Proposition [Poisson Thinning]: Let $\{N(t) : t \geq 0\}$ be a Poisson process and suppose that its events are independently labeled:

- type 1 with probability p, and
- type 2 with probability $1-p$.

Let $N_1(t)$ = # type 1 events by time t, and
$N_2(t)$ = # type 2 events by time t.

Then, N_1 and N_2 are independent Poisson processes of rates λp and $\lambda(1-p)$, respectively.

Proof: **Exercise:** show that N_1 and N_2 satisfy the second definition, i.e. properties (1)–(4). Also, cf. Ross Prop. 5.5.

Example: Immigrants arrive at a Poisson process with rate 10/week. Each immigrant is of English descent w.p. $\frac{1}{12}$. What is the probability that no people of English descent will emigrate in February (4 weeks)?

Solution: By the previous proposition, # of people of English descent who immigrate is a Poisson process with rate $\frac{1}{12} \times 10$ / week $\Rightarrow$

$$P\Big(\underbrace{(\text{\# of people of English descent immigrate in Feb})}_{\text{Poisson}\left(\frac{10}{12} \overset{4\text{ weeks}}{t}\right) = \text{Poisson}\left(\frac{10}{12} \times 4\right) = \text{Poisson}\left(\frac{10}{3}\right)} = 0\Big)$$

$$= \frac{\left(\frac{10}{3}\right)^0 e^{-\frac{10}{3}}}{0!} = e^{-\frac{10}{3}}.$$

Remark: The previous proposition easily generalizes to $k \geq 3$ types of events.

Remark: The labeling here follows a distribution independent of time. But we may also be interested in applications where the classification of an event depends on the time it occurs:

Example (Ross Example 5.20): Suppose individuals contract a virus as a rate-λ Poisson process. Once an individual is infected, there is a incubation period before symptoms appear, such that the incubation time is random and independent for different individuals.

At time t we can thus classify:

$N_1(t) = \#$ people infected with symptoms

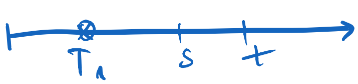
$N_2(t) = \#$ people infected with no symptoms.

An important question is thus, can we estimate $N_2(t)$?

Before solving such types of problems, we need more results on how arrival times of Poisson processes are jointly distributed.

- Conditional distribution of arrival times

Let $N(t)$ be a rate-λ Poisson process.



Q: Suppose we know $N(t)=1$. When did the event in $[0,t]$ occur?

A: $P(T_1 < s \mid N(t)=1) = \dfrac{P(T_1 < s, N(t)=1)}{P(N(t)=1)} = \dfrac{P(N(s)=1, N(t)-N(s)=0)}{P(N(t)=1)}$

$= \dfrac{P(N(s)=1)\,P(N(t)-N(s)=0)}{\,} = \dfrac{\cancel{\lambda} s\, \cancel{e^{-\lambda s}} \cdot (\lambda(t-s))^0 \cancel{e^{-\lambda(t-s)}}}{\,} = \dfrac{s}{t}.$

$P(N(t)=1)$ $\qquad \cancel{\lambda} t \; \cancel{e^{-\lambda t}}$

$\Rightarrow$ The conditional distribution of T_1 given $N(t)=1$

is $\frac{d}{ds} P(T_1 < s \mid N(t)=1) = \frac{1}{t}$, $\forall s \in [0,t]$.

$\Rightarrow$ $(T_1 \mid N(t)=1)$ is Uniform$([0,t])$!!!

Q: More generally, let $S_1, S_2, \dots, S_n$ be the first n arrival times. What is their joint pdf conditional on $N(t)=n$?

Theorem: Given that $N(t)=n$, the n arrival times $S_1, \dots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $[0,t]$.

Definition: Let $Y_1, \dots, Y_n$ be r.v.'s Their order statistics $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ are the values of $Y_1, \dots, Y_n$ in increasing order.

Example: If $Y_1 = 5$, $Y_2 = -3$, $Y_3 = 1$, $(Y_{(1)}, Y_{(2)}, Y_{(3)}) = (-3, 1, 5)$

The joint density of the order statistics $Y_{(1)}, \dots, Y_{(n)}$ is

$$f(y_1, \dots, y_n) = n! \prod_{i=1}^{n} f(y_i), \qquad y_1 < y_2 < \dots < y_n$$

Because

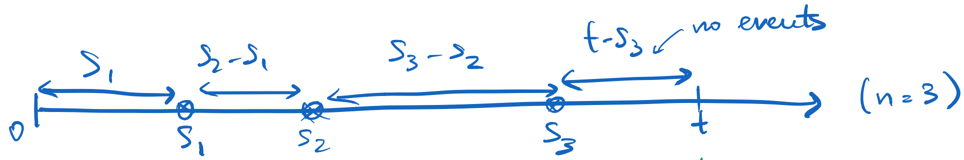
- $(Y_{(1)}, \dots, Y_{(n)}) = (y_1, \dots, y_n) \Leftrightarrow \exists$ a permutation σ s.t. $(Y_1, \dots, Y_n) = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$.
- The probability density that $(Y_1, \dots, Y_n) = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$ is $\prod_{i=1}^{n} f(y_{\sigma(i)})$.

Since there are $n!$ permutations, $f(y_1, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i)$.

If the y_i are uniformly distributed over $[0, t]$, then

$f(y_1, \ldots, y_n) = n! \frac{1}{t^n} = \frac{n!}{t^n}$

<u>Proof of Theorem</u>: We will show that the joint density of $S_1, \ldots, S_n$ given $N(t)=n$ is $f(s_1, \ldots, s_n | N(t)=n) = \frac{n!}{t^n}$.



The event $\{S_1 = s_1, \ldots, S_n = s_n, N(t) = n\}$ for $s_1 < \cdots < s_n \leq t$ is the same as the event that the first $n+1$ interarrival times $T_1, \ldots, T_{n+1}$ satisfy $T_1 = s_1$, $T_2 = s_2 - s_1, \ldots, T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$.

Thus, $f(s_1, \ldots, s_n | N(t)=n) = \dfrac{f(s_1, \ldots, s_n, T_{n+1} > t - s_n)}{P(N(t)=n)}$

$$= \frac{f(T_1 = s_1, T_2 = s_2 - s_1, \ldots, T_n = s_n - s_{n-1}, T_{n+1} > t - s_n)}{P(N(t)=n)}$$

$$= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \cdots \lambda e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t - s_n)}}{(\lambda t)^n e^{-\lambda t}/n!} = \frac{\lambda^n e^{-\lambda t} n!}{\lambda^n t^n e^{-\lambda t}} = \frac{n!}{t^n}$$