Special thanks to the organizing committee.
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Outline

1. Disjoint Cycles
   - Corrádi-Hajnal
     - Tolerance for some low-degree vertices
     - Ore condition (minimum degree-sum of nonadjacent vertices)
     - Generalized Degree-Sum Conditions
     - Connectivity
     - Neighborhood Union

2. Chorded Cycles
   - Degree conditions
   - Neighborhood Union
   - Multiply Chorded Cycles

3. Equitable Coloring
   - Definition
   - Connection to Cycles
Corrádi-Hajnal Theorem

Corrádi-Hajnal, 1963

If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.
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Examples:

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**Dirac-Erdős, 1963**

If $V_{\geq 2k} - V_{\leq 2k-2} \geq k^2 + 2k - 4$, $k \geq 3$, then $G$ contains $k$ disjoint cycles.
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**Kierstead-Kostochka-McConvey, 2016 (link)**

Let $k \geq 3$ be an integer and $G$ be a graph such that $G$ does not contain two disjoint triangles. If $V_{\geq 2k} - V_{\leq 2k-2} \geq 2k$, then $G$ contains $k$ disjoint cycles.
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Let $k \geq 2$ be an integer and $G$ be a graph with $|G| \geq 19k$ and $V_{\geq 2k} - V_{\leq 2k-2} \geq 2k$. Then $G$ contains $k$ disjoint cycles.
Dirac-Erdős Type Problems

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Open

Characterize graphs $G$ with $V_{\geq 2k} - V_{\leq 2k-2} \geq 2k$ and no $k$ disjoint cycles.
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Low degree vertices OK as long as they’re in a clique
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With a little work, implies Dirac-Erdős
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If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

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Proof (Enomoto)
- Edge-maximal counterexample
  - $(k - 1)$ disjoint cycles
  - Remaining graph at least 3 vertices
- Minimize number of vertices in cycles
- Maximize longest path in remainder
If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

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**Enomoto 1998, Wang 1999**

If \( G \) is a graph on \( n \) vertices with \( n \geq 3k \) and \( \sigma_2(G) \geq 4k - 1 \), then \( G \) contains \( k \) disjoint cycles.
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\[ \alpha(G) \geq n - 2k + 1 \implies \text{no } k \text{ cycles} \]

If \( G \) is a graph on \( n \) vertices with \( n \geq 3k \) and \( \sigma_2(G) \geq 4k - 1 \), then \( G \) contains \( k \) disjoint cycles.

Kierstead-Kostochka-Yeager, 2017 (link)
For \( k \geq 4 \), if \( G \) is a graph on \( n \) vertices with \( n \geq 3k + 1 \) and \( \sigma_2(G) \geq 4k - 3 \), then \( G \) contains \( k \) disjoint cycles if and only if \( \alpha(G) \leq n - 2k \).
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[Diagram of two disjoint cycles]
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$k = 2$:

![Graphs showing the case for $k = 2$.](image-url)
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\[ \sigma_2 = 4k - 4 : \]
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Conjecture: Gould, Hirohata, Keller 2018 (link)

Let $G$ be a graph of sufficiently large order. If $\sigma_t(G) \geq 2kt - t + 1$ for any two integers $k \geq 2$ and $t \geq 1$, then $G$ contains $k$ disjoint cycles.
### Extending Enomoto-Wang

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$t = 3$: Fujita, Matsumura, Tsugaki, Yamashita 2006 (link)
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Let $G$ be a graph with $|G| \geq (2t + 1)k$. If $\sigma_t(G) \geq 2kt - t + 1$ for any two integers $k \geq 2$ and $t \geq 5$, then $G$ contains $k$ disjoint cycles.
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**Proof**

In an edge-maximal counterexample, choose $k - 1$ disjoint cycles such that

- number of vertices in cycles is minimal, and
- number of connected components in remaining graph is minimal.
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Open

What is the best possible bound on $|G|$ in the Ma-Yan Theorem?
Can we characterize graphs $G$ with $\sigma_t(G) \geq 2kt - t + 1$ but no $k$ disjoint cycles?
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Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

Dirac, 1963 (link)

What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

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\[ G \text{ is } (2k - 1) \text{ connected} \implies \delta(G) \geq 2k - 1 \]
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\(G\) is \((2k - 1)\)-connected \(\Rightarrow \delta(G) \geq 2k - 1 \Rightarrow \sigma_2(G) \geq 4k - 2\)
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KKY: Holds for \(\sigma_2(G) \geq 4k - 3\)
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

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**Answer to Dirac’s Question for Simple Graphs (KKY 2017)**

Let \(k \geq 2\). Every graph \(G\) with

\((i)\) \(|G| \geq 3k\) and

\((ii)\) \(\delta(G) \geq 2k - 1\)

contains \(k\) disjoint cycles if and only if

- if \(k\) is odd and \(|G| = 3k\), then \(G \neq 2K_k \lor \overline{K_k}\), and
- \(\alpha(G) \leq |G| - 2k\), and
- if \(k = 2\) then \(G\) is not a wheel.

\[2k - 1\]
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

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What $(2k - 1)$-connected graphs do not have $k$ disjoint cycles?

Answer to Dirac’s Question for Simple Graphs (KKY 2017)

Let $k \geq 2$. Every graph $G$ with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k - 1$ contains $k$ disjoint cycles if and only if

- if $k$ is odd and $|G| = 3k$, then $G \neq 2K_k \lor K_k$, and
- $\alpha(G) \leq |G| - 2k$, and
- if $k = 2$ then $G$ is not a wheel.

Further:

characterization for multigraphs
Simple Graphs $\rightarrow$ Multigraphs

Idea:

- Take all 1-vertex cycles
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Simple Graphs → Multigraphs

Idea:

- Take all 1-vertex cycles
- Take as many 2-vertex cycles as possible (maximum matching)
Simple Graphs $\rightarrow$ Multigraphs

Idea:

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- Take as many 2-vertex cycles as possible (maximum matching)
Simple Graphs → Multigraphs

Idea:

- Take all 1-vertex cycles
- Take as many 2-vertex cycles as possible (maximum matching)
- What’s left is a simple graph
$(2k - 1)$-connected multigraphs with no $k$ disjoint cycles

**Answer to Dirac's Question for multigraphs: Kierstead-Kostochka-Yeager 2015 (link)**

Let $k \geq 2$ and $n \geq k$. Let $G$ be an $n$-vertex graph with simple degree at least $2k - 1$ and no loops. Let $F$ be the simple graph induced by the strong edges of $G$, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Then $G$ does not contain $k$ disjoint cycles if and only if one of the following holds:

- $n + \alpha' < 3k$;
- $|F| = 2\alpha'$ (i.e., $F$ has a perfect matching) and either (i) $k'$ is odd and $G - F = Y_{k',k'}$, or (ii) $k' = 2 < k$ and $G - F$ is a wheel with 5 spokes;
- $G$ is extremal and either (i) some big set is not incident to any strong edge, or (ii) for some two distinct big sets $I_j$ and $I_{j'}$, all strong edges intersecting $I_j \cup I_{j'}$ have a common vertex outside of $I_j \cup I_{j'}$;
- $n = 2\alpha' + 3k'$, $k'$ is odd, and $F$ has a superstar $S = \{v_0, \ldots, v_s\}$ with center $v_0$ such that either (i) $G - (F - S + v_0) = Y_{k'+1,k'}$, or (ii) $s = 2$, $v_1v_2 \in E(G)$, $G - F = Y_{k'-1,k'}$ and $G$ has no edges between $\{v_1, v_2\}$ and the set $X_0$ in $G - F$;
- $k = 2$ and $G$ is a wheel, where some spokes could be strong edges;
- $k' = 2$, $|F| = 2\alpha' + 1 = n - 5$, and $G - F = C_5$. 

---

86 / 180
$k'$ odd, $F$ has a perfect matching

Example: $k = 8$, $\alpha' = 3$, $k' = 5$. 
Big independent set, incident to no multiple edges

$2^{k-1}$
Wheel, with possibly some spokes multiple

Example: \( k = 2 \)
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

Dirac, 1963 (link)
What $(2k - 1)$-connected multigraphs do not have $k$ disjoint cycles?

Kierstead-Kostochka-Yeager 2015 (link)
Characterization of multigraphs without $k$ disjoint cycles that have minimum simple degree at least $2k - 1$. That is, the underlying simple graph $G$ has $\delta(G) \geq 2k - 1$. 

Open
Do the other results in this talk generalize nicely to multigraphs?
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

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### Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

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**Open**

Do the other results in this talk generalize nicely to multigraphs?
Outline

1. Disjoint Cycles
   - Corrádi-Hajnal
   - Tolerance for some low-degree vertices
   - Ore condition (minimum degree-sum of nonadjacent vertices)
   - Generalized Degree-Sum Conditions
   - Connectivity
     - Neighborhood Union

2. Chorded Cycles
   - Degree conditions
   - Neighborhood Union
   - Multiply Chorded Cycles

3. Equitable Coloring
   - Definition
   - Connection to Cycles
If $G$ has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x, y$, then $G$ contains $k$ disjoint cycles.
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If $G$ has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x, y$, then $G$ contains $k$ disjoint cycles.

$$d(x) + d(y) = 6$$
If $G$ has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x, y$, then $G$ contains $k$ disjoint cycles.

$$d(x) + d(y) = 6$$

$$|N(x) \cup N(y)| = 4$$
If \( G \) has \( n \geq 3k \) vertices and \( |N(x) \cup N(y)| \geq 3k \) for all nonadjacent pairs of vertices \( x, y \), then \( G \) contains \( k \) disjoint cycles.

Neither stronger nor weaker than Corrádi-Hajnal.

- If \( \delta(G) = 2k \), then \( \min_{xy \notin E(G)} \{|N(x) \cup N(y)|\} \geq 2k \).
- If \( |N(x) \cup N(y)| \geq 3k \), then \( \delta(G) \geq 0 \).
If $G$ has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x, y$, then $G$ contains $k$ disjoint cycles.

**Proof**

In an edge-maximal counterexample, choose $k - 1$ disjoint cycles such that

- number of vertices in cycles is minimal, and
- number of connected components in remaining graph is minimal
Neighborhood Union

Faudree-Gould, 2005 (link)

If $G$ has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x, y$, then $G$ contains $k$ disjoint cycles.

Sharpness:

$K_{3k-4}$

$K_5$
If \( G \) has \( n \geq 3k \) vertices and \( |N(x) \cup N(y)| \geq 3k \) for all nonadjacent pairs of vertices \( x, y \), then \( G \) contains \( k \) disjoint cycles.

Let \( G \) be a graph on \( n > 30k \) vertices such that for any nonadjacent \( x, y \in V(G) \), \( |N(x) \cup N(y)| \geq 2k + 1 \). Then \( G \) contains \( k \) disjoint cycles.
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Sharpness of $|N(x) \cup N(y)| \geq 2k + 1$:

$$k = 2$$
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If $G$ has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x, y$, then $G$ contains $k$ disjoint cycles.

Gould-Hirohata-Horn, 2013 (link) (conjecture from FG’05)

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No two disjoint cycles
If $G$ has $n \geq 3k$ vertices and $|N(x) \cup N(y)| \geq 3k$ for all nonadjacent pairs of vertices $x, y$, then $G$ contains $k$ disjoint cycles.

Let $G$ be a graph on $n > 30k$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 2k + 1$. Then $G$ contains $k$ disjoint cycles.

Perhaps $n > 30k$ is not best possible—can be reduced to $4k$?
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Proof (2 pages!)

In an edge-maximal counterexample, choose $k - 1$ disjoint cycles such that

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Conjecture: Bialostocki-Finkel-Gyárfás, 2008 (link)

If $G$ is a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then $G$ contains $r + s$ cycles, $s$ of them chorded.

$s = 0$: Corrádi-Hajnal

$r = 0$: Finkel
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**Chiba-Fujita-Gao-Li, 2010 (link)**

Let $r$ and $s$ be integers with $r + s \geq 1$, and let $G$ be a graph on $n \geq 3r + 4s$ vertices. If $\sigma_2(G) \geq 4r + 6s - 1$, then $G$ contains $r + s$ disjoint cycles, $s$ of them chorded cycles.
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2r + 3s - 1 \\
n - 2r - 3s + 1
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Corollary

Let \( G \) be a graph on \( n \geq 4s \) vertices. If \( \sigma_2(G) \geq 6s - 1 \), then \( G \) contains \( s \) disjoint chorded cycles.
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Let $G$ be a graph on $n \geq 4s$ vertices. If $\sigma_2(G) \geq 6s - 1$, then $G$ contains $s$ disjoint chorded cycles.

Molla-Santana-Yeager, 2017 (link)

For $s \geq 2$, let $G$ be a graph on $n \geq 4s$ vertices. If $\sigma_2(G) \geq 6s - 2$, then $G$ does not contain $s$ disjoint chorded cycles if and only if $G \in \{K_{3s-1,n-3s+1}, K_{3s-2,3s-2,1}\}$. 
Let $r$ and $s$ be integers with $r + s \geq 1$, and let $G$ be a graph on $n \geq 3r + 4s$ vertices. If $\sigma_2(G) \geq 4r + 6s - 1$, then $G$ contains $r + s$ disjoint cycles, $s$ of them chorded cycles.

**Corollary**

Let $G$ be a graph on $n \geq 4s$ vertices. If $\sigma_2(G) \geq 6s - 1$, then $G$ contains $s$ disjoint chorded cycles.
Chorded + Unchorded Cycles: How Sharp Is It?

Corollary: If $G$ is a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then $G$ contains $r + s$ cycles, $s$ of them chorded.
Chorded + Unchorded Cycles: How Sharp Is It?

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**Corollary:** If $G$ is a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then $G$ contains $r + s$ cycles, $s$ of them chorded.

Molla-Santana-Yeager, 2018+

Let $r$ and $s$ be integers with $r + s \geq 1$, and let $G$ be a graph on $n \geq 3r + 4s$ vertices. If $\delta(G) \geq 2r + 3s - 1$, then $G$ fails to contain a collection of $r + s$ disjoint cycles, $s$ of them chorded, if and only if $G$ is one of the following:

$2r + 3s - 1$

$n - 2r - 3s + 1$

$2r + 3s - 2$

$2r + 3s - 2$
Corollary: If $G$ is a graph on $n \geq 3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$, then $G$ contains $r + s$ cycles, $s$ of them chorded.

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Let $r$ and $s$ be integers with $r + s \geq 1$, and let $G$ be a graph on $n \geq 3r + 4s$ vertices. If $\delta(G) \geq 2r + 3s - 1$, then $G$ fails to contain a collection of $r + s$ disjoint cycles, $s$ of them chorded, if and only if $G$ is one of the following:

$s = 1$: 

- $r + 1$
- $r + 2$
- $r + 1$

- $K_{t+1}$
- $2r - t + 1$
- $2r - t + 1$
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<td>We know what happens if $\sigma_2(G) \geq 6s - 2$; what if $\sigma_2(G) \geq 6s - 3$?</td>
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Let $G$ be a graph with $|G| \geq (2t + 1)k$. If $\sigma_t(G) \geq 2kt - t + 1$ for any two integers $k \geq 2$ and $t \geq 5$, then $G$ contains $k$ disjoint cycles.
Outline

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   - Connectivity
   - Neighborhood Union

2 Chorded Cycles
   - Degree conditions
   - Neighborhood Union
   - Multiply Chorded Cycles

3 Equitable Coloring
   - Definition
   - Connection to Cycles
Let $r, s$ be nonnegative integers, and let $G$ be a graph on at least $3r + 4s$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 3r + 4s + 1$. Then $G$ contains $r + s$ disjoint cycles, $s$ of them chorded.
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Sharpness ($r = 0$):
Qiao, 2012 (link)

Let \( r, s \) be nonnegative integers, and let \( G \) be a graph on at least \( 3r + 4s \) vertices such that for any nonadjacent \( x, y \in V(G) \), 
\[
|N(x) \cup N(y)| \geq 3r + 4s + 1.
\]
Then \( G \) contains \( r + s \) disjoint cycles, \( s \) of them chorded.

Gould-Hirohata-Horn, 2013 (link)

Let \( G \) be a graph on at least \( 4s \) vertices such that for any nonadjacent \( x, y \in V(G) \), 
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|N(x) \cup N(y)| \geq 4s + 1.
\]
Then \( G \) contains \( s \) disjoint chorded cycles.
Neighborhood-Union Conditions

Qiao, 2012 (link)
Let $r, s$ be nonnegative integers, and let $G$ be a graph on at least $3r + 4s$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 3r + 4s + 1$. Then $G$ contains $r + s$ disjoint cycles, $s$ of them chorded.

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Let $G$ be a graph on at least $4s$ vertices such that for any nonadjacent $x, y \in V(G)$, $|N(x) \cup N(y)| \geq 4s + 1$. Then $G$ contains $s$ disjoint chorded cycles.

Open:
Can this be improved for large $n$, like for (not-necessarily-chorded) cycles?
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Multiply Chorded Cycles

We define \( f(c) \) to be the number of chords in \( K_{c+1} \), viewed as a cycle. That is, \( f(c) = \frac{(c+1)(c-2)}{2} \).

\[
\begin{align*}
  f(2) &= 0 \\
  f(3) &= 2 \\
  f(4) &= 5
\end{align*}
\]
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$$ f(2) = 0 \quad f(3) = 2 \quad f(4) = 5 $$

Conjecture: Gould-Horn-Magnant, 2014

If $|G| \geq k(c + 1)$ and $\delta(G) \geq ck$, then $G$ contains $k$ disjoint cycles, each with at least $f(c)$ chords.
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If \( |G| \geq k(c + 1) \) and \( \delta(G) \geq ck \), then \( G \) contains \( k \) disjoint cycles, each with at least \( f(c) \) chords.

If \( c = 2 \), then \( f(c) = 0 \), so the conjecture states:

\text{If } |G| \geq 3k \text{ and } \delta(G) \geq 2k, \text{ then } G \text{ contains } k \text{ disjoint cycles}
We define $f(c)$ to be the number of chords in $K_{c+1}$, viewed as a cycle. That is, $f(c) = \frac{(c+1)(c-2)}{2}$.

Conjecture: Gould-Horn-Magnant, 2014

If $|G| \geq k(c + 1)$ and $\delta(G) \geq ck$, then $G$ contains $k$ disjoint cycles, each with at least $f(c)$ chords.

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If \(c = 3\), then \(f(c) = 2\), so the conjecture states:

*If \(|G| \geq 4k\) and \(\delta(G) \geq 3k\), then \(G\) contains \(k\) disjoint cycles, each with at least 2 chords.*
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Qiao-Zhang, 2010 (link)

Let $G$ be a graph on $n \geq 4k$ vertices with $\delta(G) \geq \lceil 7k/2 \rceil$. Then $G$ contains $k$ disjoint, doubly chorded cycles.

Gould-Hirohata-Horn, 2015 (link)

If $G$ is a graph on $n \geq 6k$ vertices with $\delta(G) \geq 3k$, then $G$ contains $k$ vertex-disjoint doubly chorded cycles.
Conjecture: (GHM 2014)
If $|G| \geq k(c + 1)$ and $\delta(G) \geq kc$, then $G$ contains $k$ disjoint cycles, each with at least $f(c)$ chords.

Chiba-Lichiardopol, 2017 (link)
Let $k$ and $c$ be integers, $c \geq 2$, $k \geq 1$. If $G$ is a graph with $\delta(G) \geq k(c + 1) - 1$, then $G$ contains $k$ disjoint cycles, each with at least $f(c)$ chords.
Multiply Chorded Cycles

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Open

Is $\delta(G) \geq k(c + 1) - 1$ the most fitting bound?
Outline

1 Disjoint Cycles
   - Corrádi-Hajnal
   - Tolerance for some low-degree vertices
   - Ore condition (minimum degree-sum of nonadjacent vertices)
   - Generalized Degree-Sum Conditions
   - Connectivity
   - Neighborhood Union

2 Chorded Cycles
   - Degree conditions
   - Neighborhood Union
   - Multiply Chorded Cycles

3 Equitable Coloring
   - Definition
   - Connection to Cycles
An *equitable k-coloring* of a graph $G$ is a proper coloring of $V(G)$ such that any two color classes differ in size by at most one.
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What’s Really Going On

- If $G$ has $3k$ vertices and $k$ cycles, those cycles are cliques
- If $G$ has $4k$ vertices and $k$ doubly chorded cycles, those cycles are cliques
- The complement of a clique is an independent set (color class)
If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

(minimum degree sum of nonadjacent vertices)

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Kierstead-Kostochka, 2008 (link)

If $G$ is a graph such that $d(x) + d(y) \leq 2k - 1$ for every edge $xy$, then $G$ has an equitable $k$-coloring.

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$n = 3k$

Equivalent when $n = 3k$: $2(3k-1)-(2k-1)=4k-1$
If $k \geq \Delta(G) + 1$, then $G$ is equitably $k$-colorable.

Chen-Lih-Wu

Conjecture, 1994 (link)

A connected graph $G$ is equitably $\Delta(G)$ colorable if $G$ is different from $K_{m}$, $C_{2m} + 1$ and $K_{2m} + 1$ for every $m \geq 1$.

Many special cases proved; still open in general.
Hajnal-Szemerédi, 1970

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Chen-Lih-Wu Conjecture Re-stated

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then $G$ is equitably $k$-colorable.
Chen-Lih-Wu **Conjecture** Re-stated

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then $G$ is equitably $k$-colorable.

Kierstead-Kostochka-Molla-Yeager, 2016 (link)

If $G$ is a $3k$-vertex graph such that for each edge $xy$, $d(x) + d(y) \leq 2k + 1$, then $G$ is equitably $k$-colorable, or is one of several exceptions.
### Ore Conditions

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KKY, 2017

For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 

Exceptions

$|G| = 3k$, $\chi(G) \leq k$, $\sigma_2(G) \geq 4k - 3$, no $k$ disjoint cycles.

- $k = 3$

*Equitable coloring:*

*Cycles:*
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$|G| = 3k$, $\chi(\overline{G}) \leq k$, $\sigma_2(G) \geq 4k - 3$, no $k$ disjoint cycles.

- **Equitable coloring:**

\[ 2k - c \]

\[ c \]

---

**Cycles:**
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Slides available at:
http://www.math.ubc.ca/~elyse/Talk_Sendai18.pdf

Thanks!