

02/07/2020

irreducible and positive recurrent

\Rightarrow

$\bar{\pi}$ exists and is unique, $\bar{\pi}_j > 0$
 $\bar{\pi}_j = \text{proportion of time } X_n \text{ spends in } j$
 $= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{ij}^{(k)} = \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{1}{n} \sum_{k=0}^{n-1} I(X_k=j) \mid X_0=i \right]$
 $\frac{1}{\bar{\pi}_j} = m_j = \text{mean time to return to } j$
 $+ \text{aperiodic} \Rightarrow \bar{\pi}_j = \lim_{n \rightarrow \infty} P_{ij}^{(n)}$

irreducible and transient or null-recurrent

\Rightarrow

$\bar{\pi}$ does not exist and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{ij}^{(k)} = 0$

$$P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \bar{\pi} = \left(\frac{1}{2}, \frac{1}{2} \right)$$

$$P^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad P(X_n=0 \mid X_0=0) = \begin{cases} 1 & \text{n even} \\ 0 & \text{n odd} \end{cases}$$

$$\frac{1}{2n} \sum_{k=0}^{2n-1} P_{01}^{(k)} = \frac{1}{2n} \cdot n = \frac{1}{2} = \bar{\pi}_1$$

$$\frac{1}{2n+1} \sum_{k=0}^{2n} P_{01}^{(k)} = \frac{1}{2n+1} \cdot n \approx \frac{1}{2}$$

$$P_{01}^{(k)}$$

$$\lim P(B_n=3)$$

①

Two random variables with the same generating functions have the same laws.

Theorem: If X and Y are independent r.v.'s on \mathbb{N} , then $G_{X+Y}(s) = G_X(s) \cdot G_Y(s)$ $\forall s \in [-1, 1]$.

Proof: $G_{X+Y}(s) = \mathbb{E}[s^{X+Y}] = \mathbb{E}[s^X \cdot s^Y]$

$$= \mathbb{E}[s^X] \mathbb{E}[s^Y] = G_X(s) G_Y(s).$$

$\stackrel{\substack{X \perp\!\!\!\perp Y \\ X, Y \text{ independent}}}{=}$

Proposition: Let X_1, X_2, \dots be i.i.d. with generating function G_X . Let N be a r.v. on \mathbb{N} with generating function G_N , independent of X_1, X_2, \dots .

let $T = X_1 + X_2 + \dots + X_N$. Then, $G_T(s) = G_N(G_X(s))$.

Proof: $G_T(s) = \mathbb{E}[s^T] = \mathbb{E}[s^{X_1 + \dots + X_N}]$

$$= \mathbb{E}(\mathbb{E}[s^{X_1 + \dots + X_N} | N])$$

$$= \sum_{n=0}^{\infty} P(N=n) \mathbb{E}[s^{X_1 + \dots + X_N} | N=n]$$

$$= \sum_{n=0}^{\infty} P(N=n) \underbrace{\mathbb{E}[s^{X_1 + \dots + X_n}]}_{= \mathbb{E}[s^{X_1} \dots s^{X_n}] = \mathbb{E}[s^{X_1}] \dots \mathbb{E}[s^{X_n}] = (G_X(s))^n}$$

$$= \sum_{n=0}^{\infty} P(N=n) \cancel{(G_X(s))^n} = G_N(G_X(s))$$

(2)

Application to the Branching process

$$Z_{n+1} = Y_{n,1} + Y_{n,2} + \dots + Y_{n,Z_n}. \quad \text{Gy gen. fct. of } Y_{n,i}$$

~~Z_{n+1}~~ generating function of Z_{n+1} .

$$G_{n+1}(s) := G_{Z_{n+1}}(s) = G_{Z_n}(G_y(s)) = G_{Z_{n-1}}(G_y(G_y(s)))$$

$$= G_{Z_{n-1}} \circ G_y \circ G_y(s) = \dots = \underbrace{G_y \circ G_y \circ \dots \circ G_y(s)}_{n+1}$$

$$\Rightarrow \boxed{G_n(s) = \underbrace{G_y \circ \dots \circ G_y(s)}_n} \quad G_1(s) = G_y(s).$$

We will show $P(\text{extinction}) = \lim_{n \rightarrow \infty} G_n(0)$.

Proof:

$$\text{extinction event} = \bigcup_{n=0}^{\infty} \{Z_n = 0\} =$$

$$\text{and } \{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}$$

$$\{Z_1 = 0\} \subseteq \{Z_2 = 0\} \subseteq \dots \subseteq \{Z_n = 0\} \subseteq \dots \supseteq \bigcup_{n=0}^{\infty} \{Z_n = 0\}$$

$$P(\text{extinction}) \leq P(Z_1 = 0) \leq \dots \leq P(Z_n = 0) \leq \dots \rightarrow P(\text{extinction})$$

$$P(\text{extinction}) = \lim_{n \rightarrow \infty} \frac{P(Z_n = 0)}{G_n(0)} = \lim_{n \rightarrow \infty} G_n(0)$$

For a given r.v. X with gen. fct. $G_X = G$, we have

- $G(1) = \mathbb{E}[1^X] = 1.$

- $G'(1) = \mathbb{E}[X]$

(recall $G'(s) = \sum_{k=1}^{\infty} k s^{k-1} P(X=k)$)
 $\Rightarrow G'(1) = \sum_{k=0}^{\infty} k P(X=k) = \mathbb{E}[X]$

- $G''(1) = \text{Var}(X) + (\mathbb{E}[X])^2 - \mathbb{E}[X]$

(recall $G''(s) = \sum_{k=1}^{\infty} k(k-1) s^{k-2} P(X=k)$)

$$= \sum_{k=0}^{\infty} k^2 s^{k-2} P(X=k) - \sum_{k=0}^{\infty} k s^{k-2} P(X=k)$$

$$\Rightarrow G''(1) = (\mathbb{E}[X^2])^2 - \mathbb{E}[X]$$

$$= (\mathbb{E}[X^2] - (\mathbb{E}[X])^2) \text{Var}(X)$$

$$+ (\mathbb{E}[X])^2 - \mathbb{E}[X]$$

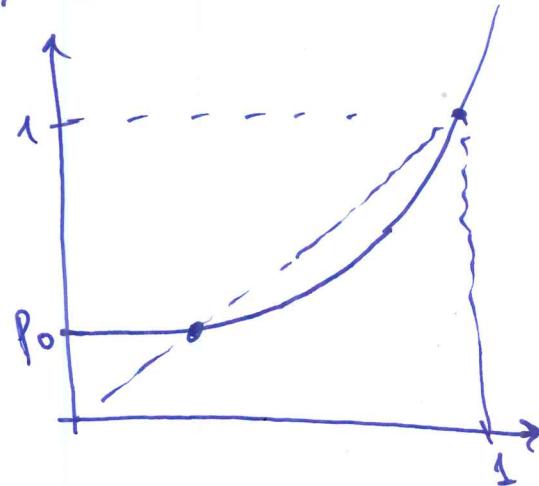
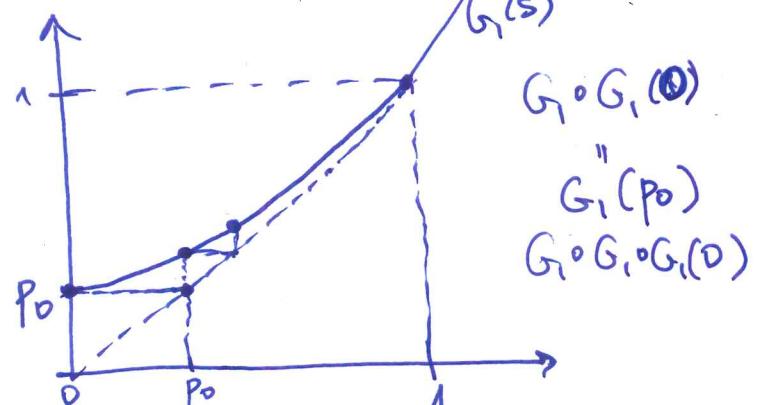
- G is non-decreasing and convex on $[0, 1]$

$\hookrightarrow G' > 0$

$\hookrightarrow G'' > 0$

$$G_1(0) = P(Y=0) = p_0$$

$$G_1(1) = 1$$



④