# The Exponential Distribution

$X \sim Exp(\lambda)$ if pdf is $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

CDF $F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$

<u>Remark</u>: When studying the lifetime of an object (e.g. in Physics → decay) we often refer to the "exponential decay rate". From a probabilistic point of view we can also associate the exponential distribution $Exp(\lambda)$ as a measure of lifespan with $\lambda$ being the <u>failure rate</u>.

Suppose $X$ is the lifespan of some object (e.g. phone battery)
pdf $f$, CDF $F$

Then $P(X \in (t, t+dt) \mid X > t) = \dfrac{P(X \in (t, t+dt))}{P(X > t)}$

probability that it dies between $t$ and $t+dt$ given that it has survived until $t$.

$$= \frac{\int_t^{t+dt} f(x)\,dx}{1 - F(t)} \overset{dt \approx 0}{\approx} \frac{f(t)\,dt}{1 - F(t)} = r(t)\,dt,$$

where $r(t) = \dfrac{f(t)}{1 - F(t)}$ is the failure (hazard) rate function.

→ If $X \sim Exp(\lambda)$, then $r(t) = \dfrac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$.

→ So the failure rate function is constant for the exponential r.v. (i.e., it doesn't depend on time).
↳ this is another version of the no memory property.

**Remark:** By the definition of the no memory property, if a r.v. is memoryless, its failure rate is constant!

**Proof:** $P(X \in (t, t+dt) \mid X > t) = r(t)\,dt$ for $dt$ small.

$$\|$$

$$1 - P(X \geq t+dt \mid X > t) \underset{\substack{\uparrow \\ \text{memoryless}}}{=} 1 - P(X \geq dt)$$ does not depend on $t$!

**Remark:** • $r(t) = \dfrac{f(t)}{1-F(t)} = \dfrac{F'(t)}{1-F(t)}$ Failure rate uniquely determines the distribution

$$\Rightarrow \quad \ln(1-F(t)) = -\int_0^t r(u)\,du + C$$

$$\Rightarrow \quad 1-F(t) = e^C \cdot e^{-\int_0^t r(u)\,du}$$

$$F(0) = 0 \quad \Rightarrow \quad 1 = e^C e^0 \quad \Rightarrow \quad C = 0.$$

$$\Rightarrow \quad F(t) = 1 - \exp^{-\int_0^t r(u)\,du}$$

If $r$ is constant, then $F(t) = 1 - e^{-t\lambda}$
$r(s) = \lambda$.

$\Rightarrow$ $X$ is exponential r.v.! Thus,

**Theorem:** The exponential distribution is the only real memoryless r.v.

The parameter $\lambda$ is the rate of the distribution.

**Example:** $X_1, \dots, X_n$ independent exponential r.v.'s with respective rates $\lambda_1, \dots, \lambda_n$, $\lambda_i \neq \lambda_j \ \forall i \neq j$.
Suppose that a bin contains $n$ different types of batteries with a type $j$ battery lasting $Exp(\lambda_j)$ time.

Suppose further that $p_j$ is the proportion of type $j$ batteries in the bin, $\sum_{j=1}^{n} p_j = 1$.

If a battery is randomly chosen, what is the distribution of the lifetime of this battery?

$T \in \{1, \dots, n\}$ $\quad P(T=j) = p_j$.

Battery lifetime is the r.v. $X_T$ "hyperexponential random variable".

To obtain the distribution function $F$ of $X_T$:

$$1-F(t) = P(X_T > t) = \sum_{i=1}^{n} P(X_i > t \mid T=i)\, P(T=i)$$
$$= \sum_{i=1}^{n} p_i e^{-\lambda_i t}$$

To get $f(t)$:

$$f(t) = F'(t) = -(1-F(t))' = \boxed{\sum_{i=1}^{n} \lambda_i p_i e^{-\lambda_i t}.}$$

$\Rightarrow$ The failure rate function is

$$r(t) = \frac{f(t)}{1-F(t)} = \frac{\sum_{i=1}^{n} \lambda_i p_i e^{-\lambda_i t}}{\sum_{i=1}^{n} p_i e^{-\lambda_i t}}.$$

Let $\lambda_1 < \lambda_2 < \dots < \lambda_n$.

Compute $P(T=1 \mid X > t)$ as $t \to \infty$.

$$P(T=1 \mid X>t) = \frac{P(X>t \mid T=1)\, P(T=1)}{P(X>t)}$$

$$= \frac{p_1 e^{-\lambda_1 t}}{\sum_{i=1}^{n} p_i e^{-\lambda_i t}} = \frac{p_1 e^{-\lambda_1 t}}{p_1 e^{-\lambda_1 t} + \sum_{i=2}^{n} p_i e^{-\lambda_i t}}$$

$$= \frac{p_1}{p_1 + \sum_{i=2}^{n} p_i e^{-\underbrace{(\lambda_i - \lambda_1)}_{>0} t}} \to 1 \quad \text{as } t \to \infty .$$

$P(T=i \mid X>t) \to 0$ as $t \to \infty$ for $i > 1$

$$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \frac{\lambda_1 p_1 e^{-\lambda_1 t} + \sum_{i=2}^{n} \lambda_i p_i e^{-\lambda_i t}}{p_1 e^{-\lambda_1 t} + \sum_{i=2}^{n} p_i e^{-\lambda_i t}}$$

$$= \lim_{t\to\infty} \frac{\lambda_1 p_1 + \sum_{i=2}^{n} \lambda_i p_i e^{-(\lambda_i - \lambda_1)t}}{p_1 + \sum_{i=2}^{n} p_i e^{-(\lambda_i - \lambda_1)t}} \to \lambda_1 \quad \text{as } t \to \infty .$$

As the randomly chosen battery ages its failure rate converges to the smallest $\lambda_i$!

The longer the battery ~~lasts~~ lasts, the more likely it is a battery type with the smallest failure rate.