## Assignment 4: Due Thursday, February 6 at start of class

## Problems to be handed in

## Problem 1

We consider the Markov chain $\left(X_{n}\right)$ on the state space $\{1,2,3,4\}$ with transition matrix

$$
P=\left(\begin{array}{cccc}
0 & 1 / 2 & 1 / 4 & 1 / 4 \\
1 / 4 & 0 & 1 / 2 & 1 / 4 \\
1 / 4 & 1 / 4 & 0 & 1 / 2 \\
1 / 2 & 1 / 4 & 1 / 4 & 0
\end{array}\right) .
$$

1. Check that the uniform distribution on $\{1,2,3,4\}$ is stationary for this chain. In the other questions, we will assume that $X_{0}$ is picked according to this distribution.
2. Compute $\mathbb{P}\left(X_{N-1}=1 \mid X_{N}=2\right)$.
3. Compute $\mathbb{P}\left(X_{N-2}=3 \mid X_{N}=4\right)$.

## Solution

1. The transition matrix is doubly stochastic, so the uniform distribution is stationary.
2. We denote by $\left(Y_{n}\right)_{0 \leq n \leq N}$ the reverse chain, i.e. $Y_{n}=X_{N-n}$. By the result in the course, $Y$ is a Markov chain with transitions $q_{i j}=\frac{\pi_{j}}{\pi_{i}} p_{j i}=p_{j i}$, so the transition matrix of $Y$ is

$$
Q=\left(\begin{array}{cccc}
0 & 1 / 4 & 1 / 4 & 1 / 2 \\
1 / 2 & 0 & 1 / 4 & 1 / 4 \\
1 / 4 & 1 / 2 & 0 & 1 / 4 \\
1 / 4 & 1 / 4 & 1 / 2 & 0
\end{array}\right)
$$

In particular, we have

$$
\mathbb{P}\left(X_{N-1}=1 \mid X_{N}=2\right)=\mathbb{P}\left(Y_{1}=1 \mid Y_{0}=2\right)=q_{21}=\frac{1}{2}
$$

3. We apply the Chapman-Kolmogorov to the chain $Y$ :

$$
\mathbb{P}\left(X_{N-2}=3 \mid X_{N}=4\right)=\mathbb{P}\left(Y_{2}=3 \mid Y_{0}=4\right)=q_{43}^{2}=\sum_{i=1}^{4} q_{4 i} q_{i 3}=\frac{1}{4} \times \frac{1}{4}+\frac{1}{4} \times \frac{1}{4}=\frac{1}{8}
$$

## Problem 2

We fix $p_{A}, p_{B}, p_{C}>0$ such that $p_{A}+p_{B}+p_{C}=1$. Alice owns 3 books entitled $A, B$ and $C$, that she keeps arranged in a pile. Each day, she reads one of the books at random and put it back at the top of the pile, without to touch the other two. She chooses the book $A$ with probability $p_{A}$, book $B$ with probability $p_{B}$ and book $C$ with probability $p_{C}$.
We denote by $X_{n}$ the order of the pile on the $n$-th day (for example $X_{n}=A B C$ if $A$ is at the top and $C$ at the bottom).

1. Give the transitions of the Markov chain $\left(X_{n}\right)$.
2. Show that $\left(X_{n}\right)$ is irreducible and ergodic.
3. What is the limiting probability that $X_{n}=A B C$ ? Is the chain reversible? Hint: first compute $\pi_{B A C}+\pi_{B C A}$.

## Solution

1. The state space is $\{A B C, A C B, B A C, B C A, C A B, C B A\}$. The transition matrix is:

$$
P=\left(\begin{array}{cccccc}
p_{A} & 0 & p_{B} & 0 & p_{C} & 0 \\
0 & p_{A} & p_{B} & 0 & p_{C} & 0 \\
p_{A} & 0 & p_{B} & 0 & 0 & p_{C} \\
p_{A} & 0 & 0 & p_{B} & 0 & p_{C} \\
0 & p_{A} & 0 & p_{B} & p_{C} & 0 \\
0 & p_{A} & 0 & p_{B} & 0 & p_{C}
\end{array}\right) .
$$

2. If Alice reads book $B$ and book $A$ on the next day, the order of the pile will be $A B C$ whatever it was in the beginning, so $A B C$ is accessible from all other states. The same reasoning applies to any ordering of the books, so the chain is irreducible.
The chain has finite state space, so it has at least one positive recurrent state. Since it has only one communicating class, all states are positive recurrent. Moreover, we have $p_{i i}>0$ for every state $i$, so the chain is also aperiodic, so it is ergodic.
3. Since the chain is irreducible and ergodic, there is a unique stationary distribution $\pi$. If we write the equation $\pi=\pi P$, we obtain one equation for each of the 6 possible orderings. In particular, we have

$$
\begin{aligned}
& \pi_{B A C}=p_{B} \pi_{B A C}+p_{B} \pi_{A B C}+p_{B} \pi_{A C B}, \\
& \pi_{B C A}=p_{B} \pi_{B C A}+p_{B} \pi_{C B A}+p_{B} \pi_{C A B} .
\end{aligned}
$$

Summing up these two equations, we obtain

$$
\pi_{B A C}+\pi_{B C A}=p_{b}\left(\pi_{B A C}+\pi_{A B C}+\pi_{A C B}+\pi_{B C A}+\pi_{C B A}+\pi_{C A B}\right)=p_{B}
$$

We also have

$$
\pi_{A B C}=p_{A} \pi_{A B C}+p_{A} \pi_{B A C}+p_{A} \pi_{B C A}=p_{A} \pi_{A B C}+p_{A} p_{B},
$$

so $\left(1-p_{A}\right) \pi_{A B C}=p_{A} p_{B}$, which gives $\pi_{A B C}=\frac{p_{A} p_{B}}{1-p_{A}}$. By the theorem from the course we conclude

$$
\lim _{n \rightarrow+\infty} \mathbb{P}\left(X_{n}=A B C\right)=\pi_{A B C}=\frac{p_{A} p_{B}}{1-p_{A}}
$$

Finally, we have $p_{A B C, C A B}=p_{C}>0$ but $p_{C A B, A B C}=0$ so the chain is not reversible.

## Problem 3

An urn contains 5 black or white balls. At time 0 , all the balls are white. At each step, we pick a ball uniformly in the urn, remove it and replace it with a ball of the opposite colour. Let $B_{n}$ be the number of black balls at time $n$.

1. Compute the long-run fraction of time where $B_{n}=3$. Hint: Use reversibility.
2. What is the expected value of the first time where the urn contains 5 white balls again?
3. Is it true that $\mathbb{P}\left(B_{n}=3\right)$ converges?

## Solution

1. The transition diagram of the chain $\left(B_{n}\right)$ is the following:


This is an irreducible chain and all states are positive recurrent, so the long-term fractions of time spent on each state are given by the unique stationary distribution $\pi$. To find $\pi$, we try to use reversibility. We need:

$$
\begin{aligned}
\pi_{0} & =\frac{1}{5} \pi_{1} \\
\frac{4}{5} \pi_{1} & =\frac{2}{5} \pi_{2} \\
\frac{3}{5} \pi_{2} & =\frac{3}{5} \pi_{3} \\
\frac{2}{5} \pi_{3} & =\frac{4}{5} \pi_{4} \\
\frac{1}{5} \pi_{4} & =\pi_{5}
\end{aligned}
$$

From here, we obtain

$$
\begin{aligned}
\pi_{1} & =5 \pi_{0} \\
\pi_{2} & =2 \pi_{1}=10 \pi_{0} \\
\pi_{3} & =\pi_{2}=10 \pi_{0} \\
\pi_{4} & =\frac{1}{2} \pi_{3}=5 \pi_{0} \\
\pi_{5} & =\frac{1}{5} \pi_{4}=\pi_{0}
\end{aligned}
$$

Finally, we have

$$
1=\pi_{0}+\pi_{1}+\pi_{2}+\pi_{3}+\pi_{4}+\pi_{5}=(1+5+10+10+5+1) \pi_{0}=32 \pi_{0},
$$

so $\pi_{0}=\frac{1}{32}$ and $\pi_{3}=\frac{10}{32}=\frac{5}{16}$, so the long-term fraction of time spent with $B_{n}=3$ is $\frac{5}{16}$.
2. Let $T_{0}$ be the first time of return of $\left(B_{n}\right)$ to 0 . Using a result from the course, we have $\mathbb{E}\left[T_{0}\right]=m_{0}=\frac{1}{\pi_{0}}=32$.
3. The chain $\left(B_{n}\right)$ has period 2 , so it is not aperiodic and we cannot use the result from the course to guarantee the convergence. More precisely, since $B_{0}=0, B_{n}$ will always be even for $n$ even and odd for $n$ odd. In particular $\mathbb{P}\left(B_{n}=3\right)=0$ for $n$ even. On the other hand, $\left(B_{n}\right)$ spends a fraction $\frac{5}{16}$ of time at 3 , so it spends a fraction $\frac{5}{8}$ of the odd times at 3 , so $\mathbb{P}\left(B_{n}=3\right)$ does not converge.

## Problem 4

Bob gambles in the following way: he starts with $i \geq 0$ dollars. At each step, he wins a dollar with probability $\frac{1}{3}$ and loses a dollar with probability $\frac{2}{3}$. However, if he has 0 dollar and loses, he stays at 0 dollar and can keep gambling (i.e. Bob cannot have debts). For example, if Bob has one
dollar, loses twice and then wins, then he will have 1 dollar again. We are interested in the Markov chain $\left(X_{n}\right)$ describing the fortune of Bob at time $n$.

1. Give the transitions of $\left(X_{n}\right)$.
2. Find, with proof, the limiting probability that Bob owns $i$ dollars at time $n$.

## Solution

1. The transitions are $p_{i, i-1}=\frac{2}{3}$ for all $i \geq 1, p_{i, i+1}=\frac{1}{3}$ for all $i \geq 0, p_{0,0}=\frac{2}{3}$ and $p_{i j}=0$ everywhere else.
2. The chain $\left(X_{n}\right)$ is irreducible. Moreover $p_{00}>0$, so the state 0 has period 1 , so all the states are aperiodic. However, since the state space is infinite, it is not immediate that the chain is positive recurrent. To prove it, we look for a stationary distribution $\pi$, and try to do so using reversibility. We need, for every $i \geq 1$ :

$$
\pi_{i} \times \frac{1}{3}=\pi_{i+1} \times \frac{2}{3},
$$

so $\pi_{i+1}=\frac{1}{2} \pi_{i}$ for all $i$, so $\pi_{i}=\frac{1}{2^{2}} \pi_{0}$. Since the sum of $\pi_{i}$ must be 1 , we obtain $\pi_{0}=\frac{1}{2}$ and $\pi_{i}=\frac{1}{2^{2+1}}$. In particular $\left(X_{n}\right)$ has a stationary distribution, so it is positive recurrent. By the theorem from the course, the limiting probabilities are given by $\pi_{i}=\frac{1}{2^{i+1}}$.

## Recommended Problems

These provide additional practice but are not to be handed in. Textbook Chapter 4 (12th ed.): Exercises 42, 52, 54, 68, 71, 73.

