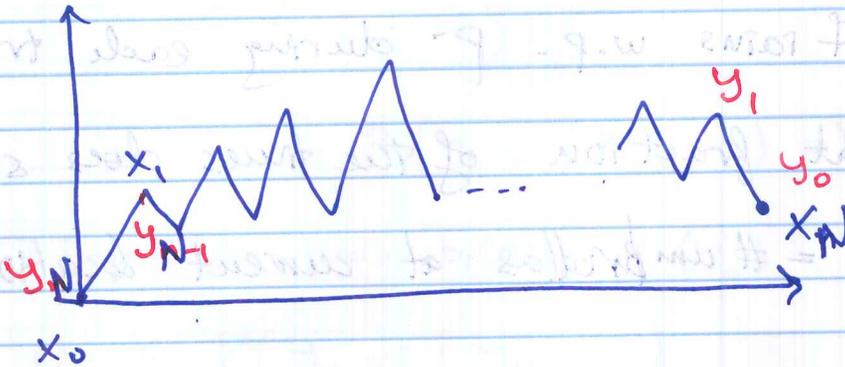


Time Reversibility

M.C. X_0, X_1, \dots, X_N



Theorem: Given a M.C. $(X_n)_{0 \leq n \leq N}$ with stationary distribution π (we assume it exists) and with $P(X_0=j) = \pi_j$, let $Y_n = X_{N-n}$.

Then, $(Y_n)_{0 \leq n \leq N}$ is a M.C. with stationary distr. π , and transition probabilities $Q_{ij} = \frac{P_{ji} \pi_j}{\pi_i}$.

Remark: The existence of the stationary distr. π of (X_n) is necessary for the reversed process to be homogeneous.

Proof: 1) Markov property for Y_n :

$$P(Y_n=j | Y_{n-1}=i, Y_{n-2}=i, \dots) = P(Y_n=j | Y_{n-1}=i) \quad (*)$$

$$\text{LHS} = \frac{P(Y_n=j, Y_{n-1}=i, \overbrace{Y_{n-2}=i, \dots}^E)}{P(Y_{n-1}=i, \overbrace{Y_{n-2}=i, \dots}^E)} = \frac{P(Y_n=j, Y_{n-1}=i, E)}{P(Y_{n-1}=i, E)}$$

$$= \frac{P(X_{N-n}=j, X_{N-n+1}=i, E)}{P(X_{N-n+1}=i, E)} \quad \begin{matrix} \rightarrow Y_{n-2}=i, \dots, Y_0= \\ X_{N-n+2}=i, \dots, X_N= \end{matrix}$$

(2)

$$\cancel{P(E | Y_n = j)} = \frac{P(E | X_{N-n} = j, X_{N-n+1} = i) P(X_{N-n} = j, X_{N-n+1} = i)}{P(X_{N-n+1} = i)}$$

$$\frac{P(E | X_{N-n+1} = i) P(X_{N-n+1} = i)}{P(X_{N-n+1} = i)}$$

By Markov property for (X_n)

$$\frac{P(X_{N-n} = j, X_{N-n+1} = i)}{P(X_{N-n+1} = i)} = \frac{P(Y_n = j, Y_{n-1} = i)}{P(Y_{n-1} = i)}$$

$$= P(Y_n = j | Y_{n-1} = i) = \text{RHS}$$

2) Transition probabilities of (Y_n) .

$$Q_{ij} = P(Y_n = j | Y_{n-1} = i)$$

$$\hookrightarrow = \frac{P(X_{N-n} = j | X_{N-n+1} = i)}{P(X_{N-n+1} = i)}$$

$$= \frac{P(X_{N-n} = j, X_{N-n+1} = i)}{P(X_{N-n+1} = i)}$$

$$= \frac{P(X_{N-n+1} = i | X_{N-n} = j) P(X_{N-n} = j)}{P(X_{N-n+1} = i)}$$

$$= \cancel{\frac{P_{ji} \pi_j}{\pi_i}} \quad \boxed{\frac{P_{ji} \pi_j}{\pi_i} = Q_{ij}}$$

3) π is a stationary distr for (Y_n) .

$$\text{Want: } \pi Q = \pi. \quad (\pi Q)_j = \sum_i \pi_i Q_{ij} = \sum_i \frac{\pi_i P_{ji} \pi_j}{\pi_i}$$

$$= \sum_i \pi_j P_{ji} = \pi_j \sum_i P_{ji} = \pi_j. \Rightarrow \boxed{\pi Q = \pi}$$

$$Q_{ij} = P_{ji} \frac{\pi_j}{\pi_i}$$

Definition: A M.C. is **time-reversible** if $Q_{ij} = P_{ij}$.

Remark: It is always true that $Q_{ii} = P_{ii}$.

• For a time-reversible M.C., we have

$$P_{ij} = Q_{ij} = P_{ji} \frac{\pi_j}{\pi_i} \quad \overline{QP} = \overline{\pi}$$

$$\Rightarrow \boxed{P_{ij} \pi_j = P_{ji} \pi_i} \quad \forall i, j.$$

The set of equalities $\{P_{ij} \pi_j = P_{ji} \pi_i\}$ are called the "detailed balance equalities"

Interpretation: At stationarity

proportion of jumps $i \rightarrow j = \pi_i P_{ij}$

proportion of jumps $j \rightarrow i = \pi_j P_{ji}$

Examples: 1) 2-state M.C.

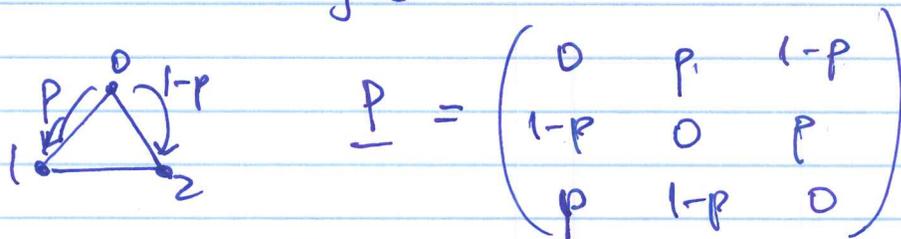
$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

We know $\pi = \left(\frac{q}{p+q}, \frac{p}{p+q} \right)$

$$\pi_0 P_{01} = \frac{1}{p+q} \cdot p = \pi_1 P_{10}$$

~~Therefore~~ \Rightarrow it is time-reversible.

2) R.W. on triangle



$$P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

P is doubly-stochastic, irreducible and ergodic

$$\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

M.C. is true-reversible $\Leftrightarrow \pi_i P_{ij} = \pi_j P_{ji} \quad \forall i \neq j$

$$\text{i.e. } P_{ij} = P_{ji} \quad \forall i \neq j$$

$$\text{i.e. } p = 1-p, \quad \boxed{p = \frac{1}{2}}$$

Proposition: Let X_n be an irreducible and ergodic M.C. If we can find $x_i \geq 0$ s.t.

$$x_i P_{ij} = x_j P_{ji} \quad \forall i, j$$

$$\text{and } \sum_{i=1}^n x_i = 1,$$

then $x_i = \pi_i$ and the M.C. is true-reversible.

Proof: If $\forall i, j \quad x_i P_{ij} = x_j P_{ji}$, then

$$(xP)_j = \sum_i x_i P_{ij} \stackrel{\downarrow}{=} \sum_i x_j P_{ji} = x_j \underbrace{\sum_i P_{ji}}_1 = x_j.$$

$$\Rightarrow xP = x \Rightarrow \pi = x \text{ and M.C. is true-reversible!}$$

Example: Alice has 3 umbrellas (total)
at home and at work.

- takes an umbrella if it is raining, if there is one
- doesn't take an umbrella if not raining
- it rains w.p. p during each trip

Q: What fraction of the times does she get wet?

X_n = # umbrellas at current location.