

irreducible
and
positive recurrent

π exists and is unique, $\pi > 0$
 $\pi_j = \text{proportion of time } X_n \text{ spends in state } j$.
 $\frac{1}{\pi_j} = u_j = \text{mean time to return to state } j$.
 + aperiodic $\Rightarrow \pi_j = \lim_{n \rightarrow \infty} P_{ij}^{(n)}$.

irreducible
and
transient or null-recurrent

π does not exist and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P_{ij}^{(k)} = 0$
w.p.1.

Last time we defined branching processes. Recall:

Definition: A branching process (or Galton-Watson process) is a sequence of random variables $(Z_n)_{n \geq 0}$ defined by

$$\begin{cases} Z_0 = 1 \\ Z_{n+1} = \sum_{i=1}^{Z_n} Y_{n,i}, \end{cases}$$

where $Y_{n,i}$ are i.i.d. random variables on \mathbb{N} whose law is called the reproduction law of the process. Z_n is the # of individuals in the n -th generation, and $Y_{n,i}$ is the # of offspring of the i -th individual of the n -th generation.

Q: The original question asked by Galton and Watson was: what is the probability of extinction of the population?
 More generally, how can we find $P(Z_n = k)$?

Remark: . $\{0\}$ is a recurrent class, also absorbing. Assuming $P(Y_{n,i} > 0) = p_i > 0$, $P_{00} = p_0 > 0$, so 0 is accessible from all $i > 0$.
 $\Rightarrow i$ is transient $\forall i > 0$.

. Since any finite set of transient states can only be visited finitely many times, either Z_n is eventually 0, or $Z_n \rightarrow \infty$.

To study the extinction probability (or the survival probability), we will use **Generating functions**.

Definition: Let X be a random variable on \mathbb{N} . We call the **generating function** of X the power series:

$$G_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} P(X=k) s^k.$$

Remarks: The radius of convergence of G_X is ≥ 1 since

$$G_X(1) = \sum_{k=0}^{\infty} P(X=k) = 1. \Rightarrow G_X \text{ is well-defined } \forall s \in [-1, 1].$$

$$\bullet G_X(0) = P(X=0), \quad G_X'(s) = \sum_{k=1}^{\infty} P(X=k) k s^{k-1} \Rightarrow G_X'(0) = P(X=1).$$

$$G_X''(s) = \sum_{k=2}^{\infty} P(X=k) k(k-1) s^{k-2} \Rightarrow G_X''(0) = \frac{P(X=2)}{2!}$$

$$\dots \quad G_X^{(k)}(0) = \frac{P(X=k)}{k!}.$$

\Rightarrow Two random variables with the same generating functions have the same laws.

Theorem: If X and Y are independent random variables on \mathbb{N} , then $\forall t \in [-1, 1]$, $G_{X+Y}(t) = G_X(t) \cdot G_Y(t)$.

Proof: $G_{X+Y}(t) = \mathbb{E}[t^{X+Y}] = \mathbb{E}[t^X \cdot t^Y] = \mathbb{E}[t^X] \mathbb{E}[t^Y]$
 $= G_X(t) G_Y(t)$.

Proposition: Let X_1, X_2, \dots be i.i.d. r.v.'s with generating function G_X . Let N be a r.v. with generating function G_N , independent of X_1, X_2, \dots . Let $T = X_1 + \dots + X_N$. Then, $G_T(s) = G_N(G_X(s))$.

$$\begin{aligned}
 \text{Proof: } G_T(s) &= \mathbb{E}[s^T] = \mathbb{E}[\mathbb{E}[s^T | N]] \\
 &= \sum_{n=0}^{\infty} \mathbb{E}[s^T | N=n] P(N=n) = \sum_{n=0}^{\infty} \mathbb{E}[s^{X_1+\dots+X_n}] P(N=n) \\
 &= \sum_{n=0}^{\infty} (G_X(s))^n P(N=n) = G_T(G_X(s)).
 \end{aligned}$$

Application to the Branching process:

$$\begin{aligned}
 Z_{n+1} &= Y_{n,1} + Y_{n,2} + \dots + Y_{n,n}. \\
 G_{Z_{n+1}}(s) &\stackrel{\text{generating function of } Z_{n+1}}{=} G_{Z_n}(G_Y(s)) = G_{Z_{n-1}}(G_Y(G_Y(s))) \\
 \Rightarrow G_{Z_{n+1}}(s) &= G_{Z_n}(G_Y \circ G_Y(s)) = \dots = G_{Z_1}(\underbrace{G_Y \circ \dots \circ G_Y}_{n}(s)) \\
 Z_1 &= Y_{0,1} \Rightarrow \quad = \underbrace{G_Y \circ G_Y \circ \dots \circ G_Y}_{n+1}(s).
 \end{aligned}$$

$$\text{Will show } P(\text{extinction}) = \lim_{n \rightarrow \infty} G_n(0).$$

The generating function of Z_n , $G_n(s)$ is the n -th iterate of the generating function of $Z_1 = Y_{0,1}$.

Let $\eta = P(\text{extinction})$. (Then, survival probability = $1-\eta$.)

$$\begin{aligned}
 \text{extinction event} &= \bigcup_{n=0}^{\infty} \{Z_n = 0\} \quad \text{and} \quad \{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}. \\
 \Rightarrow P(Z_1 = 0) &\leq P(Z_2 = 0) \leq \dots \leq P(Z_n = 0) \leq \dots \\
 \text{and } \eta &= \lim_{n \rightarrow \infty} P(Z_n = 0) = \lim_{n \rightarrow \infty} G_n(0).
 \end{aligned}$$

We now study the generating function of the reproductive law G_1 .

For a given random variable X with generating function $G_X = G$, we have the following properties:

- $G(1) = \mathbb{E}[1^X] = 1.$

- $G'(1) = \mathbb{E}[X]$ (exercise)
 (recall $G'(s) = \sum_{k=1}^{\infty} ks^{k-1} P(X=k) = \mathbb{E}[X]$).

- $G''(1) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \text{Var}(X) + (\mathbb{E}[X])^2 - \mathbb{E}[X].$

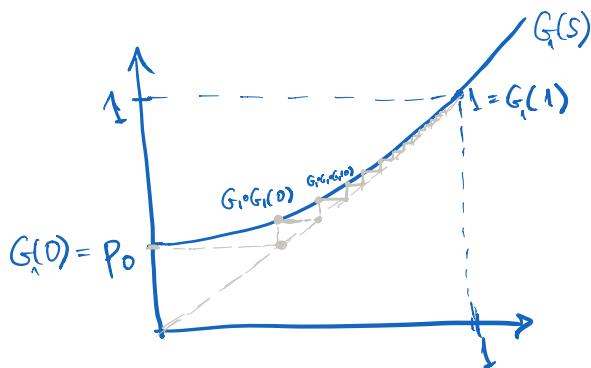
$$\Rightarrow \text{Var}(X) = G''(1) - G'(1)^2 + G'(1).$$

$$\begin{aligned} \text{(recall } G''(s) &= \sum_{k=2}^{\infty} k(k-1)s^{k-2} P(X=k) \\ &= \sum_{k=2}^{\infty} k^2 s^{k-2} P(X=k) - \sum_{k=2}^{\infty} k s^{k-2} P(X=k) \end{aligned}$$

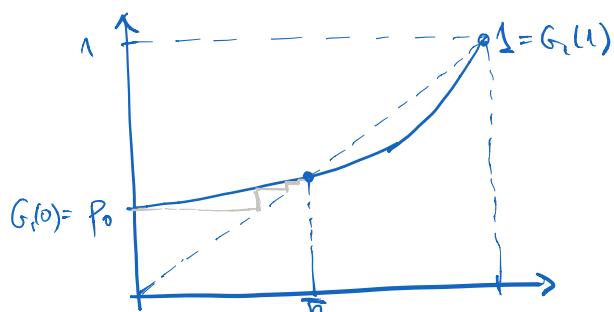
$$= \mathbb{E}[X^2] - \mathbb{E}[X] - \mathbb{E}[X] + \mathbb{P}(X \neq 1).$$

- G is non-decreasing and convex on $[0, 1]$.
 $\hookrightarrow G' \geq 0$ $\hookrightarrow G'' \geq 0$

We can thus visualize $G_n(0)$ from the graph of G :



$$\eta = \lim_{n \rightarrow \infty} G_n(0) = G_0 \circ G_1 \circ \dots \circ G_n(0)$$



$$\eta = \lim_{n \rightarrow \infty} G_n(0) = \bar{\eta}$$

$\Rightarrow \eta$ is the smallest fixed point of G on $[0, 1]$.

Remark: if $p_0 = 0$, the population doesn't decrease
 $\Rightarrow \eta = 0$.

Assuming $p_0 > 0$:

$$\text{By convexity, } \eta = \lim G_n(0) = \begin{cases} 1 & \text{if } G'(1) < 1 \text{ or if } G'(1) = 1 \\ & \quad \text{and } G''(1) > 0 \\ < 1 & \text{if } G'(1) > 1. \end{cases}$$

Since $G'(1) = \mathbb{E}[Y_{0,1}]$ and $G''(1) = \text{Var}(Y_{0,1}) + \mathbb{E}[Y_{0,1}^2] - \mathbb{E}[Y_{0,1}]^2$

we conclude

Theorem: let $\mu = \mathbb{E}[Y]$ and $\sigma^2 = \text{Var}(Y)$, where Y is the reproductive law of the branching process. Then, the probability of extinction η is the smallest nonnegative root of $s = G(s)$

and:

$$\eta = 1 \quad \text{if } \mu < 1$$

$$\eta = 1 \quad \text{if } \mu = 1 \text{ & } \sigma^2 > 0$$

$$\eta < 1 \quad \text{if } \mu > 1$$