2. Further properties.

Theorem (Superposition): Let $\{N_1(t),\ t\geq 0\}$ & $\{N_2(t),\ t\geq 0\}$ be independent Poisson processes of rates λ_1, λ_2. Let $N(t) = N_1(t) + N_2(t)$. Then, $N(t)$ is a Poisson process of rate $\lambda_1 + \lambda_2$.

Proof: We use the second definition.

(1). $N(0) = N_1(0) + N_2(0)$.

(2). independent increments $s_1 < s_2 \leq t_1 < t_2$

$$+\left|\begin{array}{lll} N_1(s_2) - N_1(s_1) & \text{indep of} & N_1(t_2) - N_1(t_1) \\ N_2(s_2) - N_2(s_1) & \text{indep of} & N_2(t_2) - N_2(t_1) \end{array}\right.$$

$$\Rightarrow N(s_2) - N(s_1) \text{ indep of } N(t_2) - N(t_1)$$

(3). $P(N(t+h) - N(t) = 1) = P(N_1(t+h) - N_1(t) = 1 \ \& \ N_2(t+h) - N_2(t) = 0)$

$+ P(N_1(t+h) - N_1(t) = 0 \ \& \ N_2(t+h) - N_2(t) = 1)$

$$\stackrel{\text{indep of } N_1 \text{ and } N_2}{=} P(N_1(t+h) - N_1(t) = 1) P(N_2(t+h) - N_2(t) = 0)$$

$$+ P(N_1(t+h) - N_1(t) = 0) P(N_2(t+h) - N_2(t) = 1)$$

$$= (\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) + (\lambda_2 h + o(h))(1 - \lambda_1 h + o(h))$$

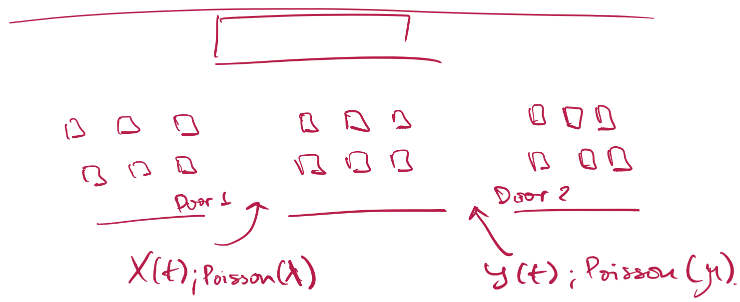
$$= \lambda_1 h - \lambda_1\lambda_2 h^2 + o(h) + o(h) + \lambda_2 h - \lambda_1\lambda_2 h^2 + o(h)$$

$$= (\lambda_1 + \lambda_2) h + o(h). \checkmark$$

(4) Similarly, show (exercise) that $P(N(t+h) - N(t) = 0) =$

$$= 1 - (\lambda_1 + \lambda_2) h + o(h).$$

Example: Students came to class by either using door 1 or door 2.



Suppose the entrance processes $X(t)$ and $Y(t)$ for the 2 doors are independent and Poisson (unlikely in reality...)

Find: (a) $P(1^{st}$ 2 students in class took door 1$)$

(b) Distribution of total # of students by time t.

(c). Distribution of total # of students by time t who took door 1, given that $(X+Y)(t)=n$.

Solution:

(a). $P(1^{st}$ student took door $1) =$

$$= P(\underbrace{\text{first arrival time of } X}_{Exp(\lambda)} < \underbrace{\text{first arrival time of } Y}_{Exp(\mu)}) = \frac{\lambda}{\lambda+\mu}.$$

By the no memory property (If first arrival of X is $T_1 = t$, then starting at time t, next arrival of X is $Exp(\lambda)$ and next arrival of Y is $Exp(\mu)$)

$\Rightarrow P(2^{nd}$ student took door $1) = \frac{\lambda}{\lambda+\mu}$.

$\Rightarrow P(1^{st}$ two students took door $1) = \left(\frac{\lambda}{\lambda+\mu}\right)^2.$

(b). $X+Y$ is a Poisson process of rate $\lambda+\mu$.

$\Rightarrow$ # students by time $t \sim \text{Poisson}((\lambda+\mu)t)$

(c). $P(X(t)=k \mid X(t)+Y(t)=n)$

$$= \frac{P(X(t)=k,\ X(t)+Y(t)=n)}{P(X(t)+Y(t)=n)} = \frac{P(X(t)=k,\ Y(t)=n-k)}{P(X(t)+Y(t)=n)}$$

$$= \frac{P(X(t)=k)\,P(Y(t)=n-k)}{\text{''} \quad \text{''} \quad \text{''}} = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \cdot \frac{(\mu t)^{n-k} e^{-\mu t}}{(n-k)!} \cdot \frac{n!}{((\lambda+\mu)t)^n e^{-(\lambda+\mu)t}}$$

$$= \frac{\lambda^k \cancel{t^k}\, \cancel{e^{-\lambda t}}\, \mu^{n-k} \cancel{t^{n-k}}\, \cancel{e^{-\mu t}}}{(\lambda+\mu)^n \cancel{t^n}\, \cancel{e^{-(\lambda+\mu)t}}} \cdot \frac{n!}{k!(n-k)!} = \frac{\binom{n}{k}\lambda^k \mu^{n-k}}{(\lambda+\mu)^n}$$

$$= \binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{n-k} \sim \text{Binomial}\left(n, \frac{\lambda}{\lambda+\mu}\right)$$

$\Rightarrow$ distribution of $(X(t) \mid X(t)+Y(t)=n)$ is

$\text{Binomial}\left(n, \frac{\lambda}{\lambda+\mu}\right)$.

(consistent with (a) where the next student takes door 1 with "success rate" $\frac{\lambda}{\lambda+\mu}$).

The following is a kind of converse of the previous theorem.