MATH 303 **Introduction to Stochastic Processes** **Jan-Apr, 2020**

Assignment 4: Due Friday, February 7 at start of class

Problems to be handed in

Problem 1

We consider the Markov chain (X_n) on the state space $\{1,2,3,4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 & 0 \end{pmatrix}.$$

1. Check that the uniform distribution on $\{1,2,3,4\}$ is stationary for this chain. In the other questions, we will assume that X_0 is picked according to this distribution.
2. Compute $\mathbb{P}(X_{N-1} = 1 | X_N = 2)$.
3. Compute $\mathbb{P}(X_{N-2} = 3 | X_N = 4)$.

Problem 2

We fix $p_A, p_B, p_C > 0$ such that $p_A + p_B + p_C = 1$. Alice owns 3 books entitled A, B and C, that she keeps arranged in a pile. Each day, she reads one of the books at random and put it back at the top of the pile, without to touch the other two. She chooses the book A with probability p_A, book B with probability p_B and book C with probability p_C.
We denote by X_n the order of the pile on the n-th day (for example $X_n = ABC$ if A is at the top and C at the bottom).
1. Give the transitions of the Markov chain (X_n).
2. Show that (X_n) is irreducible and ergodic.
3. What is the limiting probability that $X_n = ABC$? Is the chain reversible? *Hint*: first compute $\pi_{BAC} + \pi_{BCA}$.

Problem 3

An urn contains 5 black or white balls. At time 0, all the balls are white. At each step, we pick a ball uniformly in the urn, remove it and replace it with a ball of the opposite colour. Let B_n be the number of black balls at time n.
1. Compute the long-run fraction of time where $B_n = 3$. *Hint*: Use reversibility.
2. What is the expected value of the first time where the urn contains 5 white balls again?
3. Is it true that $\mathbb{P}(B_n = 3)$ converges?

Problem 4

Bob gambles in the following way: he starts with $i \geq 0$ dollars. At each step, he wins a dollar with probability $\frac{1}{3}$ and loses a dollar with probability $\frac{2}{3}$. However, if he has 0 dollar and loses, he stays at 0 dollar and can keep gambling (i.e. Bob cannot have debts). For example, if Bob has one dollar, loses twice and then wins, then he will have 1 dollar again. We are interested in the Markov chain (X_n) describing the fortune of Bob at time n.

1. Give the transitions of (X_n).
2. Find, with proof, the limiting probability that Bob owns i dollars at time n.

Recommended Problems

These provide additional practice but are not to be handed in. Textbook Chapter 4 (12th ed.): Exercises 42, 52, 54, 68, 71, 73.