MATH 303 **Introduction to Stochastic Processes** **Jan-Apr, 2020**

Assignment 1: Due Wednesday, January 15 at start of class

Problems to be handed in

Problem 1

1. We consider the 2-state Markov chain $(X_n)_{n\geq 0}$ on $\{0,1\}$, such that $P_{01}=p$ and $P_{10}=q$, where $0<p,q<1$. Write the transition matrix of $(X_n)_{n\geq 0}$ and draw its transition diagram.
2. Find the stationary distribution Π of $(X_n)_{n\geq 0}$.
3. Show that $P(X_{n+1}=0)=(1-p-q)P(X_n=0)+q$.
4. **(bonus)** Show that $P(X_n)$ converges to Π when $n\to+\infty$.

Problem 2

We consider the random walk $(X_n)_{n\geq 0}$ on $\{-1,0,1\}$ such that at each step, the walker moves from 0 to -1 or 1 with equal probability, and from -1 and 1, the walker moves to 0 only.
1. Draw the transition diagram and write the transition matrix of $(X_n)_{n\geq 0}$.
2. Compute the 2- and 3-step transition matrix. Deduce the n-step transition matrix.
3. We modify $(X_n)_{n\geq 0}$ as follows: At each step the walker first decides to move or stay by flipping a fair coin first and follows the same rules as above if the decision is to move. Draw the transition diagram and write the transition matrix of this new Markov chain $(Y_n)_{n\geq 0}$.
4. By induction, prove that the n-step transition matrix of $(Y_n)_{n\geq 0}$ is

$$\frac{1}{2}\begin{pmatrix} \frac{2^{n-1}+1}{2^n} & 1 & \frac{2^{n-1}-1}{2^n} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2^{n-1}-1}{2^n} & 1 & \frac{2^{n-1}+1}{2^n} \end{pmatrix}$$

5. We assume that $(Y_n^2)_{n\geq 0}$ is a Markov chain (on $\{0,1\}$). Find the transition matrix of $(Y_n^2)_{n\geq 0}$ and study its asymptotic behavior (remark: you can use Problem 1).
6. We modify this walk again so at position 1, the walker directly moves back to 0. Draw the transition diagram of the new chain $(Z_n)_{n\geq 0}$. Show that $(Z_n^2)_{n\geq 0}$ is not a Markov chain. (*hint*: Compare $P(Z_2^2=0\mid Z_1^2=1)$ and $P(Z_2^2=0\mid Z_1^2=1\ ,\ Z_0^2=0)$).
7. **(bonus)** Let $(X_n)_{n\geq 0}$ be a Markov chain on $\mathbb{Z}$. Show that if $\forall x,y\ P_{-x,y}=P_{x,y}$, then $(X_n^2)_{n\geq 0}$ is a Markov chain.

Problem 3

We consider an urn with two blue balls and two red balls.
1. What is the probability of picking 2 balls of the same color?
2. We repeat the process of picking 2 balls (and place them back) 4 times. What is the probability of having picked 2 balls of the same color 2 times in a row? (*hint*: Example 4.12 in Ross)

Recommended Problems

These provide additional practice but are not to be handed in. Textbook Chapter 4 (12th ed.): Exercises 1, 3, 4, 6, 7, 8, 10.