- HW7 is posted online, due on Wed, Mar 18.
- Reminder: midterm is on Wed, Mar 25, in class!!!

Example: Let's get back to our example: individuals independently get infected following a λ-Poisson process, s.t. symptoms appear after time T since infection where T has CDF G.

$N_1(t)$ = # individuals infected with symptoms

$N_2(t)$ = # individuals infected with no symptoms.

$N(t) = N_1(t) + N_2(t)$ = total # of individuals infected.

Goal: Find $\mathbb{E}[N_2(t)]$, which is an estimate of $N_2(t)$.

Solution: Fix t and use previous proposition with $k=2$.

$$P(N_1(t)=n_1, N_2(t)=n_2) = P(N_1(t)=n_1, N_2(t)=n_2 \mid N(t)=n_1+n_2)\, P(N(t)=n_1+n_2).$$

For $0 \le s \le t$, an individual infected at time s has symptoms by time t w.p. $\underbrace{G(t-s)}_{p_1(s)}$ and has no symptoms by time t w.p. $\underbrace{1-G(t-s)}_{p_2(s)}$.

Conditional on $N(t)=n$, arrival times (i.e. infection times) are i.i.d. Unif$([0,t])$. $\Rightarrow$ For each of the n infected people by time t:

$$P(\text{shows symptoms by time } t) = \int_0^t P(\text{shows symptoms by time } t \mid \text{infected at time } s)\, \underbrace{\frac{1}{t}}_{\text{Prob of infected by time } s.}\, ds$$

$$= \frac{1}{t}\int_0^t G(t-s)\,ds =: \bar{p}_1$$

$$P(\text{no symptoms by time } t) = \frac{1}{t}\int_0^t (1-G(t-s))\,ds =: \bar{p}_2$$

Like the proof last time we then get:

$$P(N_1(t)=n_1, N_2(t)=n_2 \mid N(t)=n)P(N(t)=n) = \cdots = \left[\frac{(\lambda t\bar{p}_1)^{n_1}}{n_1!}e^{-\lambda t\bar{p}_1}\right]\left[\frac{(\lambda t\bar{p}_2)^{n_2}}{n_2!}e^{-\lambda t\bar{p}_2}\right]$$

$$\Rightarrow \quad N_1(t) \sim \text{Poisson}\big(\overbrace{\lambda \int_0^t G(t-s)\,ds}^{\lambda t \bar{p}_1}\big)$$

$$N_2(t) \sim \text{Poisson}\big(\underbrace{\lambda \int_0^t 1-G(t-s)\,ds}_{\lambda t \bar{p}_2}\big)$$

$$\Rightarrow \quad \mathbb{E}[N_1(t)] = \lambda \int_0^t \underbrace{G(t-s)}\,ds$$

$$\mathbb{E}[N_2(t)] = \underbrace{\lambda}_{\text{need to estimate.}} \int_0^t \overbrace{(1-G(t-s))}^{\text{known}}\,ds$$

To estimate λ, since we know $N_1(t)$, say $N_1(t) = n_1$ (number of people infected with symptoms by time t), then use

$$n_1 \approx \mathbb{E}[N_1(t)] = \lambda \int_0^t G(t-s)\,ds \quad \text{to get an estimate}$$

$$\hat{\lambda} = \frac{n_1}{\int_0^t G(t-s)\,ds} \Rightarrow \hat{n}_2 = \hat{\lambda} \int_0^t (1-G(t-s))\,ds = \frac{n_1 \int_0^t (1-G(t-s))\,ds}{\int_0^t G(t-s)\,ds}.$$

e.g., $t = 16$ days, $G \sim Exp\left(\frac{1}{10\,\text{days}}\right)$, $n_1 = 2000$, then $\hat{n}_2 = 2000 \dfrac{\int_0^{16} e^{-\frac{(16-s)}{10}}\,ds}{\int_0^{16}\left(1-e^{-\frac{(16-s)}{10}}\right)ds}$

$$= 2000 \cdot \frac{10(1-e^{-16/10})}{16 - 10(1-e^{-16/10})}, \quad \hat{n}_2 \approx 114.$$

This concludes our chapter on the Poisson process. For further details and generalization, see 5.4 in Ross, and, in particular, how (iv) in the definition can be modified to define non-homogeneous processes. Also, for counting processes with inter-arrival times that are not necessarily exponential, see Chapter 7 (Ross) and renewal processes.

3. Continuous-time Markov Chains (Ross, Chapter 6)

We now consider a class of stochastic processes that contains the Poisson process, but is also a continuous-time analogue of the discrete time Markov chain, which has many applications.

The continuous-time Markov chains are characterized by the following continuous Markov property.

"Future is independent of past given present."

Definition: Let $\{X(t), t \geq 0\}$ be a collection of r.v.'s, each taking values in $\{0, 1, 2, \ldots\}$. This process is a continuous-time Markov chain if

discrete state space

$$P(X(s+t) = j \mid X(s) = i, X(u) = x(u) \text{ for } 0 < u < s)$$
$$= P(X(s+t) = j \mid X(s) = i), \quad \forall s, t \geq 0, \forall \text{ states } i, j, \forall x(u) \ 0 \leq u < s.$$

Remark: • We will always assume stationarity, i.e. $P(X(t+s) = j \mid X(s) = i)$ is independent of s, i.e., we consider only homogeneous continuous-time Markov Chains

• Stationarity has the following consequence on interarrival times:

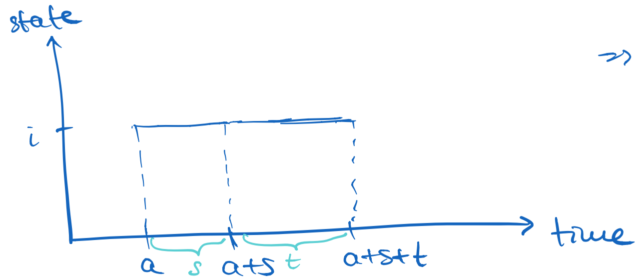
Suppose the MC is in state i at time a and let

T_i = additional time until it leaves i

Then $P(T_i > s+t \mid T_i > s) = P(X(r) = i \ \forall r \in [a+s, a+s+t] \mid X(u) = i \ \forall u \in [a, a+s])$

$= P(X(r) = i \ \forall r \in [a+s, a+s+t] \mid X(a+s) = i)$ → independent of $a+s$ by stationarity

$= P(T_i > t)$ ⇒ same value for all a, s.



$\Rightarrow$ $\boxed{T_i \text{ has the memoryless property}}$

$\Rightarrow$ $T_i \sim Exp(v_i)$ for some $v_i > 0$

- Also, T_i must be independent of the next state the chain jumps to (otherwise the Markov property would be violated, because the time waiting in a state would affect the next state)

$$P(X(s+t) = j \mid X(s) = i,\ X(u) = i \quad \text{for } 0 < u < s)$$
$$= P(X(s+t) = j \mid X(s) = i)$$

$\Rightarrow$ We can fully describe a CTMC by:

- $T_i \sim Exp(v_i)$ with $v_i > 0$
- P_{ij} = transition probabilities from i to j, which satisfy

$$P_{ii} = 0 \ \& \ \sum P_{ij} = 1.$$

Remark: • the sequence of states visited by the process can thus be described by the discrete-time MC with transition probabilities P_{ij}. This discrete-time MC is called the embedded chain.

- We will later see more precisely how the CTMC behaves locally, i.e., $P(X(t+h) = j \mid X(t) = i)$ as $h \to 0$ (cf. Poisson process)
- Before that, we consider an important family of examples:

Birth and death Processes