- Midterm on Thursday: 6-7PM in IBLC 182
  alternative Thursday: 9:30-10:30 in Buchanan A102.
- Homeworks 1-3 have been returned to the MLC, grades entered in Canvas.
- See website for practice problems.

→ how to derive and use transition matrices and diagrams
→ how to apply those to compute probabilities and expectations
  prob visit state i before state j... $\mathbb{E}[X_n \mid X_0 = 1]$
→ how to classify a M.C. and its states.
→ how to study limiting probabilities and their applications.
→ the main results from class and the (simpler) proofs.

<u>Last time:</u> Branching process

$Z_0 = 1.$

$Z_{n+1} = Y_{n,1} + \dots + Y_{n,Z_n}$

$Y_{n,i}$ i.i.d. same law called the reproductive law.

<u>Q</u>: What is $\mathbb{P}(\text{extinction}) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{Z_n = 0\}\right)$

$$= \lim_{n\to\infty} \mathbb{P}(Z_n = 0) = \lim_{n\to\infty} G_n(0)$$

showed last time

where $G_n(s) = \mathbb{E}[s^{Z_n}]$ generating function of Z_n.

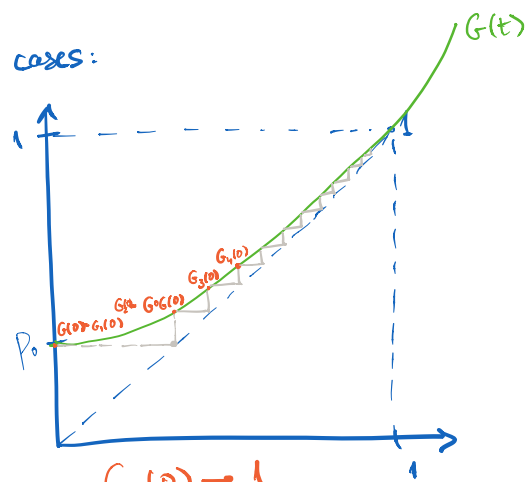
We showed that $G_n(s) = \underbrace{G \circ \dots \circ G}_{n\text{-times}}(s)$,

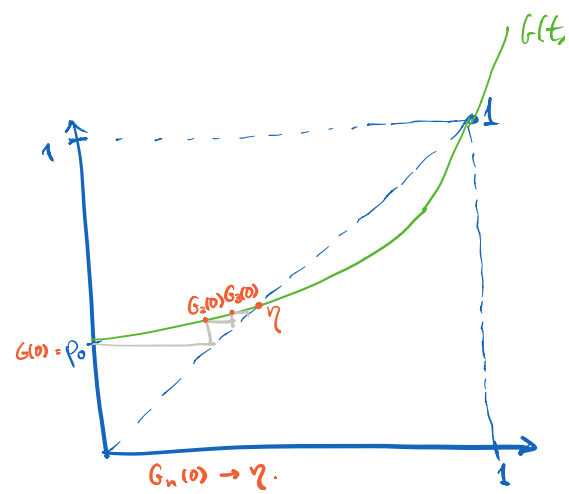
where G is the generating function of $Y_{n,i}$.

Also, G is non-decreasing and convex on $[0,1]$.

$G(0) = p_0$ = probability of 0 offspring.

Two cases:





$G_n(0) \to$

$\Rightarrow P(\text{extinction}) = 1.$

$\Rightarrow P(\text{extinction}) = \eta.$

Here η = smallest fixed point of G on $[0,1]$.

<u>Remark</u>: if $p_0 = 0$, the population doesn't decrease

$\Rightarrow \eta = 0.$

Assuming $p_0 > 0$:

By convexity, $\eta = \lim G_n(0) = \begin{cases} 1 & \text{if } G'(1) < 1 \text{ or if } G'(1) = 1 \text{ and } G''(1) > 0 \\ < 1 & \text{if } G'(1) > 1. \end{cases}$

Since $G'(1) = \mathbb{E}[Y_{0,1}]$ and $G''(1) = Var(Y_{0,1}) + \mathbb{E}[Y_{0,1}^2] - \mathbb{E}[Y$

we conclude

<u>Theorem</u>: Let $\mu = \mathbb{E}[Y]$ and $\sigma^2 = Var(Y)$, where Y is the reproductive law of the branching process. Then, the probability of extinction η is the smallest nonnegative root of $s = G(s)$

and:

$P(\text{extinction}) = 1$ if $\mu < 1$

$P(\text{extinction}) = 1$ if $\mu = 1$ & $\sigma^2 > 0$

$P(\text{extinction}) < 1$ if $\mu > 1$

<u>Example</u>: Recall the Poisson random variable

$Y \sim \text{Poisson}(\lambda)$, $Y \in \mathbb{N}$

$P(Y = k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$, $\mathbb{E}[Y] = \lambda$, $Var(Y) = \lambda$.

Suppose the reproductive law follows a Poisson distribution with parameter $\lambda > 0$. What is the probability of extinction?