MATH 303 **Introduction to Stochastic Processes** **Jan-Apr, 2020**

Assignment 6: Due Monday, March 9 at start of class

Problems to be handed in

Problem 1

Let $N(t)$ be a Poisson process with rate λ, with S_n the time of the nth event. Find the following quantities (no mark awarded if no justification is provided).
1. $\mathbb{E}(N(4))$ and $Var(N(4))$.
2. $\mathbb{E}(S_5)$ and $Var(S_5)$.
3. $P(N(4) < 2)$ and $P(S_2 > 4)$.
4. $P(S_3 > 5 \mid N(2) = 1)$ and $P(N(3) < 2)$.

Problem 2

1. Let X_1, X_2 and X_3 be independent exponential r.v.'s with respective rates λ_1, λ_2 and λ_3 (not necessarily equal). Find the probability that $X_1 = \min(X_i,\ 1 \le i \le 3)$ (*hint*: Compute $P(X_1 < \min(X_2, X_3))$).
2. Bob enters a bank as it is closing. There are three tellers 1, 2 and 3, independently serving three clients with exponentially distributed time with respective rate λ_1, λ_2 and λ_3. The first teller who finishes serves Bob with probability 0.5 (and closes otherwise). If this teller closes, the next one who finishes serves Bob.
2a. What is the probability that Bob is served by teller 1?
2b (bonus) What is the expected time Bob spends at the bank and the probability he leaves last?

Problem 3

8 runners R_i ($i = 1, \ldots, 8$) enter a race, with a time to complete respectively $\sim Exp(i)$ (and independent from the other runners).
1. Suppose the winner of the race (with the shortest time) earns 10 dollars and other runners lose 1. What is the expected gain of runner 1?
2. If instead the winner of the race earns e^{-at}, where t is the time of the winner and $a > 0$ (losers make 0), what is the expected gain of runner 1?

Problem 4

We consider a process Z_n, where $Z_0 = 1$, and at each generation $n \ge 1$,

$$Z_{n+1} = I_n + \sum_{i=1}^{Z_n} X_{n,i},$$

where the $X_{n,i}$'s are i.i.d. with common distribution X, and the I_n's are i.i.d and independent from the $X_{n,i}$'s, with common distribution I (this can be seen as a branching process where an immigrant population arrives at each generation and independently reproduces at the next one).
1. Find the probability generating function (pgf) of Z_n, as a function of the pgf's of I, X, and n.
2. Let $X \sim Binomial(1, p)$ and $I \sim Poisson(\mu)$, where $0 < p < 1$ and $\mu > 0$. Compute the pgf of Z_n as a function of μ, p and n.

3. Compute the expectation and variance of Z_n.
4. (bonus) For the same process as in (2), Identify the distribution associated with the pgf of Z_n as $n \to +\infty$. Conclude about the limiting behaviour of the process (transience, positive or noll-recurrence).

These provide additional practice but are not to be handed in. Textbook Chapter 5 examples 5.2-5.5 and 5.8- 5.10, exercices 12, 15, 25, 26, 32, 34.