02/26/2020

Last time we showed that if X and Y are independent with pdf's f_X and f_Y, the pdf of $Z = X+Y$ is

$$f_{X+Y}(t) = (f_X * f_Y)(t) = \int_{-\infty}^{\infty} f_Y(s)\, f_X(t-s)\, ds,$$

the "convolution of f_X and f_Y"

Furthermore, if $X_1 \sim Exp(\lambda_1)$ and $X_2 \sim Exp(\lambda_2)$ are independent, then, the pdf of their sum is:

$$f_{X_1+X_2}(t) = \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 t}$$

→ This generalizes by induction (see Ross 5.2.4) to

$$f_{X_1+\cdots+X_n}(t) = \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i t}$$

where $C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$

→ What happens when $\lambda_1 = \lambda_2 = \lambda$?

Going back to • $f_{X_1+X_2}(t) = \lambda_1 \lambda_2 e^{-\lambda_1 t} \int_0^t e^{-(\lambda_2-\lambda_1)s} ds$

$$= \lambda^2 e^{-\lambda t} \cdot t$$

• $f_{X_1+X_2+X_3}(t) = (f_{X_3} * f_{X_1+X_2})(t)$

$$= \int_{-\infty}^{\infty} \underbrace{f_{X_3}(t-s)}_{=0 \text{ when } s \geq t} \underbrace{f_{X_1+X_2}(s)}_{=0 \text{ when } s \leq 0} ds$$

$$= \int_0^t \lambda e^{-\lambda(t-s)} \cdot \lambda^2 e^{-\lambda s} \cdot s\, ds = \lambda e^{-\lambda t} \frac{(\lambda t)^2}{2}$$

By induction, one can show

$$f_{X_1+\dots+X_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

<u>Proposition</u>: If $X_1, \dots, X_n$ are independent exponential r.v.'s with common rate λ, then $\sum_{i=1}^{n} X_i \sim \Gamma(n, \lambda)$. That is, its density is

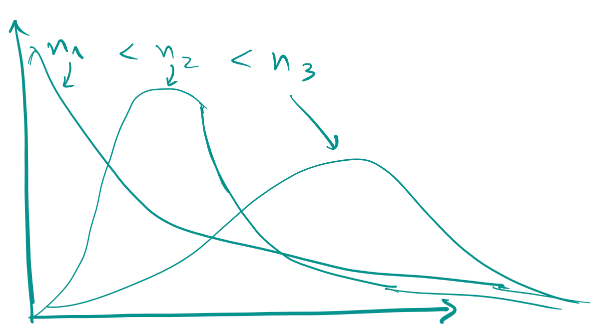
$$f(t)\ \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}.$$

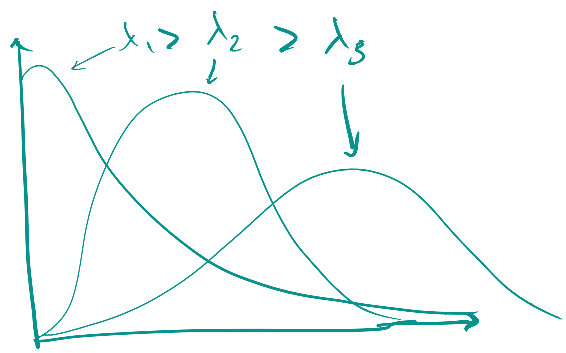
<u>Remark</u>: The mean and variance of the Gamma distribution.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \frac{n}{\lambda}, \quad \mathrm{Var}(X) \underset{\text{independence.}}{=} \sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{n}{\lambda^2}.$$

→ When n increases, mean & variance increase

→ When λ increases, mean & variance decrease.





By considering a sum of i.i.d. exponential r.v.'s, we'll obtain a first description of a Poisson process.

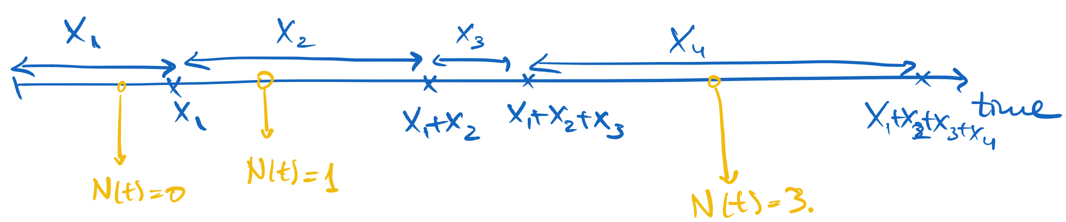
II. The Poisson Process

→ We will see 2 equivalent definitions, the 1st one will follow from the use of exponential variables to define arrival times, the 2nd will put emphasis on its key property as a counting process, i.e., a stochastic process $N(t)$ that represents the total # of events by time t.

<u>First definition</u>: • A Poisson process with rate λ is a counting process $N(t)$ such that the time between 2 events is $Exp(\lambda)$.

i.e. $\left(\min\{t : N(t) = n+1\} - \min\{t : N(t) = n\}\right) \sim Exp(\lambda) \ \forall n.$

• The time for n events to occur (i.e., $\min\{t : N(t) = n\}$) in a Poisson process of rate λ is therefore $\sum_{i=1}^{n} X_i$, where the X_i's are n i.i.d. $Exp(\lambda)$. The X_i's represent interarrival times.



<u>Examples</u>: • job completion in a busy queue; • earthquakes; • pizza orders; • cars passing counter on a highway; • customers visiting bank etc. ...

<u>Remark</u>: λ represents the rate of job completion

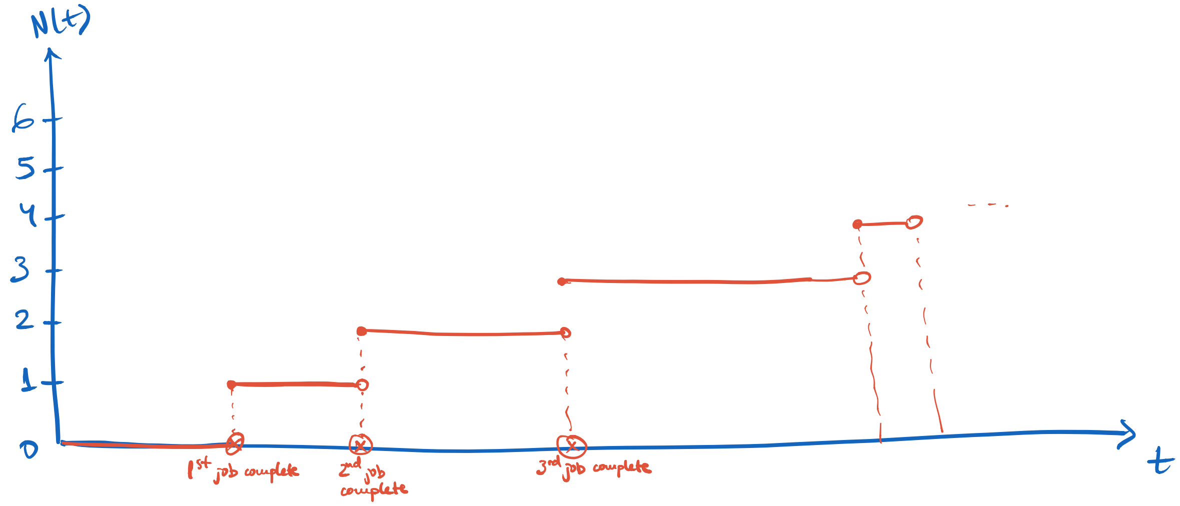
Average job takes $\frac{1}{\lambda}$ minutes $\Rightarrow$ $\frac{1}{\lambda}$ min/job

$\Rightarrow$ The rate of job completion is $1/_{1/\lambda} = \lambda$ jobs/min.

For a counting process, $N(t) = \#$ jobs completed by time $t \geq 0$

$$N(t) \in \{0,1,2,\dots\} = \mathbb{N}.$$

Problem: For N = Poisson Process (λ), and given t, what is the pmf of $N(t)$, i.e. what is $P(N(t)=k)$ for $k \in \mathbb{N}$?



Remark: Intuition tells us that we should have

$$\mathbb{E}[N(t)] = \underbrace{\lambda}_{\#\text{jobs}/\text{min}} \underbrace{t}_{\text{min}}$$

Solution: For $n \geq 1$, $\underbrace{\{N(t) \geq n\}}_{\text{at least } n \text{ jobs by time } t} = \underbrace{\{X_1 + \cdots + X_n \leq t\}}_{n\text{-th job completed by time } t.}$

From the previous section we know that $X_1 + \cdots + X_n \sim \Gamma(n, \lambda)$

$$\Rightarrow P(N(t) \geq n) = P(X_1 + \cdots + X_n \leq t) = \int_0^t \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} dx$$

$$= \frac{1}{(n-1)!} \int_0^t y^{n-1} e^{-y} dy \quad \left(\begin{array}{l} y = \lambda x \\ dy = \lambda dx \end{array} \right)$$