MATH 303 **Introduction to Stochastic Processes** **Jan-Apr, 2020**

Assignment 5: Due Monday, March 2 at start of class

Problems to be handed in

Problem 1

Compute the probability generating function of X, and the probability of extinction of the branching process associated with the following reproduction laws X:
1. $P(X=0)=\frac{1}{3}$ and $P(X=2)=\frac{2}{3}$.
2. $X \sim$ Binomial $\left(2, \frac{1}{4}\right)$.
3. $X \sim$ Geometric $\left(\frac{1}{4}\right)$.

Problem 2

We recall that for a random variable X on $\mathbb{N}=\{0,1,\ldots\}$, we can associate the probability generating function $G_X(s)=\sum_{k=0}^{+\infty} P(x=k)s^k$, for all $s \in [-1,1]$.

1. Find expressions of $\mathbb{E}(X)$ and $Var(X)$ using G' and G'' (hint: use the formulas shown in class for $G'(s)$ and $G''(s)$ and look at $s=1$).
2. Let $S_N=\sum_{i=1}^{N} X_i$, where X_i are i.i.d random variables and N is a random variable independent of the X_i's. Show that

$$\mathbb{E}(S_N)=\mathbb{E}(N)\mathbb{E}(X) \text{ , and } Var(S_N)=Var(N)(\mathbb{E}(X))^2+\mathbb{E}(N)Var(X),$$

where $\mathbb{E}(X)$ and $Var(X)$ respectively are the common expectation and variance of the X_i's. (*hint*: use the fact -shown in class- that $G_{S_N}(s)=G_N(G_X(s)))$
3. We now consider the branching process Z_n, with μ being the mean of the reproduction law and σ^2 its variance. Show that $\mathbb{E}(Z_n)=\mu\mathbb{E}(Z_{n-1})$ for all $n \geq 1$, and deduce $\mathbb{E}(Z_n)$ as a function of μ and n.
4 (bonus). Show that

$$Var(Z_n)=\begin{cases} \mu^{n-1}\sigma^2\frac{1-\mu^n}{1-\mu} & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

5. Application: Compute the expectation and variance of Z_n for the reproduction laws introduced in Problem 1.

Problem 3

Let X be an exponential variable with unknown parameter λ. We know that $\mathbb{P}(X \leq 10)=\frac{5}{9}$.
1. Compute $\mathbb{P}(X \geq 15)$.
2. Compute $\mathbb{P}(X \geq 15 | X \geq 10)$.
3. Compute $\mathbb{P}(X \geq 15 | 10 \leq X \leq 20)$.
4. Find the number $y>0$ for which $\mathbb{P}(y \leq X \leq 2y)$ is the largest.

Recommended Problems

These provide additional practice but are not to be handed in. Textbook Chapter 4 (12th ed.): Exercises 55, 64-66. Chapter 5: Exercises 1-6.