

- HW 5 due March 2
- midterm 1 will be available Wed at MLC
- midterm 2 will in class March 25 Wed

Recall: Exponential distribution $X \sim \text{Exp}(\lambda)$

pdf $p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

CDF $F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Properties: • memoryless; exponential r.v.'s are the only ≥ 0 real r.v.'s with this property

- failure rate function is constant $= \lambda$
- Minimum of 2 independent Exp r.v.'s.

$$X_1 \sim \text{Exp}(\lambda_1), X_2 \sim \text{Exp}(\lambda_2)$$

$$X = \min(X_1, X_2), \quad X \sim ?$$

(Remark: in the example with 2 tellers, X would be the 1st time one of the 2 customers leaves).

Idea: look at the event $E = \{X > x\}$

$$X > x \Leftrightarrow X_1 > x \text{ and } X_2 > x$$

$$\Rightarrow E = \{X_1 > x, X_2 > x\}$$

$$\text{So, } P(X > x) = P(X_1 > x, X_2 > x) \stackrel{\substack{\uparrow \\ X_1 \text{ indep. of } X_2}}{=} P(X_1 > x) P(X_2 > x) \quad (2)$$

$$= e^{-\lambda_1 x} \cdot e^{-\lambda_2 x} = e^{-(\lambda_1 + \lambda_2)x}$$

CDF of $X = \min(X_1, X_2)$ is

$$F(x) = P(X \leq x) = 1 - e^{-(\lambda_1 + \lambda_2)x}$$

$$\Rightarrow X \sim \text{Exp}(\lambda_1 + \lambda_2).$$

We can generalize (proof by induction)

Proposition: Let X_1, \dots, X_n be independent, $X_i \sim \text{Exp}(\lambda_i)$. Then
 $X = \min(X_1, \dots, X_n) \sim \text{Exp}(\lambda_1 + \dots + \lambda_n)$.

Example: Recall the previous example:

What is the expected time until all 3 customers have left?

↪ What is the expected time until 1 customer leaves? $\rightarrow \frac{1}{2\lambda}$

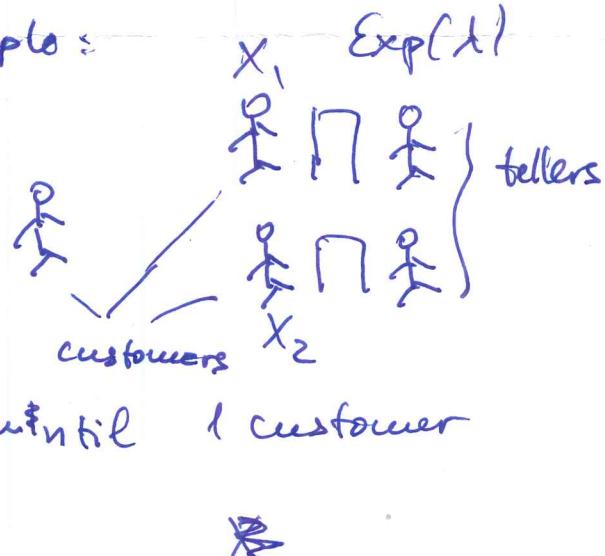
$$X_1 \sim \text{Exp}(\lambda), \quad X_2 \sim \text{Exp}(\lambda)$$

$\min(X_1, X_2)$ = the time it takes for 1st customer to leave.

$$\text{Exp}(2\lambda)$$

↪ Once one customer leaves, clock "restarts" and we have 2 tellers with 1 customer at each teller

$$\text{Exp}(\lambda)$$



So, expected time until next customer leaves (3)
is $\frac{1}{2\lambda}$

↪ Clock restarts again and we have 1 teller with
1 customer $\stackrel{2}{\text{Exp}(\lambda)}$.

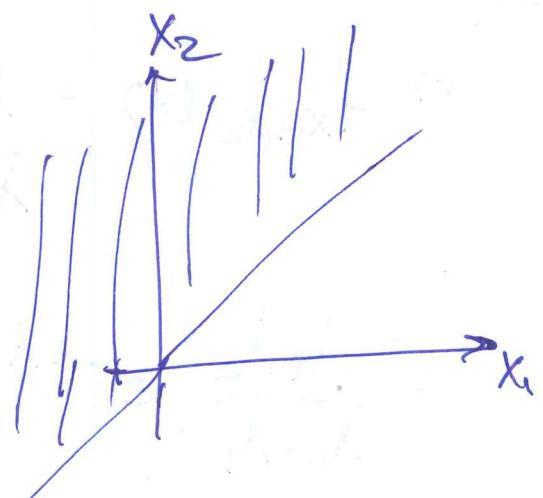
⇒ it takes $\frac{1}{\lambda}$ for the last customer to leave
(on average)

$$\Rightarrow \text{In total } \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda}$$

• Probability that $X_1 < X_2$, where $X_1 \sim \text{Exp}(\lambda_1)$
 $X_2 \sim \text{Exp}(\lambda_2)$
independent.

Proposition: If X_1, X_2 are indep., $\text{Exp}(\lambda_1)$ and $\text{Exp}(\lambda_2)$ respectively

then $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.



Proof: $P(X_1 < X_2) =$

$$= \iint_{X_1 < X_2} p(x_1, x_2) dx_1 dx_2$$

$p(x_1) \cdot p(x_2)$

$$= \iint_{X_1 < X_2} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} dx_1 dx_2$$

$$= \int_0^{\infty} \int_0^{x_1} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} dx_1 dx_2 = \int_0^{\infty} \lambda_2 e^{-\lambda_2 x_2} \left[\int_0^{x_2} \lambda_1 e^{-\lambda_1 x_1} dx_1 \right] dx_2$$

$$= 1 - e^{-\lambda_1 x_2}$$

$$= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2} (1 - e^{-\lambda_1 x_2}) dx_2 = \underbrace{\int_0^\infty \lambda_2 e^{-\lambda_2 x_2} dx_2}_1 -$$

$$= 1 - \int_0^\infty \lambda_2 e^{-(\lambda_1 + \lambda_2) x_2} dx_2$$

$$= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_0^\infty (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) x_2} dx_2$$

$$= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

- Sum of independent exponential r.v.'s

Suppose X, Y independent r.v.'s, and $Z = X + Y$
 \uparrow continuous

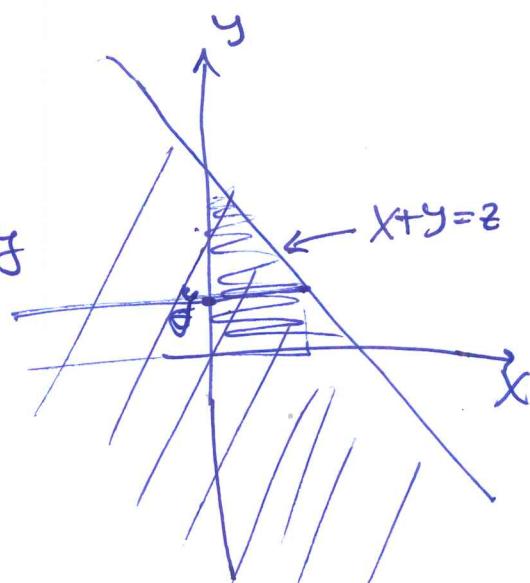
pdf's: f_X, f_Y, f_Z

$$F_Z(z) = P(Z \leq z) = \iint_{x+y \leq z} f_X(x) f_Y(y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y) dx dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \left[\int_{-\infty}^{z-y} f_X(x) dx \right] dy = \int_{-\infty}^{\infty} f_Y(y) F_X(z-y) dy$$

$F_X(z-y)$



$$F_Z(z) = \int_{-\infty}^{\infty} f_Y(y) \underbrace{F_X(z-y)}_{\text{differentiate}} dy \quad (5)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y) dy = "f_X * f_Y"(z)$$

the convolution of f_X and f_Y .

let $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ indep. and $\lambda_2 \geq \lambda_1$

$$\begin{aligned} \text{Then } f_{X_1+X_2}(z) &= \int_{-\infty}^{\infty} \lambda_2 e^{-\lambda_2 y} \cdot \lambda_1 e^{-(z-y)\lambda_1} dy \\ &= \int_0^z \cancel{\lambda_1 \lambda_2 e^{-z\lambda_1}} \cdot e^{-(\lambda_2 - \lambda_1)y} dy \end{aligned}$$

Case 1: $\lambda_1 \neq \lambda_2$, i.e., $\lambda_2 > \lambda_1$

$$\Rightarrow f_{X_1+X_2}(z) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-z\lambda_1} \underbrace{\int_0^z (\lambda_2 - \lambda_1) e^{-(\lambda_2 - \lambda_1)y} dy}_{1 - e^{-(\lambda_2 - \lambda_1)z}}$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-z\lambda_1} (1 - e^{-(\lambda_2 - \lambda_1)z})$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-z\lambda_1} - \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_2 z}$$

$$= \boxed{\frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-z\lambda_1} + \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 z}}$$