Disjoint Cycles and Equitable Coloring

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Disjoint Cycles
Corrádi-Hajnal Theorem

If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.
Corrádi-Hajnal, 1963

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Sharpness:
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Sharpness:

\[ \begin{align*}
  & \begin{array}{c}
  k \\
  k \\
  k \\
  \end{array} \\
  \text{2k - 1}
\end{align*} \]
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Minimum degree sum of nonadjacent vertices:

$$\sigma_2(G) := \min \{ d(x) + d(y) : xy \not\in E(G) \}$$

That is, low vertices form a clique.
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Implies Corrádi-Hajnal
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Sharpness:
If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 
Dirac’s Question
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

**Dirac, 1963**

What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

**Answer to Dirac's Question for Simple Graphs**

(Kierstead-Kostochka-Yeager, 2015+)

Let \(k \geq 2\). Every graph \(G\) with (i) \(|G| \geq 3k\) and (ii) \(\delta(G) \geq 2k - 1\) contains \(k\) disjoint cycles if and only if \(\alpha(G) \leq |G| - 2k\), and if \(k\) is odd and \(|G| = 3k\), then \(G \neq 2K_k \cup K_k\), and if \(k = 2\) then \(G\) is not a wheel.

Further:

Characterization for multigraphs

Kierstead-Kostochka-Yeager

Combinatorica, to appear.

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- \(\alpha(G) \leq |G| - 2k\), and
- if \(k\) is odd and \(|G| = 3k\), then \(G \neq 2K_k \lor K_k\), and
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**Further:**

Characterization for *multigraphs*
The Case $n = 3k$

If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

Kierstead-Kostochka-Yeager, 2015+

For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 
If $G$ is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles, or is one of several exceptions, or $\overline{G}$ is not $k$-colorable.
Chen-Lih-Wu Conjecture

**Hajnal-Szemerédi, 1970**

If \( k \geq \Delta(G) + 1 \), then \( G \) is equitably \( k \)-colorable.
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If \( \chi(G), \Delta(G) \leq k \), and if \( k \) is odd \( K_{k,k} \not\subseteq G \), then \( G \) is equitably \( k \)-colorable.
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If \( \chi(G), \Delta(G) \leq k \), and if \( k \) is odd \( K_{k,k} \not\subseteq G \), then \( G \) is equitably \( k \)-colorable.

Kierstead-Kostochka-Molla-Yeager, 2015+
If \( G \) is a \( k \)-colorable \( 3k \)-vertex graph such that for each edge \( xy \), \( d(x) + d(y) \leq 2k + 1 \), then \( G \) is equitably \( k \)-colorable, or is one of several exceptions.
Thanks for Listening!