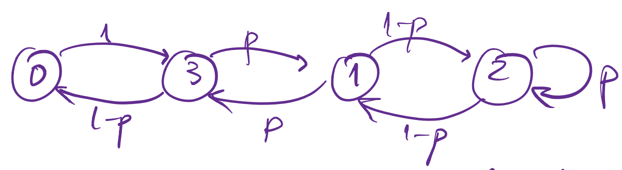
Example: Alice has 3 umbrellas (total) at home and at work.

- takes an umbrella if it is raining, if there is one
- doesn't take an umbrella if it is not raining
- it rains each trip w.p. p (independently of the other trips)

Q: What fraction of time does Alice get wet?

Solution: X_n = # umbrellas at current location

state space $= \{0,1,2,3\}$.



Chain is irreducible and ergodic (aperiodic, positive recurrent)

Answer is $\pi_0 \cdot p$ ← probability it rains, where π is the stationary distribution.

π_0 ↓ time with no umbrella

To find π, we can try to find a solution to:

$$\begin{cases} \pi_0 P_{03} = \pi_3 P_{30} \\ \pi_3 P_{31} = \pi_1 P_{13} \\ \pi_1 P_{12} = \pi_2 P_{21} \end{cases} \iff \begin{cases} \pi_0 = \pi_3 (1-p) \\ \pi_3 = \pi_1 \\ \pi_1 = \pi_2 \end{cases}$$

so $\pi_1 = \pi_2 = \pi_3$, $\sum_i \pi_i = 1$, $\pi_0 = (1-p)\pi_3$

$\Rightarrow \pi_3(1-p) + 3\pi_3 = 1$, $\Rightarrow \pi_1 = \pi_2 = \pi_3 = \dfrac{1}{4-p}$

$\Rightarrow \pi_0 = \frac{1-p}{4-p}$ $\qquad \pi = \left(\frac{1-p}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}\right)$

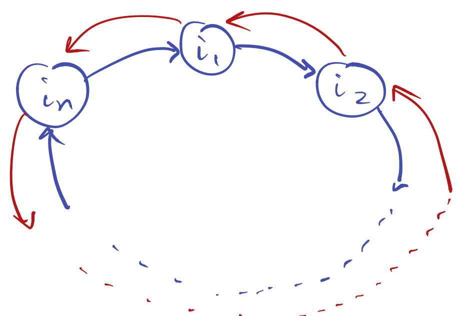
$\Rightarrow \boxed{\pi_0 \cdot p = \frac{1-p}{4-p} \cdot p}$.

Material for Midterm 1 ends here.

A criterion for time-reversibility.

Proposition (Kolmogorov's criterion): An irreducible ergodic M.C. is reversible if and only if for all finite sequences of states $(i_1, \dots, i_n)$,

$$P_{i_1 i_2} P_{i_2 i_3} \cdots P_{i_{n-1} i_n} P_{i_n i_1} = P_{i_1 i_n} P_{i_n i_{n-1}} \cdots P_{i_2 i_1}.$$



Proof: $\boxed{\Rightarrow}$ $n=2$: $\quad P_{i_1 i_2} P_{i_2 i_1} = P_{i_2 i_1} P_{i_1 i_2}$ always true!

$n=3$: $\quad P_{jk} P_{ki} = P_{jk} P_{ki} \quad \Big| \cdot \pi_i P_{ij} = \pi_j P_{ji}$

$\Leftrightarrow \quad \pi_i P_{ij} P_{jk} P_{ki} = \underbrace{\pi_j P_{ji}}_{} P_{jk} P_{ki}$

$\Leftrightarrow \quad \pi_i P_{ij} P_{jk} P_{ki} = P_{ji} \underbrace{\pi_k P_{kj}}_{} P_{ki}$

$\Leftrightarrow \quad \cancel{\pi_i} P_{ij} P_{jk} P_{ki} = P_{ji} P_{kj} P_{ik} \cancel{\pi_i}$

$$\Rightarrow \quad \boxed{p_{ij}\, p_{jk}\, p_{ki} = p_{ji}\, p_{ik}\, p_{kj}}$$

and the reasoning is the same when $n \geq 3$.

(i.e., start with $(\pi_{i_1} p_{i_1 i_2})(p_{i_2 i_3} \cdots p_{i_n i_1}) = \underbrace{(\pi_{i_2} p_{i_2 i_1})(p_{i_2 i_3}}_{\text{combine}} \cdots p_{i_n i_1})$

$\vdots$

until $\cancel{\pi_{i_1}}\, p_{i_1 i_2}\, p_{i_2 i_3} \cdots p_{i_n i_1} = p_{i_2 i_1}\, p_{i_3 i_2} \cdots p_{i_1 i_n} \cancel{\pi_{i_1}}$.

$\boxed{\Leftarrow}$: For any couple of states (i,j) and any sequence of states $i_1, \dots, i_k$:

$$p_{i i_1}\, p_{i_2 i_3} \cdots p_{i_k j}\, p_{ji} = p_{ij}\, p_{j i_k} \cdots p_{i_1 i}$$

Summing over all possible sequences $i_1, \dots, i_d$:

$$\sum_{i_1, \dots, i_d} (p_{i i_1}\, p_{i_2 i_3} \cdots p_{i_k j})\, p_{ji} = p_{ij} \sum_{i_1, \dots, i_d} (p_{j i_k} \cdots p_{i_1 i})$$

$$\Rightarrow \quad p_{ij}^{(k)}\, p_{ji} = p_{ij}\, p_{ji}^{(k)} \quad \forall k$$

k-step transition probability from i to j.

$$\downarrow k \to \infty \qquad \downarrow k \to \infty$$

$$\pi_j\, p_{ji} = p_{ij}\, \pi_i$$

A classical example of time-reversible M.C.:

the Ehrenfest chain (1907 Paul & Tatiana Ehrenfest) to describe the movements of molecules

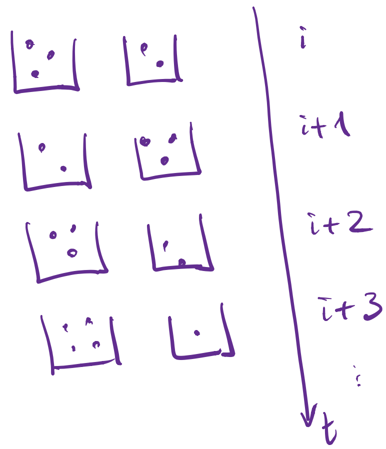
→ Toy model of gas behaviour in two containers

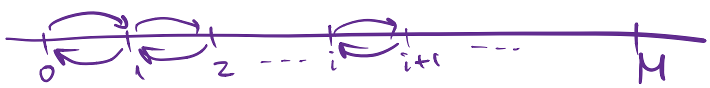
→ Consider M balls distributed in 2 urns.

At each step select a ball at random and move it to the other urn.

X_n = # balls in urn #1.

$S = \{0, 1, \dots, M\}$.





The chain is ergodic (why?)

finite state space, aperiodic, irreducible, positive recurrent $\Rightarrow$ ergodic

What is the stationary distribution?

$$P(X_n = i+1 \mid X_{n-1} = i) = \frac{M-i}{M}.$$

$$P(X_n = i-1 \mid X_{n-1} = i) = \frac{i}{M}.$$

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$$

$$\pi_i \frac{M-i}{\cancel{M}} = \pi_{i+1} \cdot \frac{i+1}{\cancel{M}}$$

$$\frac{\pi_i}{\pi_{i+1}} = \frac{i+1}{M-i}$$