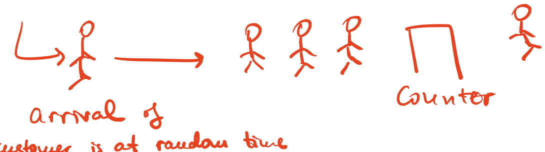
Chapter 5: The exponential Distribution and the Poisson process.

After studying processes in discrete time, we now focus on modeling and studying processes in continuous time.

Example: • Waiting line:



→ How to model and study the length of the line at time t? or the waiting time of a customer?

↳ We can see this example as a counting process. (counting # of customers at time t)

↳ Applications in many domains: communication, physics, neuroscience, etc, etc.

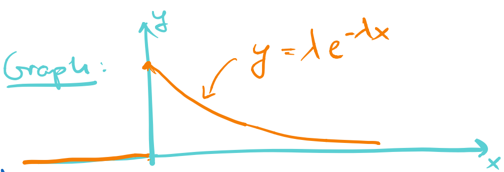
Before studying this type of process, we will first review the exponential distribution, which is intimately connected with the processes of this chapter.

5.2. The Exponential Distribution:

A continuous r.v. X has the exponential distribution with parameter λ, $\lambda > 0$ if its pdf is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Graph:



CDF: $$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Mean: $$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\infty} \lambda x e^{-\lambda x}\, dx$$

integrating by parts:
$u = x$, $dv = \lambda e^{-\lambda x} dx$

$$= -x e^{-\lambda x}\Big]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, dx$$

$du = 1$, $v = -e^{-\lambda x}$

$$= 0 + \left(-\frac{1}{\lambda} e^{-\lambda x}\right)\Big|_0^\infty = \frac{1}{\lambda}.$$

Moment generating function: $\Phi(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$
$\Phi^{(k)}(0) = \mathbb{E}[X^k]$

$$= \begin{cases} \frac{\lambda}{\lambda - t}, & t < \lambda \\ \infty, & t \geq \lambda \end{cases}$$

Moments: $\mathbb{E}[X] = \frac{d}{dt}\Phi(t)\Big|_{t=0} = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0}$

$$\mathbb{E}[X^2] = \frac{d^2}{dt^2}\Phi(t)\Big|_{t=0} = \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0} = \frac{2}{\lambda^2}$$

Probability generating function is $f(s) = \mathbb{E}[s^X]$.

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Properties: memoryless $P(X > s+t \mid X > t) = P(X > s)$ $\forall s,t \geq 0.$

probability of survival for s+t years given survival for t years is probability of survival for s years.

$e^{-\lambda(s+t)}/e^{-\lambda t} = e^{-\lambda t}$ ✓

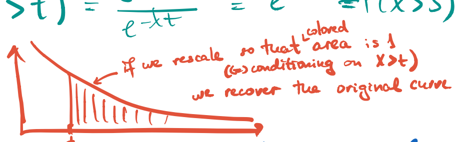
Equivalently, $P(X > s+t,\ X > t) = P(X > s)P(X > t).$

Example: Suppose the service time for a customer at a bank follows $Exp(\lambda)$.

Suppose customer 2 has now waited for t minutes. What is the probability they'll wait for another s minutes?

$$P(X > s+t \mid X > t) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$$

Graphical interpretation:



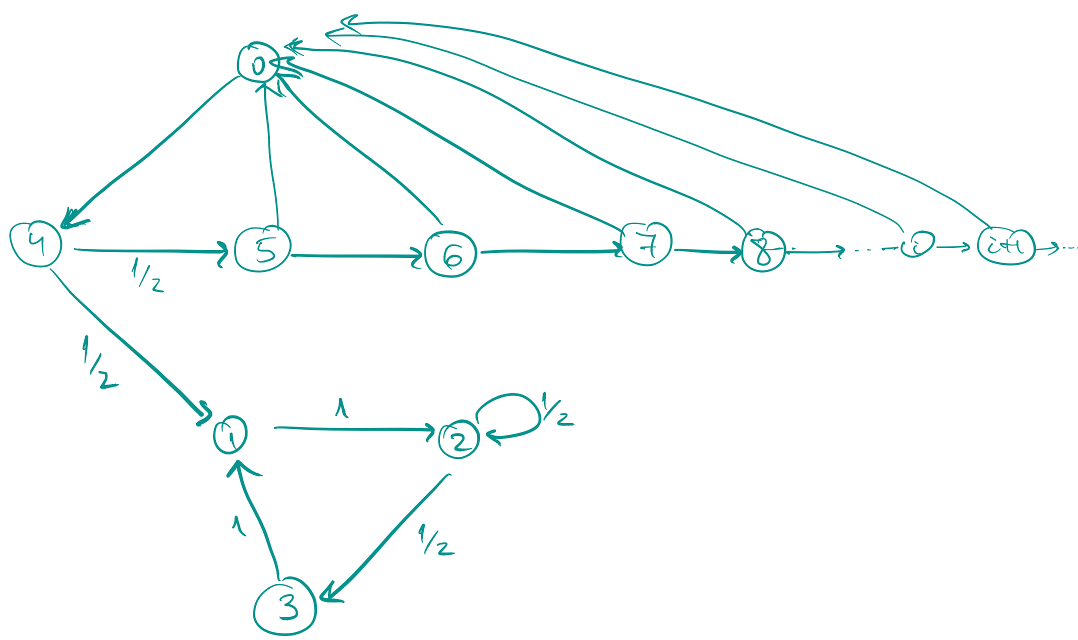
Example: Consider a post office with two clerks. The amount of time a clerk spends with a customer is $\sim Exp(\lambda)$.

Cameron enters the post office and there are already two customers Alice and Bob being helped by Clerk 1 and Clerk 2.

What is the probability that of the 3 customers Cameron is the last to leave the office?

Solution: Suppose Cameron finds a free clerk at time t.
At this point either Alice or Bob would have just left the office. By the memoryless property, the amount of time that whoever remains will still be $Exp(\lambda)$
$\Rightarrow$ it is the same as if they were just starting to be helped.
$\Rightarrow$ by symmetry prob(Cameron leaves last) $= \frac{1}{2}$.

Example:



1. What are the different communicating classes and what types are they?

2. Consider the closed class $\{1,2,3\}$. Write down the transition matrix, and compute $P(X_3=i \mid X_0=1)$, for $i=1,2,3$.

3. What is $\lim_{n\to\infty} p_{23}^{(n)}$?

4. What is the mean # of steps to revisit 3?

1. $\{1,2,3\}$ positive recurrent

$\{0,4,5,6,\dots\}$ transient $\qquad P(\text{return to } 4 \mid X_0=4) \le \frac{1}{2} < 1.$

2.
$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \end{array}.$$

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 5/8 & 1/8 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

3. Not reversible.

$$(\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix} = \left(\pi_3,\ \pi_1 + \tfrac{1}{2}\pi_2,\ \tfrac{1}{2}\pi_2\right) = (\pi_1, \pi_2, \pi_3)$$

$$\pi_1 = \pi_3 = \tfrac{1}{2}\pi_2 \qquad \tfrac{5}{2}\pi_2 = 1$$

$$\pi_2 = \frac{2}{5}, \quad \pi_1 = \pi_3 = \frac{1}{5}.$$

$$\Rightarrow \lim_{n\to\infty} P_{23}^{(n)} = \pi_3 = \frac{1}{5}$$

4. Mean # of steps to revisit 3 is 5.