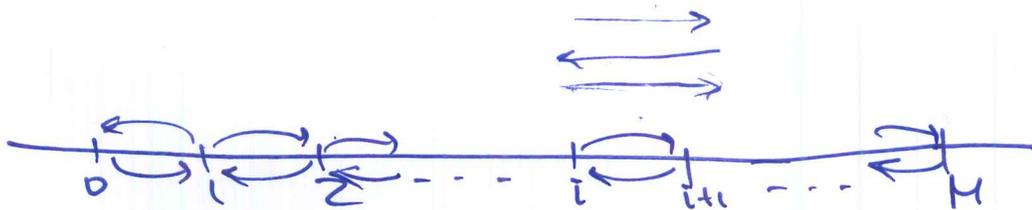
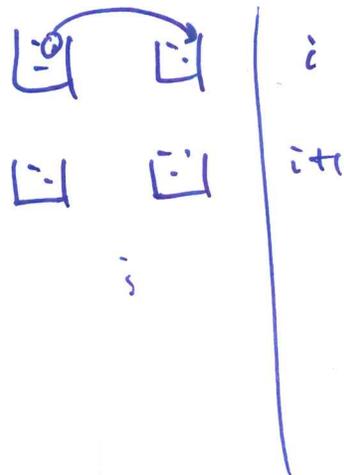


# The Ehrenfest chain

$X_n = \# \text{ balls in urn 1}$

$S = \{0, 1, \dots, M\}$



$$P(X_n = i+1 \mid X_{n-1} = i) = \frac{M-i}{M}$$



$$P(X_n = i-1 \mid X_{n-1} = i) = \frac{i}{M}$$

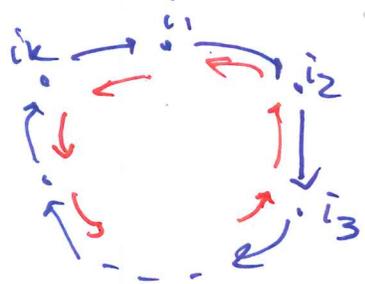
Q: What is  $\pi$ ?

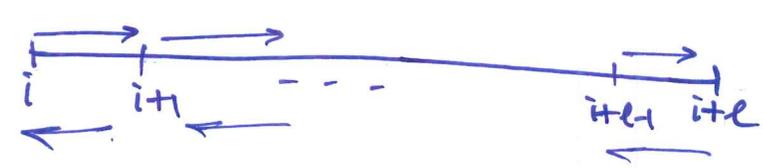
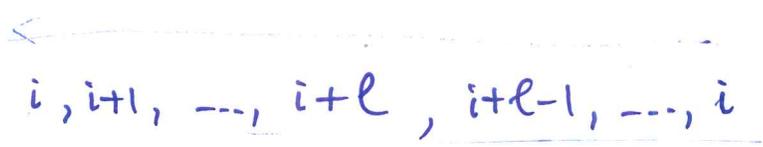
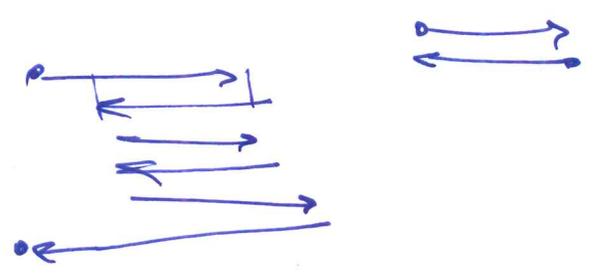
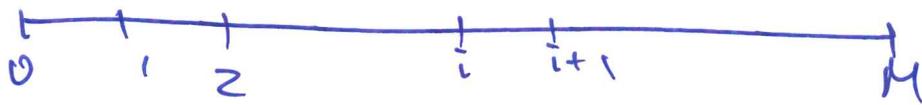
time-reversible:  $\pi_i P_{ij} = \pi_j P_{ji}$

Criterion for time-reversibility:

$$\begin{aligned} & P_{i_1 i_2} P_{i_2 i_3} \dots P_{i_{k-1} i_k} P_{i_k i_1} \\ &= P_{i_1 i_k} P_{i_k i_{k-1}} \dots P_{i_2 i_1} \end{aligned}$$

$$\forall i_1, \dots, i_k$$





$i+1$   $i$   $i+1$   
 $i+2$   
 $i+l-1$   $i+l$

CW  $P_{i,i+1} \dots P_{i+l-1,i+l} \cdot P_{i+l,i+l-1} \dots P_{i+1,i} =$   
 $= P_{i,i+1} \dots P_{i+l-1,i+l} \cdot P_{i+l,i+l-1} \dots P_{i+1,i}$  CCW

Now, let's solve  $\pi_i P_{ij} = \pi_j P_{ji}$ .

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$$

$$\pi_i \frac{M-i}{M} = \pi_{i+1} \frac{i+1}{M}, \quad \forall i = 0, 1, \dots, M-1$$

$$\sum_{i=0}^M \pi_i = 1$$

$$\pi_0 \cdot 1 = \pi_1 \cdot \frac{1}{M}, \quad \pi_1 \cdot \frac{M-1}{M} = \pi_2 \cdot \frac{2}{M}$$

$$\pi_2 \cdot \frac{M-2}{M} = \pi_3 \cdot \frac{3}{M}$$

$$\pi_1 = \frac{M\pi_0}{1}, \quad \pi_2 = \frac{M-1}{2} \cdot \pi_1 = \frac{M-1}{2} \cdot \frac{M}{1} \pi_0$$

$$\pi_3 = \frac{M-2}{3} \pi_2 = \frac{M-2}{3} \frac{M-1}{2} \frac{M}{1} \cdot \pi_0$$

⋮

$$\pi_i = \frac{M-i+1}{i} \cdot \frac{M-i+2}{i-1} \cdots \frac{M-1}{2} \cdot \frac{M}{1} \pi_0$$

$$= \frac{M(M-1) \cdots (M-i+1)}{i!} \pi_0 = \frac{M!}{i!(M-i)!} \pi_0 = \binom{M}{i} \pi_0$$

$$\boxed{\pi_i = \binom{M}{i} \pi_0}$$

$$\sum_{i=0}^M \pi_i = 1 \quad \text{i.e.} \quad \left[ \sum_{i=0}^M \binom{M}{i} \right] \pi_0 = 1$$

$$\left[ \sum_{i=0}^M \binom{M}{i} 1^i \cdot 1^{M-i} \right] \pi_0$$

$$\stackrel{\text{"}}{=} 2^M \pi_0$$

$$\pi_0 = \frac{1}{2^M}$$

$$\pi_i = \binom{M}{i} \frac{1}{2^M}$$

$$\pi \sim \text{Bin}(M, \frac{1}{2})$$

# Branching process

(Ross 4.7, but we go in more detail)

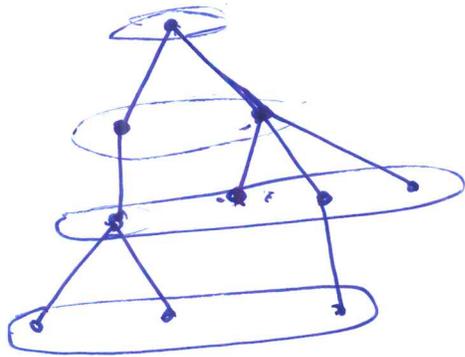
$$z_0 = 1$$

$$z_1 = 2$$

$$z_2 = 4$$

$$z_3 = 3$$

$$z_n = 0, n \geq 4$$



•  $z_n$  = size of population at time  $n$

• Each individual has a random # of offspring (with same distribution)

• The process can be modeled as follows.

Definition: A **branching process** (or **Galton-Watson process**) is a sequence of random variables  $(Z_n)_{n \geq 0}$  defined by

$$z_0 = 1$$
$$z_{n+1} = \sum_{i=1}^{z_n} y_{n,i} \quad \left( \text{e.g. } y_{2,1} = 1 \right)$$

where  $y_{n,i}$  are i.i.d. random variables on  $\mathbb{N}$  whose law is called the **reproduction law** of the process. ~~the~~  $z_n$  is the # of individuals in the  $n$ -th generation, and  $y_{n,i}$  is the # of offspring of the  $i$ -th individual of the  $n$ -th generation.

Q: What is the probability of extinction of the population?

In general, what is  $P(Z_n = k)$ ?

(4) ~~(4)~~

Remarks: •  $\{0\}$  is a recurrent class, also absorbing.

Assuming  $P(Y_{n,i} = 0) = p_0 > 0$ ,  $P_{i0} = p_0^i > 0 \Rightarrow$

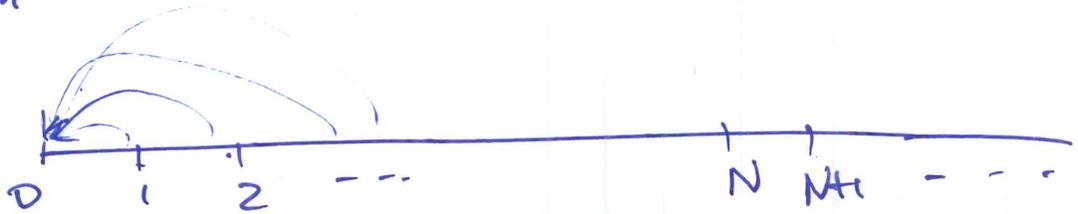
0 is accessible from all  $i > 0$

$\Rightarrow i$  is transient  $\forall i > 0$

• Since any finite set of transient states can only be visited finitely many times, either

•  $Z_n$  is eventually 0, or

•  $Z_n \rightarrow \infty$  as  $n \rightarrow \infty$ .



To study the extinction probability (or the survival probability), we will use **Generating functions**.

Definition: Let  $X$  be a r.v. on  $\mathbb{N}$ . We call the **generating function** of  $X$  the power series:

$$G_X(s) = \mathbb{E}[s^X] = \sum_{i=0}^{\infty} s^i P(X=i).$$

Remarks: • The radius of convergence of  $G_X$  is  $\geq 1$  since  $G_X(1) = \sum_{i=0}^{\infty} P(X=i) = 1 \Rightarrow G_X$  is well-defined for  $s \in [-1, 1]$ .

•  $G_X(0) = P(X=0)$ ,  $G_X'(s) = \sum_{i=1}^{\infty} s^{i-1} i P(X=i)$

$G_X'(0) = P(X=1)$

$$G_X''(s) = \sum_{i=2}^{\infty} s^{i-2} i(i-1) P(X=i)$$

$$G_X''(0) = 2 \cdot 1 \cdot P(X=2)$$

$\vdots$

In general,  $G_X^{(k)}(s) = \sum_{i=k}^{\infty} s^{i-k} i(i-1)\dots(i-k+1) P(X=i)$

$$\Rightarrow G_X^{(k)}(0) = \cancel{i(i-1)\dots(i-k+1)} \cdot k(k-1)\dots 1 \cdot P(X=k)$$

$$\Rightarrow \boxed{P(X=k) = \frac{G_X^{(k)}(0)}{k!}}$$