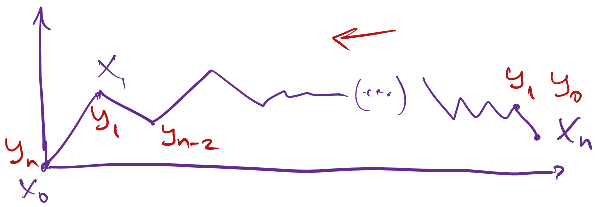
Time reversibility (Ross 4.8)

We see one last theoretical concept useful to compute limiting probabilities.

Given a M.C. $X_0, X_1, \dots, X_n$, we can observe it backwards in time.



The Markov property (past and future are independent given present) is symmetric under this reversal, so it allows to see y_k as a M.C.

Q: What is the relation to the original process? Can we say something about limiting probabilities?

Theorem: Given a M.C. $(X_n)_{0 \le n \le N}$ with stationary distribution π (we assume it exists) and with $P(X_0 = j) = \pi_j$, let $y_n = X_{N-n}$.
Then, $(y_n)_{0 \le n \le N}$ is a M.C. with stationary distribution π and transition probabilities $Q_{ij} = \dfrac{P_{ji}\pi_j}{\pi_i}$.

- Remark: the existence of the stationary distribution is necessary for the reversed process to be homogeneous. (will see this later)

Proof: 1) Markov property for y_n:

$$P(y_n = j \mid y_{n-1} = i, \underbrace{y_{n-2} = \dots}_{E}) = P(y_n = j \mid y_{n-1} = i)$$

$$\text{LHS} = \frac{P(X_{N-n} = j, X_{N-n+1} = i, \overbrace{X_{N-n+2} = \dots}^{E})}{P(X_{N-n+1}, E)} = \frac{P(E \mid \cancel{X_{N-n} = j,}\, X_{N-n+1} = i)}{P(E \mid X_{N-n+1} = i)} \cdot \frac{P(X_{N-n} = j, X_{N-n+1} = i)}{P(X_{N-n+1} = i)}$$

by Markov property

$$= P(X_{N-n} = j \mid X_{N-n+1} = i) = P(y_n = j \mid y_{n-1} = i) = \text{RHS}.$$

2) Transition probabilities: $Q_{ij} = P(Y_n = j \mid Y_{n-1} = i) \overset{\text{see 1)}}{=}$

$$= \frac{P(X_{N-n} = j \mid X_{N-n+1} = i)\, P(X_{N-n} = j)}{P(X_{N-n+1} = i)}$$

$= P_{ji}\dfrac{\pi_j}{\pi_i}$ (using the assumption that $X_0 \sim \pi$)

3) π is stationary: $\displaystyle\sum_i \pi_i Q_{ij} = \sum_i \cancel{\pi_i} P_{ji} \frac{\pi_j}{\cancel{\pi_i}} = \sum_i P_{ji}\pi_j$

$$= \pi_j \underbrace{\sum_i P_{ji}}_{=1} = \pi_j .$$

$\Rightarrow \pi Q = \pi$.

<u>Definition</u>: A M.C. is time-reversible if $Q_{ij} = P_{ij}$.

$\| \; P_{ji}\dfrac{\pi_j}{\pi_i}$

<u>Remark</u>: • We always have $Q_{ii} = P_{ii}$

• In this case, $\pi_i P_{ij} = \pi_j P_{ji}$ $\forall i,j$: this forms a set of "detailed balance" equalities.

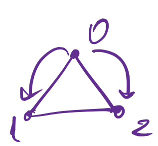
<u>Interpretation of detailed balance</u>: the proportion of jumps at statinarity $i \to j$ $= \pi_i P_{ij}$

= the proportion of jumps $j \to i$ $(= \pi_j P_{ji})$.

<u>Example</u>: 1) 2-state M.C. $\underline{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$. We know $\underline{\pi} = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$.

and $\pi_0 P_{01} = \pi_1 P_{10}$, so it is reversible.

2). R.W. on triangle (of HW2)



$$\underline{P} = \begin{bmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{bmatrix}$$

aperiodic, positive recurrent

$\underline{P}$ is doubly stochastic, irreducible and ergodic.

$\Rightarrow \pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is stationary.

M.C. is reversible $\iff \pi_i P_{ij} = \pi_j P_{ji} \ \forall i,j^{\,i\neq j}$, i.e., $P_{ij} = P_{ji}$

$\iff \boxed{p = \frac{1}{2}}$

<u>Proposition</u>: Let X_n be an irreducible ergodic M.C.

If we can find $x_i \geq 0$ s.t. $x_i P_{ij} = x_j P_{ji} \ \forall i,j$, $\sum x_i = 1$,
then $x_i = \pi_i$ and the M.C. is reversible.

<u>Proof</u>: If $\forall i,j \ x_i P_{ij} = x_j P_{ji}$, then $\sum_i x_i P_{ij} = \sum_i x_j P_{ji} = x_j \sum P_{ji} = x_j$.

and $\sum x_j = 1 \Rightarrow x = \pi$, $\quad \pi P = \pi$, and $Q_{ij} = \frac{P_{ji}\pi_j}{\pi_i} = P_{ij}$.

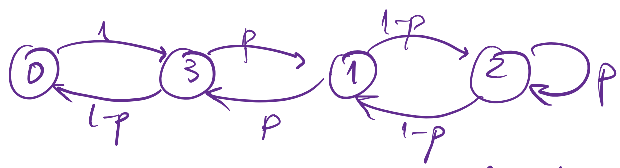
<u>Example</u>: Alice has 3 umbrellas (total) at home and at work.

- takes an umbrella if it is raining, if there is one
- doesn't take an umbrella if it is not raining
- it rains each trip w.p. p (independently of the other trips)

<u>Q</u>: What fraction of time does Alice get wet?

<u>Solution</u>: X_n = # umbrellas at current location

state space = $\{0,1,2,3\}$.



Chain is irreducible and ergodic (aperiodic, positive recurrent)

Answer is $\pi_0 \cdot p$ ← probability it rains, where π is the stationary distribution.

π_0: time with no umbrella

To find π, we can try to find a solution to:

$$\begin{cases} \pi_0 P_{03} = \pi_3 P_{30} \\ \pi_3 P_{31} = \pi_1 P_{13} \\ \pi_1 P_{12} = \pi_2 P_{21} \end{cases} \iff \begin{cases} \pi_0 = \pi_3 (1-p) \\ \pi_3 = \pi_1 \\ \pi_1 = \pi_2 \end{cases}$$

so $\pi_1 = \pi_2 = \pi_3$, $\sum_i \pi_i = 1$, $\pi_0 = (1-p)\pi_3$

$$\Rightarrow \pi_3(1-p) + 3\pi_3 = 1, \Rightarrow \pi_1 = \pi_2 = \pi_3 = \frac{1}{4-p}$$

$$\Rightarrow \pi_0 = \frac{1-p}{4-p} \qquad \pi = \left(\frac{1-p}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}\right)$$

$$\Rightarrow \boxed{\pi_0 \cdot p = \frac{1-p}{4-p} \cdot p}.$$