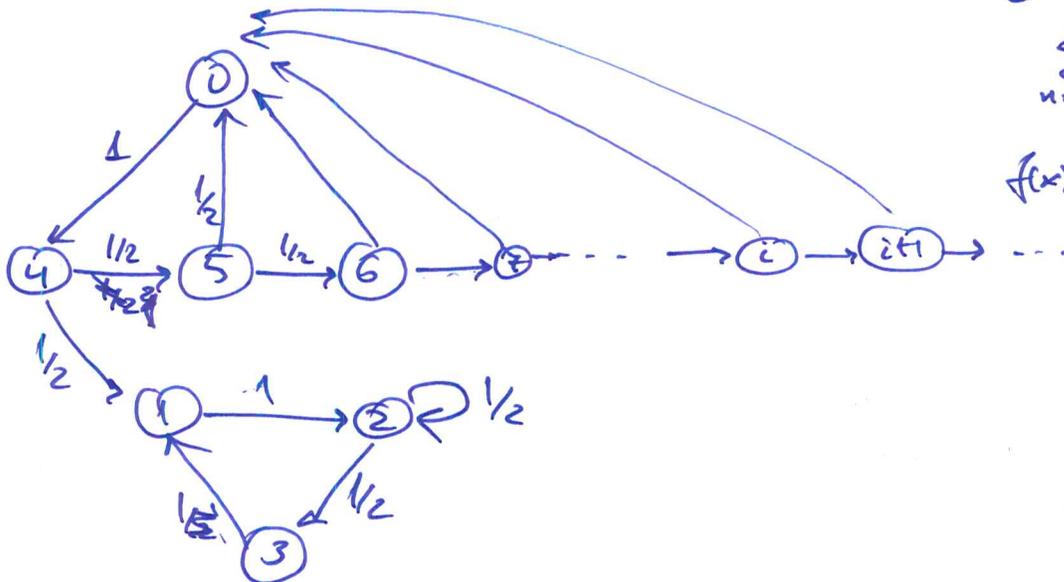


02/12/2020

Example:



$$\frac{1}{2} + \frac{1}{4}$$

$$\frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 3$$

$$\sum_{n=2}^{\infty} \frac{1}{2^{n+1}}$$

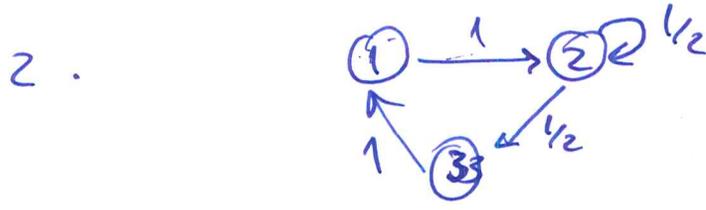
$$f(x) = \sum_{n=2}^{\infty} \frac{1}{x^{n+1}}$$

1. What are the communicating classes and what types are they (transient/recurrent, aperiodic/periodic)?
2. Consider the induced M.C. on  $\{1, 2, 3\}$ . Write down the transition matrix and compute  $P(X_3 = i | X_0 = 1)$  for  $i = 1, 2, 3$ .
3. What is  $\lim_{n \rightarrow \infty} p_{23}^{(n)}$ ?
4. What is the mean # of steps to revisit 3?

Solutions:

1.  $\{1, 2, 3\}$  recurrent, positive-recurrent, aperiodic

$\{0, 4, 5, 6, \dots\}$  transient, aperiodic



$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

$$P^3 = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 5/8 & 1/8 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$$P(X_3 = 1 | X_0 = 1) = \frac{1}{2}$$

$$P(X_3 = 2 | X_0 = 1) = \frac{1}{4}$$

$$P(X_3 = 3 | X_0 = 1) = \frac{1}{4}$$

$$3. \quad \lim_{n \rightarrow \infty} P_{23}^{(n)} = \pi_3 = \frac{1}{4}$$

irred, ergodic

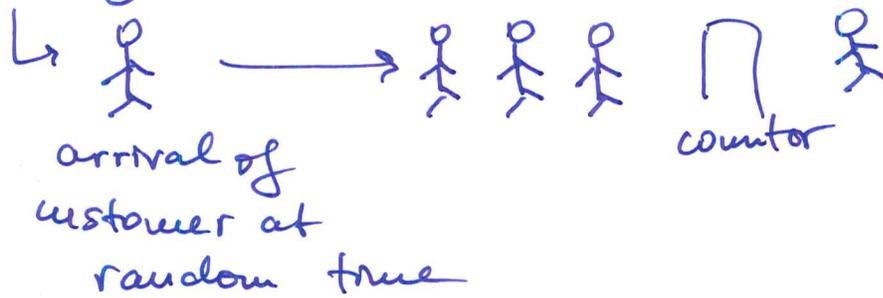
$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

$$(\pi_3, \pi_1 + \frac{1}{2}\pi_2, \frac{1}{2}\pi_2) = (\pi_1, \pi_2, \pi_3).$$

$$4. \quad m_{23} = \frac{1}{\pi_3} = 4.$$

# Chapter 5. The exponential distribution and the Poisson process

Example: Waiting line



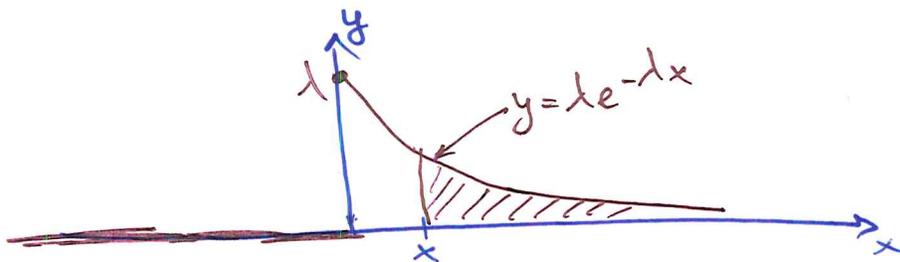
→ How to model length of line?  
↳ waiting time of a customer?

First, we review the exponential distribution.

## § 5.2. The Exponential Distribution.

Definition: A continuous r.v.  $X$  has the exponential distribution with parameter  $\lambda$ ,  $\lambda > 0$  if its

pdf is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$


CDF:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(y) dy = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\text{i.e., } P(X > x) = \begin{cases} e^{-\lambda x} & x \geq 0 \\ 1 & x < 0 \end{cases}$$

Moment generating function:  $\Phi(t) = \mathbb{E}[e^{tx}] =$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{(t-\lambda)x} dx = \begin{cases} \frac{\lambda}{\lambda-t} & , t < \lambda \\ \infty & , t \geq \lambda \end{cases}$$

(Probability)  
Generating  
Function  
 $f(s) = \mathbb{E}[s^X]$

Recall:

$$\mathbb{E}[X^k] = \Phi^{(k)}(0)$$

$$k=1: \mathbb{E}[X] = \Phi'(t) \Big|_{t=0} = \frac{\lambda}{(\lambda-t)^2} \Big|_{t=0} = \frac{1}{\lambda}$$

$$k=2: \mathbb{E}[X^2] = \Phi''(t) \Big|_{t=0} = \frac{2\lambda}{(\lambda-t)^3} \Big|_{t=0} = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

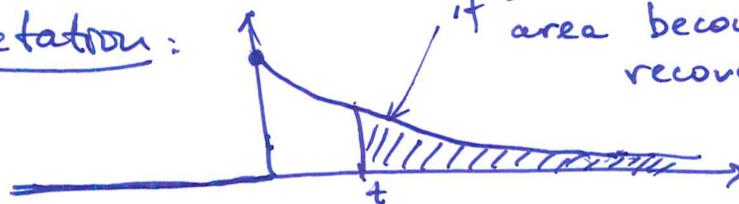
Memoryless property:

$$P(X > t+s | X > t) = P(X > s), \quad \forall s, t \geq 0.$$

Example: If  $X$  = time that clerk takes to help 1 customer  
Prob of waiting  $s$  more mins given we waited  $t$  mins  
= prob of waiting  $s$  mins

Proof: 
$$P(X > t+s | X > t) = \frac{P(X > t+s, X > t)}{P(X > t)}$$
$$= \frac{P(X > t+s)}{P(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s).$$

Graphical interpretation:

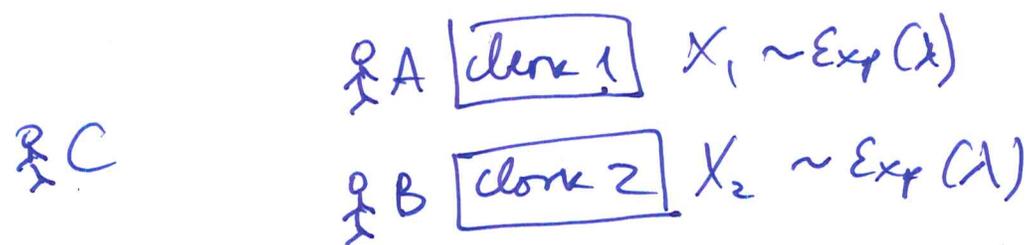


if we rescale so that this area becomes 1, we recover the original curve.

Example: Post office with two clerks. The amount of time a clerk spends with a customer is  $\sim \text{Exp}(\lambda)$

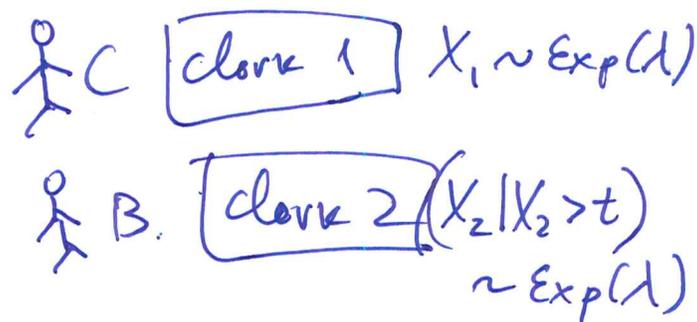
Cameron enters the post office and there are already 2 customers: Alice and Bob being helped by the 2 clerks.

What's the probability that of the 3 customers, C is the last to leave the office?



Suppose C is helped at time  $t$ . At this point either A or B ~~leaves~~ would have just left.

time  $t$



Same distribution  $\Rightarrow$  by symmetry,

$$P(B \text{ leaves first}) = P(C \text{ leaves first}) = \frac{1}{2}.$$