

# MATH 303, SECTION 201

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## 1. Syllabus

## 2. Motivating examples

A stochastic process is a sequence of random variables

$$(X_0, X_1, X_2, \dots) \text{ or } (X_t : 0 \leq t < \infty)$$

can think of it as the evolution  
of a random variable over time.

### §4. Markov Chains

- $X_n = \begin{cases} 0, \text{ sunny on day } n \\ 1, \text{ rainy on day } n \end{cases}$
- $(X_{n+1} | X_n, X_{n-1}, \dots, X_0) = X_{n+1} | X_n$ .
- $P(X_{n+1}=0 | X_n=0)=0.9$   
 $P(X_{n+1}=1 | X_n=0)=0.1$
- $P(X_{n+1}=1 | X_n=1)=0.5$   
 $P(X_{n+1}=0 | X_n=1)=0.5$
- What is  $P(X_{n+2}=0 | X_n=0)?$

### §5. The Poisson Process

- $X_t = \text{number of large earthquakes along the Cascadian fault until year } t$ .
- can calculate distribution of  $X_{2050} | X_{1971} = n$
- A counting process

### §6. Continuous time Markov Processes

- cells in gene expression space
- $X_t = \text{gene expression of a cell at time } t$
- Can study how cells go from embryonic cells to bone, skin, brain, eye, or cancer cells.

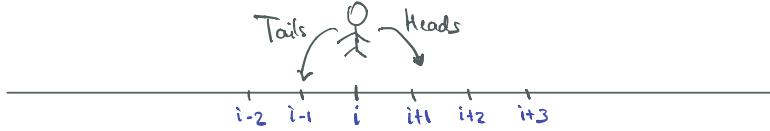
These allow us to create models of how the world works, and then predict what will happen in the future

## 3. Introductions.

### Chapter 4: Discrete Time Markov Chains

Let  $X_0, X_1, X_2, \dots$  be a sequence of random variables each taking values in the state space  $S$ .

Example: 1-D random walk



$X_n$  = position at time  $n$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{if Heads} \\ X_n - 1 & \text{if Tails} \end{cases}$$

$$X_n + \mathbb{Z} = \{-\dots, -2, -1, 0, 1, 2, \dots\}.$$

$S = \mathbb{Z}$

state space.

Example:  $X_n = \begin{cases} 0 & \text{sunny on day } n \\ 1 & \text{rainy on day } n \end{cases}$  Then,  $X_n \in \{0, 1\} = S$

In this chapter we assume that the state space  $S$  is finite or countable.

When is  $(X_n)_{n \geq 0}$  a Markov chain? Intuition: we want to study processes s.t. the probability of observing a state in the future (i.e. at time  $n+1$ ) only depends on the observed state in the present (at time  $n$ ), and not on the past (i.e. not on  $X_{n-1}, X_{n-2}, \dots$ ).

Definition: Let  $(X_n)_{n \geq 0}$  be a sequence of random variables with state space  $S$ . Then,  $(X_n)_{n \geq 0}$  is a **Markov chain** if it satisfies the

**Markov property:**  $\forall n \geq 0$  and  $\forall x_0, \dots, x_{n+1} \in S$ ,

$$P(X_{n+1} = x_{n+1} \mid X_n = x_n, \dots, X_0 = x_0) = P(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

Example:  $P(X_{n+1} = 0 \mid X_n = 0) = 0.8$

$$P(X_{n+1} = 0 \mid X_n = 1) = 0.3$$

$$P(X_{n+1} = 1 \mid X_n = 0) = 0.2$$

$$P(X_{n+1} = 1 \mid X_n = 1) = 0.7.$$

Example: 1-D random walk:  $P(X_{n+1} = a+1 \mid X_n = a) = 0.4$

$X_{n+1}$  only depends on  $X_n$

$$P(X_{n+1} = a-1 \mid X_n = a) = 0.6$$

Homework exercise: Transforming a process into a Markov chain.

Suppose that whether or not it rains today <sup>only</sup> depends on whether or not it rained in the last two days. Specifically,

Question 1: What's  $P(X_{n+1}=D | X_n=i, X_{n-1}=j)$ ?

$$P(X_{n+1}=1 | X_n=1, X_{n-1}=1) = 0.7 \Rightarrow P(X_{n+1}=0 | X_n=1, X_{n-1}=1) = 0.3$$

$$P(X_{n+1}=1 | X_n=1, X_{n-1}=0) = 0.5 \Rightarrow P(X_{n+1}=0 | X_n=1, X_{n-1}=0) = 0.5$$

$$P(X_{n+1}=1 | X_n=0, X_{n-1}=1) = 0.4 \Rightarrow P(X_{n+1}=0 | X_n=0, X_{n-1}=1) = 0.6$$

$$P(X_{n+1}=1 | X_n=0, X_{n-1}=0) = 0.2 \Rightarrow P(X_{n+1}=0 | X_n=0, X_{n-1}=0) = 0.8.$$

Question 2: Is  $(X_n)_{n \geq 0}$  a Markov Chain? No.

Question 3: How can we transform it into a Markov Chain? Hint: set  $Y_n = \begin{pmatrix} X_{n-1} \\ X_n \end{pmatrix}$ ,  
i.e.  $Y_n$  takes 4 values