

• Second definition of the Poisson process

• Preliminary (or recall): definition of  $o(h)$

Definition: A function  $f$  is  $o(h)$  if  $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 0$

Examples: •  $f(h) = h^{1+\epsilon} = o(h)$  if  $\epsilon > 0$  ( $\frac{f(h)}{h} = h^\epsilon$ ).

•  $f(h) = e^h - 1 \neq o(h)$   $\frac{e^h - 1}{h} = \frac{(1+h+\frac{h^2}{2}+\dots)}{h} \rightarrow 1 + \frac{h}{2} + \frac{h^2}{3} + \dots \not\rightarrow 0$  as  $h \rightarrow 0$ .

•  $f(h) = h(e^h - 1) = o(h)$

Remark:  $o(h) \pm o(h) = o(h)$ ;  $c \cdot o(h) = o(h)$   $\forall c$  constant.

Definition (Poisson process): A counting process  $\{N(t), t \geq 0\}$  is a

Poisson process of rate  $\lambda$  if it satisfies the following axioms:

(1)  $N(0) = 0$

(2) it has independent increments

(3)  $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$

(4)  $P(N(t+h) - N(t) \geq 2) = o(h)$

or equivalently,  $P(N(t+h) - N(t) = 0) = 1 - \lambda h + o(h)$ .

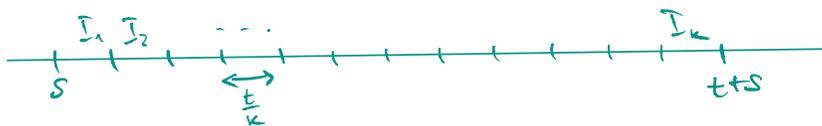
Remark: A consequence of this 2<sup>nd</sup> definition is that

$N(t+s) - N(s) \sim \text{Poisson}(\lambda t)$ .

in particular, we have the property of stationary increments.

Proof of Remark: We have that  $\text{Bin}(k, \frac{\lambda}{k}) \approx \text{Poisson}(\lambda)$  for large  $k$ .

let us divide  $(s, t+s)$  into  $k$  increments:



$$P(\text{some } I_j \text{ has } \geq 2 \text{ events}) = \sum_{j=1}^k P(I_j \text{ has } \geq 2 \text{ events})$$

$$\stackrel{(4)}{=} \sum_{j=1}^k o\left(\frac{t}{k}\right) = k \times o\left(\frac{t}{k}\right) = \underbrace{t}_{\text{const}} \times \frac{o\left(\frac{t}{k}\right)}{\frac{t}{k}} \rightarrow 0 \text{ as } k \rightarrow \infty$$

$\Rightarrow N(t) \sim \#$  intervals where 1 event occurs, as  $k \rightarrow \infty$

$$P(I_j \text{ has } \perp \text{ event}) \stackrel{(3)}{=} \lambda \frac{t}{k} + o\left(\frac{t}{k}\right)$$

$\Rightarrow \{N(t) = e\} \Leftrightarrow$  independently picking  $e$  of  $k$  intervals  
by (2) each w.p.  $\lambda \frac{t}{k} + o\left(\frac{t}{k}\right)$

$$\rightarrow N(t) \sim \text{Binomial}(k, p = \lambda \frac{t}{k} + o\left(\frac{t}{k}\right))$$

$$\approx \text{Poisson}\left(\lambda t + \underbrace{k o\left(\frac{t}{k}\right)}_{\rightarrow 0 \text{ as } k \rightarrow \infty}\right) \rightarrow \text{Poisson}(\lambda t)$$

We will use this to show the 2 definitions are equivalent.

Theorem:  $N(t)$  is a Poisson process of rate  $\lambda$  (2nd definition)

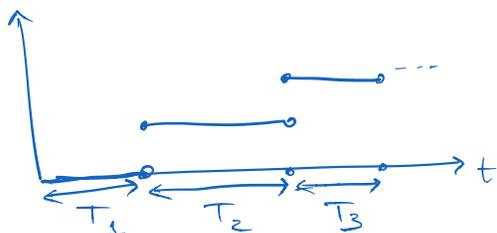
$\Leftrightarrow N(t)$  is a counting process with i.i.d. intervals  $\sim \text{Exp}(\lambda)$  (1st definition).

Proof:  $\boxed{\Leftarrow}$  We already saw that (2) indep increments holds  
(1) holds by def.

$$\begin{aligned} (3). P(N(t+h) - N(t) = 1) &= P(N(h) = 1) = \frac{\lambda h e^{-\lambda h}}{1!} = \lambda h e^{-\lambda h} \\ &= \lambda h \left(1 + (-\lambda h) + \frac{(-\lambda h)^2}{2!} + \dots\right) = \lambda h + o(h) \quad \checkmark \end{aligned}$$

$$\begin{aligned} (4) P(N(t+h) - N(t) \geq 2) &= 1 - P(N(h) = 1) - P(N(h) = 0) \\ &= 1 - (\lambda h + o(h)) - \frac{\lambda^0 h^0 e^{-\lambda h}}{0!} = -\lambda h + 1 + o(h) - (1 - \lambda h + o(h)) \\ &= \cancel{-\lambda h} + 1 + o(h) - \cancel{1} + \lambda h + o(h) \\ &= o(h). \quad \checkmark \end{aligned}$$

$\Rightarrow$ : We need to check that interarrival times  $T_1, T_2, T_3, \dots$  are i.i.d.  $\text{Exp}(\lambda)$  r.v.'s.



Recall that we showed

$$N(t+s) - N(t) \sim \text{Poisson}(\lambda s)$$

Distribution of  $T_1$ :  $P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}$   
 $\Rightarrow T_1 \sim \text{Exp}(\lambda)$ .

Distribution of  $T_2$ :  $P(T_2 > t) = \int_0^\infty P(T_2 > t | T_1 = s) f_{T_1}(s) ds$   
 ↑  
 conditioning on  $T_1$ .

•  $P(T_2 > t | T_1 = s) = P(N(s+t) - N(t) = 0) = e^{-\lambda t}$   
 $\Rightarrow T_2$  is independent of  $T_1$ .

$\Rightarrow P(T_2 > t) = \int_0^\infty e^{-\lambda t} f_{T_1}(s) ds = e^{-\lambda t}$   
 $\Rightarrow T_2 \sim \text{Exp}(\lambda)$

This argument can be extended to  $P(T_{n+1} > t | T_1 + \dots + T_n = s)$  by induction showing that  $T_1, T_2, \dots$  are i.i.d.  $\text{Exp}(\lambda)$ .

## 2. Further properties.

Theorem (Superposition): Let  $\{N_1(t), t \geq 0\}$  &  $\{N_2(t), t \geq 0\}$  be independent Poisson processes of rates  $\lambda_1, \lambda_2$ . Let  $N(t) = N_1(t) + N_2(t)$ . Then,  $N(t)$  is a Poisson process of rate  $\lambda_1 + \lambda_2$ .

Proof: We use the second definition.

$$(1). N(0) = N_1(0) + N_2(0).$$

$$(2). \text{ independent increments } s_1 < s_2 \leq t_1 < t_2$$
$$+ \begin{cases} N_1(s_2) - N_1(s_1) & \text{indep of } N_1(t_2) - N_1(t_1) \\ N_2(s_2) - N_2(s_1) & \text{indep of } N_2(t_2) - N_2(t_1) \end{cases}$$
$$\Rightarrow N(s_2) - N(s_1) \text{ indep of } N(t_2) - N(t_1)$$

$$(3). P(N(t+h) - N(t) = 1) = P(N_1(t+h) - N_1(t) = 1 \& N_2(t+h) - N_2(t) = 0)$$

$$+ P(N_1(t+h) - N_1(t) = 0 \& N_2(t+h) - N_2(t) = 1)$$

indep of  $N_1$  and  $N_2$

$$\stackrel{\downarrow}{=} P(N_1(t+h) - N_1(t) = 1) P(N_2(t+h) - N_2(t) = 0)$$

$$+ P(N_1(t+h) - N_1(t) = 0) P(N_2(t+h) - N_2(t) = 1)$$

$$= (\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) + (1 - \lambda_1 h + o(h))(\lambda_2 h + o(h))$$

$$= \lambda_1 h - \lambda_1 \lambda_2 h^2 + o(h) + o(h) + \lambda_2 h - \lambda_1 \lambda_2 h^2 + o(h)$$

$$= (\lambda_1 + \lambda_2)h + o(h). \checkmark$$

$$(4) \text{ Similarly, show (exercise) that } P(N(t+h) - N(t) = 0) = (\lambda_1 + \lambda_2)h + o(h).$$