Assignment 2: Due Friday, January 24 at start of class

Problems to be handed in

Problem 1

1. Complete the following transition matrix, associated with a Markov chain defined on $\{1,\dots 6\}$, such that transition probabilities from a given state are uniform (we also assume that each missing entry > 0, unless this is impossible):

$$\begin{pmatrix} . & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & . & 0 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & . & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & . & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & . & 0 & . & 0 \\ 0 & . & 0 & . & . & 0 \end{pmatrix}$$

2. Draw the transition diagram and find the communicating classes (briefly justify your answers).
3. Find the period of the states, and if they are transient or recurrent (briefly justify).

Problem 2

We consider the random walk on the vertices of a polygon, represented by $S = \{1, \dots N\}$, where the transition probabilities from a given vertex are given by p, q and $r > 0$ according to Figure 1.

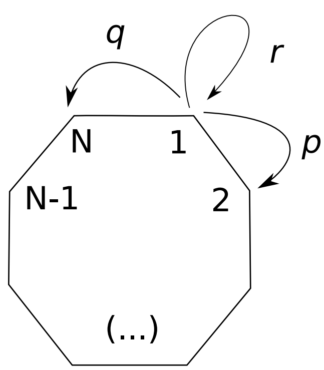

Figure 1: Random walk on the vertices of a polygon with N vertices. We only represent the transition from state 1 (same rule for the other states).

1. Write the transition matrix for $N = 6$.
2. We suppose $p, q > 0$. Show that the Markov chain is irreducible and that the uniform distribution is stationary.
3. We suppose $p, q > 0$ and $r > 0$. Show that the chain is aperiodic.
4. We suppose $p, q > 0$ and $r = 0$. Show that if N is even, then the chain has period 2, and if N is odd, then the chain is aperiodic.
5. Suppose $p, q > 0$ and $r = 0$. What is the probability to visit all the other states before returning to its initial position (hint: think of the gambler's ruin problem)?

Problem 3

We study here the Gambler's Ruin Problem (seen in class, or in Ross' 4.5.1), with same hypothesis and notations (p being the probability to win a single game). As shown in class, we know that the wealth of the gambler follows a Markov chain X_k, that either reaches 0 or N with probability 1, so we can define the random time $T = \min\{k \geq 0 \text{ s.t. } X_k = 0 \text{ or } N\}$. We want to find the mean duration of the whole game, given initial wealth k ($0 \leq k \leq N$), i.e. $\mathbb{E}_k(T) = \mathbb{E}(T|X_0 = k)$.

1. What are $\mathbb{E}_0(T)$ and $\mathbb{E}_N(T)$? (boundary conditions)

2. We write $x_k = \mathbb{E}_k(T)$. Show that for $1 \leq k \leq N-1$, x_k satisfies

$$p x_{k+1} - x_k + (1-p) x_{k-1} = -1 \qquad (*)$$

3. Solutions of $(*)$ can be written as $x_k = y_k + f(k)$, where y_k is solution of the homogeneous equation $p y_{k+1} - y_k + (1-p) y_{k-1} = 0$, and $f(k)$ is a particular solution of $(*)$.

3.a Solve the homogeneous equation.

3.b Find a particular solution of the form $f(k) = Ck$ when $p \neq \frac{1}{2}$, and $f(k) = Dk^2$ when $p = \frac{1}{2}$, where C and D are constant.

3.c Using the boundary conditions, conclude that

$$\mathbb{E}_k(T) = \begin{cases} \dfrac{1}{1-2p}\left(k - N\dfrac{1-(\alpha-1)^k}{1-(\alpha-1)^N}\right) & \text{if } p \neq \frac{1}{2} \\ k(N-k) & \text{if } p = \frac{1}{2} \end{cases}, \text{ where } \alpha = \frac{1}{p}.$$

4. **(bonus)** *Reflecting boundary:* Suppose the gambler has a wealthy relative, who guarantees them not to get ruined: The process exactly obeys the same rules, except when $X_n = 0$, as the gambler receives a dollar so $X_{n+1} = 1$. Find $\mathbb{E}_k(T)$ under these new conditions.

Recommended Problems

These provide additional practice but are not to be handed in. Textbook Chapter 4 (12th ed.): Exercises 14, 15, 16 and 17.