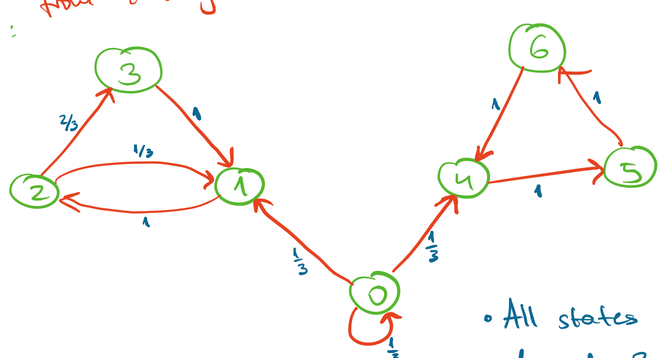
(2). We say that states i and j communicate $(i \leftrightarrow j)$ if i is accessible from j and j is accessible from i.

Remark: The transition diagram is useful to determine accessible and communicating states.

accessible: there is a directed path from i to j

communicating: there is a directed path from i to j and a directed path from j to i.

Example:



$$\underset{\sim}{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

- All states are accessible from 1.
- 1 and 2 communicate
- 4 and 6 communicate
- all states other than 0 do not communicate with 0.

Remark: Communication is an equivalence relation:

(i) $i \longleftrightarrow i$ (since $P^{(0)}_{ii} = 1 > 0$) Reflexivity

(ii) if $i \longleftrightarrow j$, then $j \longleftrightarrow i$ (by def.) symmetry

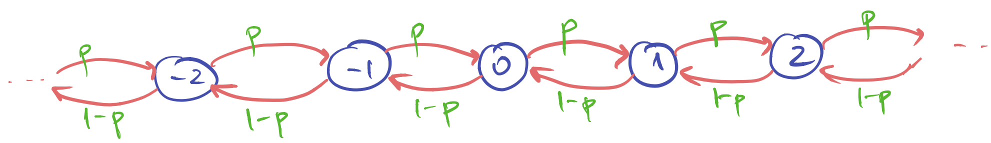
(iii) if $i \longleftrightarrow j$ and $j \longleftrightarrow k$, then $i \longleftrightarrow k$ Transitivity.

Therefore, we can partition the state space into communicating classes.

Example: In the example above, the communicating classes are $\{0\}, \{4,5,6\}, \{1,2,3\}$.

Definition: A Markov chain is irreducible if there is only one communicating class.

Example: 1-D random walk



$(X_n)_{n\geq 0}$ is irreducible if $0 < p < 1$.