

local behaviour of birth and death processes

$$\begin{array}{ll} \lambda_n & \text{birth rate} \\ \mu_n & \text{death rate} \end{array}$$

Given $X(t) = n$, what is $X(t+h)$ for h small?

$$\rightarrow P(\text{No events happen between } t \text{ and } t+h) = 1 - (\lambda_n + \mu_n)h + o(h)$$

$$T_n \sim \text{Exp}(\lambda_n + \mu_n) \quad = P(T_n > h) = e^{-(\lambda_n + \mu_n)h}$$

$$= 1 - (\lambda_n + \mu_n)h + o(h).$$



$$\rightarrow P(\text{One birth event happens in } [t, t+h]) = \lambda_n h + o(h)$$

$$= p_{n,n+1} \cdot P(T_n < h, T_{n+1} > h)$$

$$= \frac{\lambda_n}{\lambda_n + \mu_n} \cdot \int_0^h (\lambda_n + \mu_n) e^{-(\lambda_n + \mu_n)t} \cdot P(T_{n+1} > h-t) dt$$

$$= \lambda_n \int_0^h e^{-(\lambda_n + \mu_n)t} e^{-(\lambda_{n+1} + \mu_{n+1})(h-t)} dt$$

$$= \lambda_n \int_0^h (1 - (\lambda_n + \mu_n)t + o(h)) (1 - (\lambda_{n+1} + \mu_{n+1})(h-t) + o(h)) dt$$

$$= \lambda_n \int_0^h (1 - (\lambda_{n+1} + \mu_{n+1})h - (\lambda_n + \mu_n - \lambda_{n+1} - \mu_{n+1})t + o(h)) dt$$

$$= \boxed{\lambda_n h + o(h)}.$$

$$\rightarrow P(\text{One death event in } [t, t+h]) = \mu_n h + o(h)$$

$$\rightarrow P(\text{More than two events in } [t, t+h]) = o(h)$$

$$= 1 - (1 - (\lambda_n + \mu_n)h + o(h)) - (\lambda_n h + o(h)) - (\mu_n h + o(h)) = o(h).$$

Given $X(t) = n$,

$$X(t+h) = \begin{cases} n & \text{with prob. } 1 - (\lambda_n + \mu_n)h + o(h) \\ n+1 & \text{with prob. } \lambda_n h + o(h) \\ n-1 & \text{with prob. } \mu_n h + o(h) \\ \geq n+2 \text{ or} \\ \leq n-2 & \text{with prob. } o(h). \end{cases}$$

Example: Mean population size in Linear Growth model with immigration

$$\mu_n = n\mu \quad (n \geq 1), \quad \lambda_n = n\lambda + \Theta \quad (n \geq 0),$$

$$\text{So } X(t+h) = \begin{cases} X(t) - 1 & \text{w.p. } X(t)\mu h + o(h) \\ X(t) & \text{w.p. } 1 - (X(t)(\lambda + \mu) + \Theta)h + o(h) \\ X(t) + 1 & \text{w.p. } (X(t)\lambda + \Theta)h + o(h) \\ \text{something else} & \text{w.p. } o(h). \end{cases}$$

Q: Find $M(t) = \mathbb{E}[X(t)]$

A: idea: we want to use the equations above to find an ODE satisfied by $M(t)$.

$$M(t+h) = \mathbb{E}[X(t+h)] = \mathbb{E}[\mathbb{E}[X(t+h)|X(t)]]$$

$$\text{and } \mathbb{E}[X(t+h)|X(t)] = (X(t)-1)(X(t)\mu h + o(h)) + (X(t)+1).$$

$$\begin{aligned} & \cdot ((X(t)\lambda + \Theta)h + o(h)) + X(t)(1 - X(t)(\lambda + \mu) + \Theta)h + o(h) + o(h) \\ &= \cancel{X(t)^2\mu h} - X(t)\mu h + \cancel{X(t)^2\lambda h} + \cancel{X(t)\Theta h} + X(t)\lambda h + \Theta h + X(t) - \\ & \quad - \cancel{X(t)^2(\lambda + \mu)h} - \cancel{X(t)\Theta h} + o(h) \\ &= X(t) + [X(t)(\lambda - \mu) + \Theta]h + o(h) \end{aligned}$$

$$\text{So, } \frac{M(t+h) - M(t)}{h} = M(t)(\lambda - \mu) + \Theta + \underbrace{o(1)}_{\rightarrow 0 \text{ as } h \rightarrow 0}.$$

As $h \rightarrow 0$, $M'(t) = \underbrace{(\lambda - \mu)}_a M(t) + \Theta \rightarrow 1^{\text{st}}$ order linear ODE
 If you have done MATH 21S,
 solve with
 Integrating factors +
 Variation of parameters
 method

It is a so-called separable equation

$$\Rightarrow \int \frac{dM}{aM + \Theta} = \int ds + \text{const} \quad \leftarrow \text{depends on the initial condition}$$

$$\Rightarrow \frac{1}{a} \ln(aM(t) + \Theta) = t + K$$

$$\text{Assuming } M(0) = i, \quad K = \frac{1}{a} \ln(ai + \Theta),$$

$$\text{so } \frac{1}{a} \ln(aM(t) + \Theta) = t + \frac{1}{a} \ln(ai + \Theta).$$

$$\Rightarrow \frac{1}{a} \ln\left(\frac{aM(t) + \Theta}{ai + \Theta}\right) = t$$

$$aM(t) + \Theta = (ai + \Theta) e^{at}$$

$$\Rightarrow M(t) = i e^{(\lambda - \mu)t} + \frac{\Theta}{\lambda - \mu} (e^{(\lambda - \mu)t} - 1).$$

Remark: • If $\lambda > \mu$, then $M(t)$ grows exponentially
 (single birth rate > single death rate)

- If $\mu > \lambda$, then $\lim_{t \rightarrow \infty} M(t) = \frac{\Theta}{\mu - \lambda}$
 (single birth rate < single death rate)
- If $\mu = \lambda$, $M(t) = \Theta t + i$.

$$\begin{aligned}
 P(T_n < h, T_{n+1} + T_{n+1} > h) &\stackrel{\text{condition on the value of } T_n.}{=} \int_0^h p_{T_n}(t) P(T_{n+1} + T_{n+1} > h | T_n = t) dt \\
 &= \int_0^n (\lambda_n + \mu_n) e^{-t(\lambda_n + \mu_n)} \underbrace{P(T_{n+1} > h - t | T_n = t) dt}_{T_n \text{ and } T_{n+1} \text{ are independent}} \\
 &= \int_0^h (\lambda_n + \mu_n) e^{-t(\lambda_n + \mu_n)} P(T_{n+1} > h - t) dt \\
 &= \dots
 \end{aligned}$$