Recall: • Continuous-time Markov Chains

$$P(X(s+t)=j \mid X(s)=i,\ X(u)=x(u),\ 0 \le u < s)$$
$$= P(X(s+t)=j \mid X(s)=i).$$

• We assume stationarity, i.e. $P(X(s+t)=j \mid X(s)=i)$ is indep. of s.

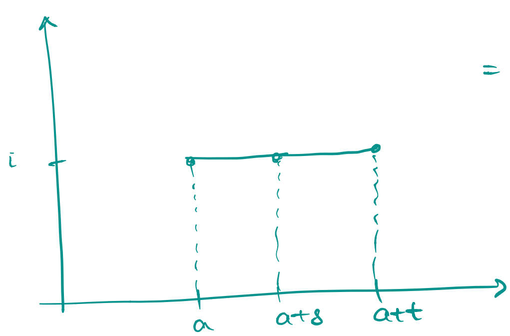
• Consequence of stationarity:

Lemma: Suppose the CTMC is in state i at time a. Let T_i = additional time spent in state i before jumping to another state. Then, T_i is memoryless.

Proof idea: $P(T_i > s+t \mid T_i > s) =$

$$= P(X(r)=i,\ r \in (a+s, a+s+t] \mid X(u)=i,\ u \in [a, a+s])$$
$$= P(X(r)=i,\ r \in (a+s, a+s+t] \mid X(a+s)=i)$$

the same for all a, s by stationarity.



$$\Rightarrow \quad = P(T_i > t).$$

Thus, T_i is $Exp(\nu_i)$ for some value of the parameters ν_i.

Lemma: The amount of time T_i that $X(t)$ spends in state i before jumping to another state is independent of the state it jumps to.

Proof idea: Information as to how long the process has been in state i should be irrelevant to the prediction of the next state by the Markov property:

$$P\left(X(s+t)=j \mid X(s)=i,\ X(u)=i,\ u\in[r,s)\right)$$
$$= P(X(s+t)=j \mid X(s)=i)$$

$\Rightarrow$ We can fully describe a CTMC by:

- $T_i \sim \mathrm{Exp}(v_i)$ with $v_i > 0$
- P_{ij} = transition probabilities from i to j, which satisfy
$$P_{ii}=0 \ \& \ \sum P_{ij} = 1.$$

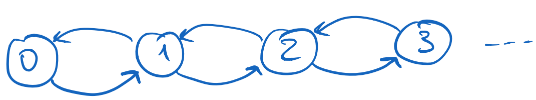
<u>Remark</u>: • the sequence of ~~states~~ visited by the process can thus be described by the discrete-time MC with transition probabilities P_{ij}. This discrete-time MC is called the <u>embedded chain</u>.

- We will later see more precisely how the CTMC behaves locally, i.e., $P(X(t+h)=j \mid X(t)=i)$ as $h\to 0$ (cf. Poisson process)
- Before that, we consider an important family of examples:

Birth and death Processes

We consider $X(t)$ = population size at time t, where:

- (arrival) birth occurs at a rate λ_n when population size is n
- (departure) death occurs at a rate μ_n when population size is n



Given $\lambda_0, \lambda_1, \lambda_2, \dots;\ \mu_0, \mu_1, \mu_2, \dots$, with $\mu_0 = 1$, this defines a CTMC

with $v_0 = \lambda_0$

$v_n = \lambda_n + \mu_n$ time of change in population $= \min(\mathrm{Exp}(\lambda_n), \mathrm{Exp}(\mu_n)) = \mathrm{Exp}(\lambda_n+\mu_n)$

and pos, $p_{n,n+1} = \frac{\lambda_n}{\lambda_n+\mu_n}$, $p_{n,n-1} = \frac{\mu_n}{\lambda_n+\mu_n}$

P(birth time < death time)   P(death time < birth time)

Examples:

(6.2) Poisson process: $\mu_n = 0 \ \forall n$; $\lambda_n = \lambda \ \forall n$

(6.3) Yule process: $\mu_n = 0 \ \forall n$; $\lambda_n = n\lambda \ \forall n$

(6.4) linear growth model with immigration

$$\mu_n = n\mu \quad (n \geq 1), \quad \lambda_n = n\lambda + \theta \quad (n \geq 0)$$

immigration rate.

(6.5) M | M | 1 queue

Markov arrivals Markov deaths # servers

→ A single server receives customers as a Poisson process. Service times are indep $\text{Exp}(\lambda)$ r.v.'s

→ $\lambda_n = \lambda, \quad n \geq 0; \qquad \mu_n = \mu, \quad n \geq 1$

$X(t)$ = # customers in system (waiting or being served) at time t.

(6.6) M | M | S queue (S servers, $S = 1, 2, 3, \ldots$)

$$\lambda_n = \lambda, \quad \mu_n = \begin{cases} n\mu, & 1 \leq n \leq S \\ S\mu, & n \geq S \end{cases}$$

• From what we saw with exponential variables and arrival times, given $X(t) = n$, $X(t+h) = \begin{cases} X(t) - 1 & \text{w.p. } \mu_n h + o(h) \\ X(t) + 1 & \text{w.p. } \lambda_n h + o(h) \end{cases}$

and $X(t+h) > X(t) + 1$ or $< X(t) - 1$ w.p. $o(h)$.

So, $X(t+h) = X(t)$ w.p. $1 - (\lambda_n + \mu_n)h + o(h)$