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①

Proposition [Poisson Thinning]:  $N(t)$  Poisson process of rate  $\lambda$ .

Suppose that its events are independently labeled as:

- type 1 with prob  $p_1$  and
- type 2 with prob  $1-p_1$ .

let  $N_1(t) = \# \text{type 1 events by time } t$

$N_2(t) = \# \text{type 2 events by time } t$ .

Then,  $N_1(t)$  and  $N_2(t)$  are independent Poisson processes of rates  $p_1 \lambda$  and  $(1-p_1)\lambda$ .

Proof: Exercise: show that  $N_1$  and  $N_2$  satisfy the 2<sup>nd</sup> definition, i.e., properties (1)-(4). Also, cf Ross Prop 5.5.

Example: Immigrants arrive at a Poisson process of rate  $10/\text{week}$ . Each immigrant is of English descent with prob.  $\frac{1}{12}$ .

What is the prob. that no people of English descent will immigrate in February (4 weeks)?

Solution: By the previous proposition, # people of English descent who immigrate is a Poisson process of rate

$$\frac{1}{12} 10/\text{week} = \frac{10}{12}/\text{week}$$

$$\Rightarrow P(\# \text{ of people of English descent immigrate in Feb} = 0) \\ \sim \text{Poisson}\left(\frac{10}{12} 4\right) = \text{Poisson}\left(\frac{10}{3}\right) \\ = \frac{\left(\frac{10}{3}\right)^0 e^{-\frac{10}{3}}}{0!} = e^{-10/3}.$$

Remark: The previous proposition easily generalizes to  $k \geq 30$  types of events.

Remark: The labelling here follows a distribution independent of time. But we might be interested in applications where labelling depends on time it occurs.

Example (Ross ex. 5-20): Suppose individuals contract a virus as a rate-1 Poisson process. Once an individual is infected, there is an incubation period before symptoms appear, incubation time is random and indep. for different individuals, and has CDF  $G$ .

$N_1(t) = \# \text{ people infected with symptoms}$

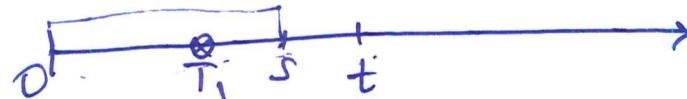
$N_2(t) = \# \text{ people infected with no symptoms}$ .

An important question is, can we estimate  $N_2(t)$ ?

### Conditional distribution of arrival times

let  $N(t)$  be a rate-1 Poisson process.

Q: Suppose we know  $N(t) = 1$ . When did the event in  $[0, t]$  occur?



$$\text{A: } P(T_1 < s \mid N(t) = 1) = \frac{P(T_1 < s, N(t) = 1)}{P(N(t) = 1)} =$$

$$= \frac{P(N(s) = 1, N(t) - N(s) = 0)}{P(N(t) = 1)} = \frac{P(N(s) = 1) P(N(t) - N(s) = 0)}{P(N(t) = 1)}$$

$$= \frac{\lambda^s e^{-\lambda s} (\lambda(t-s))^0 e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}$$

(3)

$\Rightarrow$  pdf of  $(T_1 | N(t)=1)$  is

$$\frac{d}{ds} P(T_1 < s | N(t)=1) = \frac{d}{ds} \frac{s}{t} = \frac{1}{t}$$

$\Rightarrow (T_1 | N(t)=1)$  is uniform on  $[0, t]$ .

Q: More generally, let  $s_1, \dots, s_n$  be the first  $n$  arrival times. What is their joint pdf given  $N(t)=n$ ?

Theorem: Given that  $N(t)=n$ , the  $n$  arrival times  $s_1, \dots, s_n$  have joint pdf  $f(s_1, \dots, s_n | N(t)=n) = \frac{n!}{t^n}, \text{ i.e.,}$

their joint distribution given that  $N(t)=n$  is the same as that of two order statistics of  $n$  indep.

~~Uniform~~ Uniform  $([0, t])$  r.v.'s.

Definition: Let  $y_1, \dots, y_n$  be r.v.'s. Their order statistics  $y_{(1)}, \dots, y_{(n)}$  are the values of  $y_1, \dots, y_n$  in increasing order.

Example: If  $y_1=5, y_2=-3, y_3=1, (y_{(1)}, y_{(2)}, y_{(3)})=(-3, 1, 5)$ . The joint pdf of  $y_{(1)}, \dots, y_{(n)}$  is

$$f(y_1, \dots, y_n) = n! \prod_{i=1}^n f(y_i), \quad y_1 < \dots < y_n$$

pdf of  $y_i$

Because:

•  $(y_{\sigma(1)}, \dots, y_{\sigma(n)}) = (y_1, \dots, y_n) \Leftrightarrow \exists$  a permutation  $\sigma$

s.t.  $(y_1, \dots, y_n) = \cancel{(y_{\sigma(1)}, \dots, y_{\sigma(n)})}$ .

• The ~~probabilit~~ pdf of  $(y_1, \dots, y_n)$  at  $(y_{\sigma(1)}, \dots, y_{\sigma(n)})$

if  $\prod_{i=1}^n f(y_{\sigma(i)}) = \prod_{i=1}^n f(y_i)$ .

$$\Rightarrow f(y_1, \dots, y_n) = n! \prod_{i=1}^n f(y_i) \text{ since there are } n! \text{ permutations.}$$

If the  $y_i$  are uniformly distributed over  $[0, t]$ , then

$$f(y_i) = \frac{1}{t} \Rightarrow f(y_1, \dots, y_n) = \frac{n!}{t^n}$$