Recall: Exponential distribution: $X \sim Exp(\lambda)$

pdf $\quad p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

CDF $\quad F(x) = \begin{cases} 1-e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

Properties:
- memoryless: only real ≥ 0 r.v. with this property
- failure rate function is constant $= \lambda$

- Minimum of 2 independent Exp r.v.

$X_1 \sim Exp(\lambda_1)$, $X_2 \sim Exp(\lambda_2)$ independent

$X = \min(X_1, X_2)$, $\quad X \sim ?$

(Remark: in the example with the 2 tellers, X would be the 1^{st} time one of the 2 customers to leave)

Idea: Look at the event $E = \{X > x\}$.

One can show $X > x \Leftrightarrow X_1 > x$ and $X_2 > x$

$\Rightarrow E = \{X_1 > x, X_2 > x\}$ $\quad (x \geq 0)$.

So, $P(X > x) = P(X_1 > x, X_2 > x) \underset{\substack{\uparrow \\ X_1 \text{ indep. of } X_2}}{=} P(X_1 > x) P(X_2 > x)$

$= e^{-\lambda_1 x} e^{-\lambda_2 x} = e^{-(\lambda_1 + \lambda_2)x}$.

$\Rightarrow F_X(x) = 1 - e^{-(\lambda_1+\lambda_2)x}$ and $X \sim Exp(\lambda_1 + \lambda_2)$!

We can easily generalize.

Proposition: Let $X_1, \dots, X_n$ be n independent r.v. $X_i \sim Exp(\lambda_i)$.

Then, $X = \min(X_1, \dots, X_n) \sim Exp(\lambda_1 + \dots + \lambda_n)$.

__Example:__ Recall the previous example, with 3 customers at the bank, and 2 tellers. What is the expected time until all 3 customers have left?

__Solution:__ • Time until 1^{st} leaves is min of 2 indep $Exp(\lambda)$, so is $Exp(2\lambda)$ with mean $\frac{1}{2\lambda}$.

• Then, clock effectively restarts, but now there are 2 customers, so first leaves after mean time $\frac{1}{2\lambda}$.

• Clock restarts, and last customer leaves after mean time $\frac{1}{\lambda}$.

$\Rightarrow$ Total mean time is: $\frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda}$.

• Probability that $X_1 < X_2$, where $X_i \sim Exp(\lambda_i)$.

__Proposition:__ If X_1, X_2 are independent $Exp(\lambda_i)$, then

$$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

__Proof:__ $$P(X_1 < X_2) = \iint_{x_1 < x_2} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}\, dx_1 dx_2$$

$$= \int_0^{+\infty} \int_0^{x_2} \lambda_1 \lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2}\, dx_1 dx_2$$

$$= \int_0^{+\infty} \lambda_2 e^{-\lambda_2 x_2} \underbrace{\int_0^{x_2} \lambda_1 e^{-\lambda_1 x_1} dx_1}_{= P(X_1 \le x_2) = 1 - e^{-\lambda_1 x_2}} dx_2$$

$$= \int_0^{+\infty} \lambda_2 e^{-\lambda_2 x_2} (1 - e^{-\lambda_1 x_2})\, dx_2$$

$$= \int_0^{+\infty} \lambda_2 e^{-\lambda_2 x_2} - \int_0^{+\infty} \lambda_2 e^{-(\lambda_1 + \lambda_2) x_2}\, dx_2$$

$$= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

- Sum of independent exponential r.v.

Suppose that X, Y are independent continuous r.v., and let $Z = X+Y$. Then, $F_Z(z) = P(Z \le z) = \iint_{x+y \le z} f_{X,Y}(x,y)\,dx\,dy$

$$= \iint_{x+y\le z} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} f_Y(y) \underbrace{\left(\int_{-\infty}^{z-y} f_X(x)\,dx \right)}_{F_X(z-y)} dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) F_X(z-y)\,dy$$

$$\Rightarrow f_Z(z) = F_Z'(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy$$

$= $ "$f_X * f_Y$" the convolution of f_X and f_Y

$\rightarrow$ let $X_1 \sim Exp(\lambda_1)$ and $X_2 \sim Exp(\lambda_2)$.

Then, $f_{X_1+X_2}(t) = (f_{X_1} * f_{X_2})(t) = \int_{-\infty}^{\infty} f_{X_1}(t-s) f_{X_2}(s)\,ds$

$= \int_0^t f_{X_1}(t-s) f_{X_2}(s)\,ds$ (because $f_{X_i}(u) = 0$ for $u \le 0$)

$$= \int_0^t \lambda_1 e^{-\lambda_1 (t-s)} \lambda_2 e^{-\lambda_2 s} \, ds$$

$$= \lambda_1 \lambda_2 \, e^{-\lambda_1 t} \int_0^t e^{-(\lambda_2 - \lambda_1) s} \, ds$$

$$= \lambda_1 \lambda_2 e^{-\lambda_1 t} \frac{\left(1 - e^{-(\lambda_2 - \lambda_1) t}\right)}{\lambda_2 - \lambda_1} \, , \quad \text{with } \lambda_2 \neq \lambda_1$$

$$\Rightarrow f_{X_1 + X_2}(t) = \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 t}$$

→ This generalizes by induction (see Ross 5.2.4) to

$$f_{X_1 + \cdots + X_n}(t) = \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i t}$$

where $C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$