

- HW7 posted online, due next Wed, March 18
- Midterm on Wed, March 25, in class

Example: Individuals independently get infected following a rate- λ Poisson process^{N(t)}, s.t. symptoms appear after time T since infection, where T is a r.v. with
 → incubation period CDF $G(s)$.

$N_1(t) = \# \text{ individuals infected with symptoms by time } t$
 $N_2(t) = \# \text{ individuals infected with no symptoms by time } t$

$$N(t) = N_1(t) + N_2(t).$$

Goal: Knowing G and $N_1(t)$, estimate $N_2(t)$.
 find $E[N_2(t)]$

Solution: Fix t and reason similarly

to the proposition from last time.

$$\begin{aligned} P(N_1(t) = n_1, N_2(t) = n_2) &= P(N_1(t) = n_1, N_2(t) = n_2 | N(t) = n_1 + n_2) \cdot \\ P\left(\begin{array}{l} N_1(t) = n_1, N_2(t) = n_2 \\ N(t) = n_1 + n_2 \end{array}\right) &\quad \underbrace{\cdot P(N(t) = n_1 + n_2)}_{(\lambda t)^{n_1+n_2} e^{-\lambda t}} \\ &\quad \frac{(n_1+n_2)!}{(n_1+n_2)!} \end{aligned}$$

Conditional on $N(t) = n$, arrival times (i.e., infection times) are independent $\text{Unif}([0, t])$. For each of the n infected people:

$$\begin{aligned} \text{P(shows symptoms by time } t) &= \int_0^t \text{P(shows symptoms | infected at time } s) ds \\ &\quad \circ \text{P.infected at time } s) ds \end{aligned}$$

$P(\text{shows symptoms by time } t \mid \text{infected at time } s) =$

(2)

$$= P(T \leq t-s) = G(t-s)$$

\uparrow
incubation period

$$\textcircled{*} = \int_0^t G(t-s) \cdot \frac{1}{t} ds = \frac{1}{t} \int_0^t G(t-s) ds =: \bar{P}_1$$

Similarly,

$$P(\text{shows no symptoms by time } t) = \frac{1}{t} \int_0^t (1-G(t-s)) ds =: \bar{P}_2$$

$$P(N_1(t) = n_1, N_2(t) = n_2) = \underbrace{P(N_1(t) = n_1, N_2(t) = n_2 \mid N(t) = n_1 + n_2)}_{\text{Binom}(n_1+n_2, \bar{P}_1)} \cdot P(N(t) = n_1 + n_2)$$

$$= \frac{(n_1+n_2)!}{n_1! n_2!} (\bar{P}_1)^{n_1} (\bar{P}_2)^{n_2} \frac{(\lambda t)^{n_1+n_2} e^{-(\lambda t)}}{(n_1+n_2)!} = \dots =$$

$$= \left[\frac{(\lambda t \bar{P}_1)^{n_1} e^{-\lambda t \bar{P}_1}}{n_1!} \right] \cdot \left[\frac{(\lambda t \bar{P}_2)^{n_2} e^{-\lambda t \bar{P}_2}}{n_2!} \right]$$

$$\boxed{\begin{aligned} p(x,y) &= f(x)g(y) \\ \sum_x p(x,y) &= f(x) \end{aligned}}$$

$\Rightarrow N_1(t)$ and $N_2(t)$ are independent

$$\text{and } N_1(t) \sim \text{Poisson}(\lambda t \bar{P}_1) = \text{Poisson}\left(\lambda \int_0^t \frac{G(t-s) ds}{\bar{P}_1(s)}\right)$$

$$N_2(t) \sim \text{Poisson}\left(\lambda \int_0^t \frac{(1-G(t-s)) ds}{\bar{P}_2(s)}\right)$$

$$\Rightarrow E[N_1(t)] = \lambda \int_0^t G(t-s) ds \approx n_1 \text{ know}$$

$$E[N_2(t)] = \lambda \int_0^t (1-G(t-s)) ds \text{ want } \approx \hat{n}_2$$

$$\text{Estimate } \lambda: \quad \hat{\lambda} = \frac{n_1}{\int_0^t G(t-s) ds} \xrightarrow{\# \text{ infected people by true } t \text{ we have recorded}}$$

Estimate n_2 :

$$\hat{n}_2 = \lambda \int_0^t (1 - G(t-s)) ds$$
$$= \frac{n_1}{\frac{\int_0^t (1 - G(t-s)) ds}{\int_0^t G(t-s) ds}}$$

(3)

e.g., $t=16$ days, $G \sim \text{Exp}(\frac{1}{10 \text{ days}})$ $G(s) = 1 - e^{-\lambda s}$

$n_1 = 2000$, then

$$\hat{n}_2 = 2000 \frac{\int_0^{16} e^{-\frac{(16-s)}{10}} ds}{\int_0^{16} (1 - e^{-\frac{(16-s)}{10}}) ds} \dots \approx 114.$$

Continuous-time Markov Chains (Ross, Chapter 6)

CTMC

Generalize the Poisson process

discrete-time MC

Characterized by the continuous-time Markov property:

"Future is independent of past given present!"

Definition: let $\{X(t), t \geq 0\}$ be a collection of r.v.'s, each taking values in $\{0, 1, 2, \dots\}$. This process is a

discrete state space

continuous-time Markov chain (CTMC)

if $P(X(s+t) = j \mid X(s) = i, X(u) = x(u) \text{ for } 0 \leq u < s) =$
 $= P(X(s+t) = j \mid X(s) = i) \text{ for all } s, t \geq 0$

if states i, j

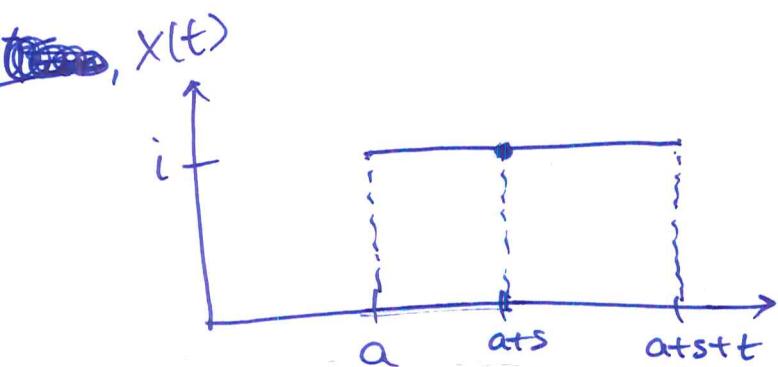
if states $x(u), 0 \leq u \leq s$.

Remarks: • We will always assume stationarity/homogeneity,
 i.e. $P(X(s+t) = j | X(s) = i)$ is independent of s (4)

• Homogeneous CTMC.

- Stationarity has the following consequence on the interarrival times: suppose the CTMC is in state i at time a and let

$T_i = \text{additional time spent in state } i \text{ before leaving state } i$



$$P(T_i > s+t | T_i > s) = P(X(r) = i, \forall r \in [ats, atst+t] \mid X(u) = i, \forall u \in [a, ats])$$

Markov Property

$$\begin{aligned} &\stackrel{\downarrow}{=} P(X(r) = i, \forall r \in [ats, atst+t] \mid X(ats) = i) \\ &\quad \text{does not depend} \\ &\quad \text{on } ats \\ &\quad \text{by stationarity.} \end{aligned}$$