

How can we compute  $P_{ij}(t)$ ?

- We will show  $P_{ij}(t)$  obey differential equations.  
To derive these, we will examine the behaviour of  $P_{ij}(t+h)$  as  $h \rightarrow 0$ .

Recall: For  $\{X(t), t \geq 0\}$  a general CTMC, we can characterize  $X(t)$  by

- $v_i$ : rate of transitions from state  $i$ ;  $T_i \sim \text{Exp}(v_i)$
- $P_{ij}$ : probability of transition from  $i$  to  $j$ ,  $i \neq j$   
$$\sum_{j \neq i} P_{ij} = 1.$$

Let  $q_{ij} = v_i P_{ij}$ .

Claim: If we know  $q_{ij}$ , we also know  $v_i$  and  $p_{ij} \forall j \neq i$ .

Proof:  $v_i = v_i \sum_{j \neq i} p_{ij} = \sum_{j \neq i} v_i p_{ij} = \sum_{j \neq i} q_{ij}$ .

Then,  $p_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_{j \neq i} q_{ij}}$ .

- We will assume the  $v_i$ 's are "small enough" (and this is always the case when the state space is finite), so that the probability on an interval  $a \rightarrow b$  contains  $\infty$ -many transitions is 0. Otherwise, the process is explosive, and may not be defined for all times  $t \geq 0$ .

Lemma (a). As  $h \rightarrow 0$ ,  $P_{ii}(h) = 1 - \nu_i h + o(h)$ , i.e.

$$\lim_{h \rightarrow 0} \frac{P_{ii}(h) - P_{ii}(0)}{h} = \lim_{h \rightarrow 0} \frac{P_{ii}(h) - 1}{h} = -\nu_i.$$

Therefore,  $\left. \frac{d}{dh} P_{ii}(h) \right|_{h=0} = -\nu_i$ .

(b). For  $i \neq j$ ,  $P_{ij}(h) = q_{ij}h + o(h)$ , i.e.

$$\lim_{h \rightarrow 0} \frac{P_{ij}(h) - P_{ij}(0)}{h} = \lim_{h \rightarrow 0} \frac{P_{ij}(h)}{h} = q_{ij}.$$

Therefore,  $\left. \frac{d}{dh} P_{ij}(h) \right|_{h=0} = q_{ij}$ .

Proof: •  $P(\text{no jumps in } [0, h] | X(0) = i) = P(T_i > h) = e^{-\nu_i h}$

$$= 1 - \nu_i h + o(h).$$

•  $\frac{P(\text{1 jump to } j \text{ in } [0, h] | X(0) = i)}{= P_{ij} P(T_i < h, T_i + T_j > h)} = \underset{j \neq i}{}$

$= \text{same proof as in Monday's lecture} =$

$$= P_{ij} (\nu_i h + o(h)) = \frac{P_{ij} \nu_i h + o(h)}{}$$

•  $\frac{P(\geq 2 jumps in [0, h] | X(0) = i)}{= \cancel{x} - (\cancel{x} - \nu_i h + o(h)) - \sum_{j \neq i} (P_{ij} \nu_i h + o(h))}$

$$= o(h).$$

Alternative proof of  $P(\geq 2 \text{ jumps} | X(0)=i)$  in lecture notes.

$$(a). P_{ii}(h) = 1 - v_i h + o(h) + o(h) = 1 - v_i h + o(h).$$

$$(b). P_{ij}(h) = p_{ij} v_i h + o(h) + o(h) = \frac{p_{ij} v_i h + o(h)}{q_{ij}} = q_{ij} h + o(h).$$

This lemma leads to the Kolmogorov Backward equations:

Theorem:  $\dot{P}_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$

we look "backward" from  $k \rightarrow j$

Proof: From the Chapman-Kolmogorov equations, we have:

$$\begin{aligned} (P_{ij}(t+h) - P_{ij}(t)) &= \sum_k P_{ik}(h) P_{kj}(t) - P_{ij}(t) \\ &= \sum_{k \neq i} \underbrace{P_{ik}(h) P_{kj}(t)}_{q_{ik}h + o(h)} + \underbrace{P_{ii}(h) P_{ij}(t)}_{1 - v_i h + o(h)} - P_{ij}(t) \end{aligned}$$

$$\begin{aligned} &= \sum_{k \neq i} (q_{ik}h + o(h)) P_{kj}(t) + P_{ij}(t) (1 - v_i h + o(h) - 1) \\ &= \sum_{k \neq i} (q_{ik}h + o(h)) P_{kj}(t) - P_{ij}(t) (v_i h + o(h)). \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} P_{ij}(t) &= \lim_{h \rightarrow 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h \rightarrow 0} \frac{\sum_{k \neq i} (q_{ik}h + o(h)) P_{kj}(t) - P_{ij}(t)(v_i h + o(h))}{h} \\ &= \sum_{k \neq i} q_{ik} P_{kj}(t) - P_{ij}(t) v_i. \quad \checkmark \end{aligned}$$

Remark: Here we assume that

$\lim_{h \rightarrow 0} \frac{\sum_{k \neq i} P_{ik}(h) P_{kj}(t)}{h} < \infty$ . More precisely, we assume a

stronger condition:  $\lim_{h \rightarrow 0} \frac{\sum_{k \neq i} P_{ik}(h)}{h} < \infty$ .

$$\sum_{k \neq i} \frac{P_{ik}(h)}{h} = \sum_{k \neq i} \frac{q_{ik}h + o(h)}{h} = \sum_{k \neq i} q_{ik} + \underbrace{\sum_{k \neq i} \frac{o(h)}{h}}_{\text{we assume this is } 0 \text{ as } h \rightarrow 0}.$$

Such a process is called **conservative**, and this is always the case when the state space is finite.

Example: Birth and death processes.

Recall:  $P_{ij} = 0$  if  $j \neq i \pm 1$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad i \geq 1.$$

$$\gamma_i = \lambda_i + \mu_i$$

$$\Rightarrow q_{ij} = \gamma_i P_{ij} = \begin{cases} \lambda_i, & \text{if } j = i+1 \\ \mu_i, & \text{if } j = i-1 \\ 0, & \text{if } j \neq i \pm 1 \end{cases}.$$

So, the backward equations are:

$$\begin{aligned} \text{If } i \neq 0, \quad p_{ij}'(t) &= \sum_{k \neq i} q_{ik} p_{kj}(t) - \gamma_i p_{ij}(t) \\ &= q_{i,i+1} p_{i+1,j}(t) + q_{i,i-1} p_{i-1,j}(t) - \gamma_i p_{ij}(t) \\ &= \lambda_i p_{i+1,j}(t) + \mu_i p_{i-1,j}(t) - (\lambda_i + \mu_i) p_{ij}(t). \end{aligned}$$

$$\text{If } i = 0, \quad p_{0j}'(t) = q_{0,1} p_{1,j}(t) - \gamma_0 p_{0j}(t) = \lambda_0 p_{1,j}(t) - \lambda_0 p_{0j}(t).$$

