

Recall: we stated the following

Theorem: Given that  $N(t)=n$ , the  $n$  arrival times  $s_1, \dots, s_n$  have the same distr. as the order statistics of  $n$  indep. r.r.'s which are  $\text{Unif}(\{0, t\})$ . In particular,  ~~$s_1, \dots, s_n | N(t)=n$~~  have pdf

$$f(s_1, \dots, s_n) = \frac{n!}{t^n} \quad \text{for any } 0 \leq s_1 \leq \dots \leq s_n \leq t.$$

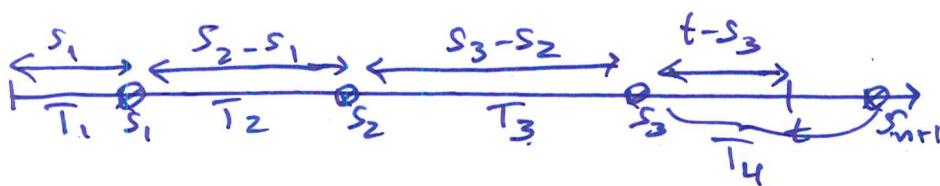


Def: let  $y_1, \dots, y_n$  be r.r.'s. Their order statistics  $y_{(1)}, \dots, y_{(n)}$  are the values of  $y_1, \dots, y_n$  in increasing order.

Last time we showed that if  $y_1, \dots, y_n \sim \text{Unif}(\{0, t\})$  are indep., then the joint pdf of  $y_{(1)}, \dots, y_{(n)}$  is precisely

$$f(y_1, \dots, y_n) = \frac{n!}{t^n} \quad \text{for all } 0 \leq y_1 \leq \dots \leq y_n \leq t.$$

Proof of Theorem: We will show that the joint density of  $s_1, \dots, s_n$  given  $N(t)=n$  is  $f(s_1, \dots, s_n | N(t)=n) = \frac{n!}{t^n}$ , for all  $0 \leq s_1 \leq \dots \leq s_n \leq t$ .



$$f(s_1=s_1, s_2=s_2, \dots, s_n=s_n | N(t)=n)$$

$$= \frac{f(s_1=s_1, \dots, s_n=s_n, \overbrace{N(t)=n}^{n+1 > t})}{P(N(t)=n)} =$$

$$\frac{f(T_1=s_1, T_2=s_2-s_1, \dots, T_n=s_n-s_{n-1}, \overbrace{T_{n+1} > t-s_n}^{\text{condition}})}{P(N(t)=n)}$$

$$\begin{aligned}
 &= \frac{f(T_1=s_1, T_2=s_2-s_1, \dots, T_n=s_n-s_{n-1}, T_{n+1} > t-s_n)}{P(N(t)=n)} \quad (2) \\
 &= \frac{f(T_1=s_1) f(T_2=s_2-s_1) \dots f(T_n=s_n-s_{n-1}) P(T_{n+1} > t-s_n)}{P(N(t)=n)} \\
 &= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2-s_1)} \dots \lambda e^{-\lambda(s_n-s_{n-1})} e^{-\lambda(t-s_n)}}{(\lambda t)^n e^{-\lambda t} / n!} \\
 &= \frac{\cancel{\lambda^n e^{-\lambda t}}}{\cancel{\lambda^n t^n e^{-\lambda t}}} \frac{n!}{t^n}.
 \end{aligned}$$

Now, suppose that we label events of a Poisson process into  $k$  possible types s.t. at each time  $s$ , there is a distribution  $P_i(s)$ ,  $P(\text{event occurring at time } s \text{ is type } i) = P_i(s)$ ,  $i=1, \dots, k$  (the type selection is indep. of what previously occurred).

Proposition: Let  $N_i(t) = \# \text{type } i \text{ events by time } t$ . Then  $N_1(t), \dots, N_k(t)$  are indep. Poisson r.v.'s with means  $\mathbb{E}[N_i(t)] = \lambda \int_0^t P_i(s) ds$ , i.e.  $N_i(t) \sim \text{Poisson}\left(\lambda \int_0^t P_i(s) ds\right)$ .

Remark: If  $P_i(s) = p$ ,  $k=2$

~~$$\mathbb{E}[N_2(t)] = \lambda \int_0^t p ds = \lambda t p.$$~~

Proof:  $(*) = P(N_1(t)=n_1, \dots, N_k(t)=n_k) = P(N_1(t)=n_1, \dots, N_k(t)=n_k, N(t)=n_1+\dots+n_k)$

$$\textcircled{*} = P(N_1(t) = n_1, \dots, N_k(t) = n_k \mid N(t) = \underbrace{n_1 + \dots + n_k}_n) P(N(t) = n, t)$$

Conditional on  $n$  events by time  $t$ , arrival times are indep. and uniform on  $[0, t]$

$P(\text{event is type } i) = \int_0^t P(\text{event is type } i \mid \text{event occurs at time } s) \frac{1}{t} ds$

$$= \int_0^t P_i(s) \frac{1}{t} ds = \frac{1}{t} \int_0^t P_i(s) ds$$

$$\Rightarrow \textcircled{*} = \frac{P(N_1(t) = n_1, \dots, N_k(t) = n_k \mid N(t) = n)}{P(N_1(t) = n_1, \dots, N_k(t) = n_k \mid N(t) = n)} P(N(t) = n)$$

$$= \frac{n!}{n_1! \dots n_k!} (\bar{P}_1)^{n_1} \dots (\bar{P}_k)^{n_k} \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= \frac{\bar{P}_1^{n_1}}{n_1!} \dots \frac{\bar{P}_k^{n_k}}{n_k!} (\lambda t)^{n_1 + \dots + n_k} e^{-\lambda t} \underbrace{(\bar{P}_1 + \dots + \bar{P}_k)}_{=1}$$

$$= \frac{(\bar{P}_1 \lambda t)^{n_1} e^{-\lambda t \bar{P}_1}}{n_1!} \dots \frac{(\bar{P}_k \lambda t)^{n_k} e^{-\lambda t \bar{P}_k}}{n_k!}$$

$\Rightarrow N_1(t), \dots, N_k(t)$  are independent and

$$N_i(t) = \text{Poisson} \left( \lambda \int_0^t P_i(s) ds \right) \frac{\lambda}{\bar{P}_i} t$$

$$f(x, y) = g(x) h(y)$$

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