Assignment 3: Due Friday, January 31 at start of class

Problems to be handed in

Problem 1

A communication class C is said to be *closed* if $P_{ij} = 0$ whenever $i \in C$ and $j \notin C$ (i.e. there is no escape from C).

1.a Consider the stochastic matrices

$$M_{1} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \qquad M_{2} = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \qquad M_{3} = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ * & 0 & * & * \end{pmatrix},$$

where all entries marked with * are greater than 0. For each of them, find the number of communicating classes, if they are closed (briefly justify).

1.b Show that if i is recurrent then it belongs to a closed class. Conclude that if the state space is finite, then there is at least one closed communicating class.

1.c Reciprocally, if the state space is finite, show that all the states of a communicating class are recurrent.

1.d Is this still true in general (i.e. for all state space)? Justify.

Problem 2

1. Consider the asymmetric random walks in \mathbb{Z} , where one moves one step to the right with probability p, and to the left with probability 1 - p, where $p \neq \frac{1}{2}$. Show that the walk is transient. (hint: use a similar method as shown in class for a 1-D symmetric random walk)

2.a We now consider the uniform random walk on \mathbb{Z}^2 . Show that the probability to return at the starting place after 2n steps is given by $p_{2n} = \frac{1}{4^{2n}} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2$. (hint: use multinomial coefficient to count all the ways to come back in 2n steps)

2.b Show that $\sum_{k=0}^{n} {n \choose k}^2 = {2n \choose n}$.

2.c Find an equivalent of p_{2n} and conclude that the walk is recurrent.

Problem 3

We consider a Markov chain $(X_n)_{n\geq 0}$ on states 1, 2, 3 and 4, with transition matrix

$$\left(\begin{array}{ccccccc}
1/5 & 2/5 & 1/5 & 1/5 \\
1/4 & 1/4 & 1/4 & 1/4 \\
* & * & * & * \\
* & * & * & * \\
\end{array}\right)$$

Assuming $X_0 = 1$,

1. what is the probability to enter state 3 before state 4?

2. what is the mean number of transitions until either state 3 or 4 is entered? (hint: for both questions, do a one-step analysis, i.e. condition on the outcome of X_1) 3. (bonus) Solve 2. using the method presented in Ross, section 4.6.

Recommended Problems

These provide additional practice but are not to be handed in. Textbook Chapter 4 (12th ed.): Exercises 17, 40, 50, 60, 61, 63.