Optimal, Integral, Likely

Optimization, Integral Calculus, and Probability
for students of commerce and the social sciences

Practice Book

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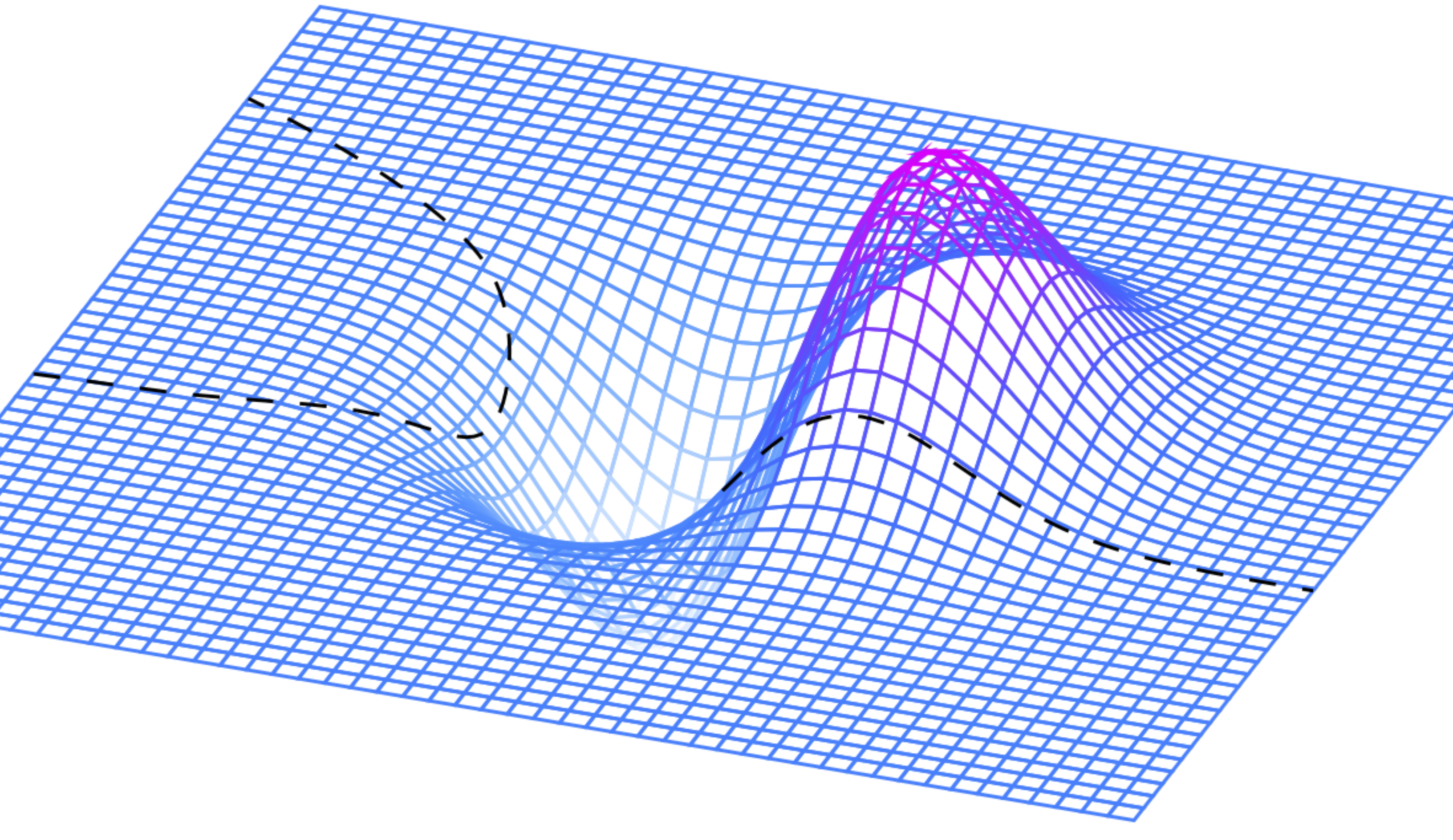
Adapted from *CLP Calculus* by Joel Feldman, Andrew Rechnitzer, and Elyse Yeager



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Source files can be found at https://gitlab.math.ubc.ca/ecyeager/OIL

This book contains new material as well as material adapted from open sources.

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- Section 4.5 contains several problems adapted from Chapter 4.2 of Introductory Statistics by Ilowsky and Dean under a Creative Commons Attribution License v4.0.

▸▸ Acknowledgements

UBC Point Grey campus sits on the traditional, ancestral and unceded territory of the xʷməθkʷəy̓əm (Musqueam). Musqueam and UBC have an ongoing relationship sharing insight, knowledge, and labour. Those interested in learning more about this relationship might start here.

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▸▸ Contact

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How to use this book

Introduction

First of all, welcome back to Calculus!

This book is a companion question book for the main textbook.

How to Work Questions

This book is organized into four sections: Questions, Hints, Answers, and Solutions. As you are working problems, resist the temptation to prematurely peek at the back! It's important to allow yourself to struggle for a time with the material. Even professional mathematicians don't always know right away how to solve a problem. The art is in gathering your thoughts and figuring out a strategy to use what you know to find out what you don't.

If you find yourself at a real impasse, go ahead and look for a hint in the Hints section. Think about it for a while, and don't be afraid to read back in the notes to look for a key idea that will help you proceed. If you still can't solve the problem, well, we included the Solutions section for a reason! As you're reading the solutions, try hard to understand why we took the steps we did, instead of memorizing step-by-step how to solve that one particular problem.

If you struggled with a question quite a lot, it's probably a good idea to return to it in a few days. That might have been enough time for you to internalize the necessary ideas, and you might find it easily conquerable. Pat yourself on the back–sometimes math makes you feel good! If you're still having troubles, read over the solution again, with an emphasis on understanding why each step makes sense.

One of the reasons so many students are required to study calculus is the hope that it will improve their problem-solving skills. In this class, you will learn lots of concepts, and be asked to apply them in a variety of situations. Often, this will involve answering one really big problem by breaking it up into manageable chunks, solving those chunks, then putting the pieces back together. When you see a particularly long question, remain calm and look for a way to break it into pieces you can handle.

▸▸ Working with Friends

Study buddies are fantastic! If you don't already have friends in your class, you can ask your neighbours in lecture to form a group. Often, a question that you might bang your head against for an hour can be easily cleared up by a friend who sees what you've missed. Regular study times make sure you don't procrastinate too much, and friends help you maintain a positive attitude when you might otherwise succumb to frustration. Struggle in mathematics is desirable, but suffering is not.

When working in a group, make sure you try out problems on your own before coming together to discuss with others. Learning is a process, and getting answers to questions that you haven't considered on your own can rob you of the practice you need to master skills and concepts, and the tenacity you need to develop to become a competent problem-solver.

▸▸ Types of Questions

Q[1]: hint answer solution

Outlined questions make up the *suggested question set*. These questions are usually highly typical of what you'd see on an exam, although some of them are atypical but carry an important moral. If you find yourself unconfident with the idea behind one of these, it's probably a good idea to practice similar questions.

This suggested question set is a *minimal* selection of questions to work on. You are highly encouraged to work on more.

Q[2](∗): hint answer solution

In addition to original problems, this book contains problems pulled from quizzes and exams given at UBC for Math 101 and 105 (second-semester calculus), Math 121 (honours second-semester calculus), and Math 317 (Calculus 4). These problems are marked with a star. The authors would like to acknowledge the contributions of the many people who collaborated to produce these exams over the years.

Instructions and other comments that are attached to more than one question are written in this font.

The questions are organized into Stage 1, Stage 2, and Stage 3.

▸▸ Stage 1

The first category is meant to test and improve your understanding of basic underlying concepts. These often do not involve much calculation. They range in difficulty from very basic reviews of definitions to questions that require you to be thoughtful about the concepts covered in the section.

▸▸ Stage 2

Questions in this category are for practicing skills. It's not enough to understand the philosophical grounding of an idea: you have to be able to apply it in appropriate situations. This takes practice!

▸▸ Stage 3

The last questions in each section go a little farther than Stage 2. Often they will combine more than one idea, incorporate review material, or ask you to apply your understanding of a concept to a new situation.

▸▸ Open-Ended Questions

We've started adding a small number of open-ended questions. These are meant to have paragraph-style answers, and solutions are not provided.

In exams, as in life, you will encounter questions of varying difficulty. A good skill to practice is recognizing the level of difficulty a problem poses. Exams will have some easy questions, some standard questions, and some harder questions.

▸▸ How to Read the Title

"Optimal, Integral, Likely" can be recited in your head with roughly the same energy as an infomercial. "It's optimal! It's integral! And that's not even all!"

Contents

Part I

THE QUESTIONS

Chapter 1

VECTORS AND GEOMETRY IN TWO AND THREE DIMENSIONS

1.1▲ Points

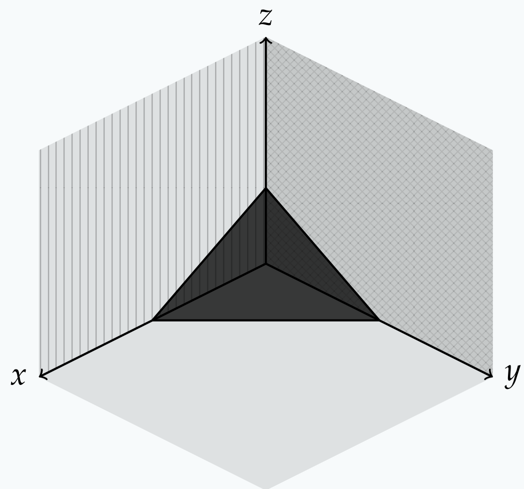
Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Part of $\mathbb{R}^3$ is sketched below, along with a triangle.



Identify the following parts of the sketch:

(a) the xy-plane
(b) the yz-plane
(c) the xz-plane
(d) the vertex of the triangle lying on $(1,0,0)$
(e) the vertex of the triangle lying on $(0,1,0)$
(f) the vertex of the triangle lying on $(0,0,1)$

Q[2]: hint answer solution

Describe the set of all points (x,y,z) in $\mathbb{R}^3$ that satisfy

(a) $x^2+y^2+z^2=2x-4y+4$

(b) $x^2+y^2+z^2<2x-4y+4$

Q[3]: hint answer solution

Describe and sketch the set of all points (x,y) in $\mathbb{R}^2$ that satisfy

(a) $x=y$

(b) $x+y=1$

(c) $x^2+y^2=4$

(d) $x^2+y^2=2y$

(e) $x^2+y^2<2y$

Q[4]: hint answer solution

Describe the set of all points (x,y,z) in $\mathbb{R}^3$ that satisfy the following conditions. Sketch the part of the set that is in the first *octant*. That is, sketch the part of the set with non-negative values of x, y, and z.

(a) $z=x$

(b) $x^2 + y^2 + z^2 = 4$

(c) $x^2 + y^2 + z^2 = 4, z = 1$

(d) $x^2 + y^2 = 4$

(e) $z = x^2 + y^2$

▸▸ Stage 2

Q[5]: hint answer solution
What is the distance from the point $(1, 2, 3)$ to the point $(4, -5, 6)$?

Q[6]: hint answer solution
What is the distance from the point $(-5, -1, -9)$ to the xy-plane?

Q[7]: hint answer solution
A bird sets off from its nest. It flies one kilometre due north, then two kilometres due east, gaining 100 metres of altitude. How far is it from its nest?

Q[8]: hint answer solution
A bird sets off from its nest on the ground. It flies two kilometres due north, then two kilometres due east, ending up at a point that is 3 km away from its nest. How high above the ground is that point?

Q[9]: hint answer solution
A giant straight wall rises from the ground, reaching high in the sky, casting a cold shadow as far as you can see. You walk straight out from the base of the wall for 2 km, ash floating in the air, catching in your throat and stinging your eyes. Tired, you sit on the ground to rest, and look around you. In the hazy distance, you see what at first you think must be an illusion: a single tree. It's the only thing standing in this desolate flatness. Curiosity overcomes your fatigue, and you wobble onto blistered feet. (Not your feet—ew. You kick them out of the way.) You turn at a right angle to your previous course, walking 1 km parallel to the looming monolith, and reach the tree. Even at this distance, the wall seems to emit a sinister hum. Except, no — you realize that sound isn't the wall at all. Three metres up the tree, a colony of murder hornets is busily expanding their nest. For the first time today, you smile.

How far are the murder hornets from the wall?

Q[10]: hint answer solution
The pressure $p(x, y)$ at the point (x, y) is determined by $x^2 - 2px + y^2 = 1$. An *isobar* is a curve with equation $p(x, y) = c$ for some constant c. Sketch several isobars.

Q[11]: hint answer solution
Show that the set of all points P that are twice as far from $(3, -2, 3)$ as from $(3/2, 1, 0)$ is a sphere. Find its centre and radius.

▸▸ Stage 3

Q[12]: hint answer solution
Consider any triangle. Pick a coordinate system so that one vertex is at the origin and a second vertex is on the positive x–axis. Call the coordinates of the second vertex $(a, 0)$ and those of the third vertex (b, c). Find the circumscribing circle (the circle that goes through all three vertices).

Q[13](∗): hint answer solution
Find an equation for the set of all points $P = (x, y, z)$ such that the distance from P to the point $(0, 0, 1)$ is equal to the distance from P to the plane $z + 1 = 0$.

Sketch the set, and also describe it in words.

1.2▲ Vectors

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution
Let $\mathbf{a} = \langle 2, 0\rangle$ and $\mathbf{b} = \langle 1, 1\rangle$. Evaluate and sketch $\mathbf{a} + \mathbf{b}$, $\mathbf{a} + 2\mathbf{b}$ and $2\mathbf{a} - \mathbf{b}$.

Q[2]: hint answer solution
Determine whether or not the given points are collinear (that is, lie on a common straight line)

(a) $(1, 2, 3),\ (0, 3, 7),\ (3, 5, 11)$

(b) $(0, 3, -5),\ (1, 2, -2),\ (3, 0, 4)$

Q[3]: hint answer solution
Determine whether the given pair of vectors are perpendicular

(a) $\langle 1, 3, 2\rangle\,,\ \langle 2, -2, 2\rangle$

(b) $\langle -3, 1, 7\rangle\,,\ \langle 2, -1, 1\rangle$

(c) $\langle 2, 1, 1\rangle\,,\ \langle -1, 4, 2\rangle$

Q[4]: hint answer solution
Does the triangle with vertices $(1, 2, 3)$, $(4, 0, 5)$ and $(3, 6, 4)$ have a right angle?

Q[5]: hint answer solution
Consider the following statement: "If $\mathbf{a} \neq \mathbf{0}$ and if $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ then $\mathbf{b} = \mathbf{c}$." If the statement is true, prove it. If the statement is false, give a counterexample.

▸▸ Stage 2

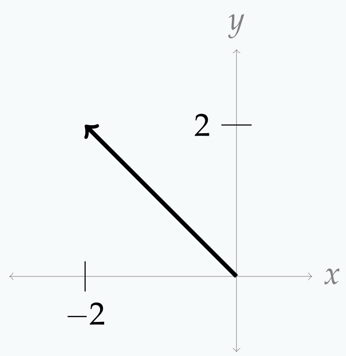
Q[6]: hint answer solution

In each part, give the vector described.

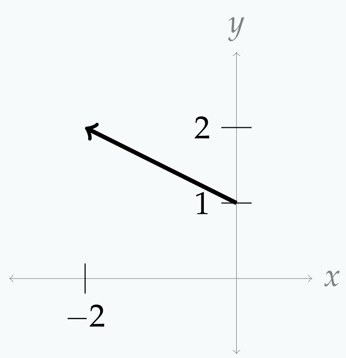
(a) Head at $(0,1)$, tail at $(1,0)$.

(b) Head at $(1,2,3)$, tail at $(4,5,4)$.

(c) Pictured below:



(d) Pictured below:



Q[7]: hint answer solution

Find the magnitude of each vector below.

(a) $\langle 1,2\rangle$

(b) $\langle -2,1\rangle$

(c) $\langle 2,4\rangle$

Q[8]: hint answer solution

For each vector below, find a unit vector in the same direction.

(a) $\langle 3,4\rangle$

(b) $\langle 1,1,1\rangle$

(c) $\langle 0,1,0\rangle$

(d) $\langle 7,7,8\rangle$

(e) $2\hat{\boldsymbol{\imath}}$

Q[9]: hint answer solution

Find two vectors parallel to the vector $\langle 9,0,7\rangle$ with length 783.

Q[10]: hint answer solution

Let $\mathbf{a} = \langle 1,2,1\rangle$ and $\mathbf{b} = \langle 3,4,5\rangle$. Compute $\mathbf{a} + 3\mathbf{b}$ and $|\mathbf{a} - \mathbf{b}|$.

Q[11]: hint answer solution

Find the equation of a sphere if one of its diameters has endpoints $(2,1,4)$ and $(4,3,10)$.

Q[12]: hint answer solution

Compute the dot product of the vectors $\mathbf{a}$ and $\mathbf{b}$. Which pairs are perpendicular? Which are parallel?

(a) $\mathbf{a} = \langle 1,2\rangle$, $\mathbf{b} = \langle -2,3\rangle$

(b) $\mathbf{a} = \langle -1,1\rangle$, $\mathbf{b} = \langle 1,1\rangle$

(c) $\mathbf{a} = \langle 1,1\rangle$, $\mathbf{b} = \langle 2,2\rangle$

(d) $\mathbf{a} = \langle 1,2,1\rangle$, $\mathbf{b} = \langle -1,1,1\rangle$

(e) $\mathbf{a} = \langle -1,2,3\rangle$, $\mathbf{b} = \langle 3,0,1\rangle$

Q[13]: hint answer solution

Determine all values of y for which the given vectors are perpendicular.

(a) $\langle 2,4\rangle$, $\langle 2,y\rangle$

(b) $\langle 4,-1\rangle$, $\langle y,y^2\rangle$

(c) $\langle 3,1,1\rangle$, $\langle 2,5y,y^2\rangle$

Q[14]: hint answer solution

Let $\mathbf{u} = -2\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}}$ and $\mathbf{v} = \alpha\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$. Find α so that

(a) $\mathbf{u} \perp \mathbf{v}$

(b) $\mathbf{u} \| \mathbf{v}$

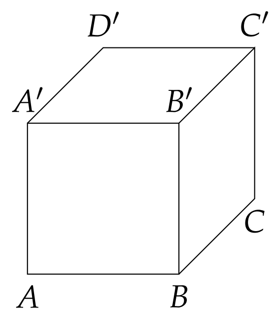
▸▸ Stage 3

Q[15]: hint answer solution

Show that the diagonals of a parallelogram bisect each other.

Q[16]: hint answer solution

Consider a cube with side length s. Name, in order, the four vertices on the bottom of the cube A, B, C, D and the corresponding four vertices on the top of the cube A', B', C', D'.



Show that all edges of the tetrahedron $A'C'BD$ have the same length.

Q[17]: hint answer solution
A prism has the six vertices

$$\begin{aligned} A &= (1,0,0) & A' &= (5,0,1) \\ B &= (0,3,0) & B' &= (4,3,1) \\ C &= (0,0,4) & C' &= (4,0,5) \end{aligned}$$

(a) Verify that three of the faces are parallelograms. Are they rectangular?

(b) Find the length of AA'.

1.3▲ Equations of Planes in 3d

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution
The vector $\hat{\mathbf{k}}$ is a normal vector (i.e. is perpendicular) to the plane $z = 0$. Find another nonzero vector that is normal to $z = 0$.

Q[2]: hint answer solution
Find equations for the planes described below.
(a) Plane P passes through the origin and has normal vector $\langle 1,4,1\rangle$.
(b) Plane Q passes through the point (1,4,1) and has normal vector $\langle 7,8,9\rangle$.
(c) Plane R passes through the point (2,2,5) and is parallel to Plane Q.

Q[3](*): hint answer solution
Find the equation of the plane that contains $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Q[4]: hint answer solution
Find the equation of the plane containing the points $(1,0,1)$, $(1,1,0)$ and $(0,1,1)$.

Q[5]: hint answer solution
What's wrong with the following exercise? "Find the equation of the plane containing $(1,2,3)$, $(2,3,4)$ and $(3,4,5)$."

Q[6]: hint answer solution
The vectors $\langle 1,4,7\rangle$ and $\langle 2,a,b\rangle$ are both normal to the same plane. What are a and b?

▸▸ Stage 2

Q[7]: hint answer solution
A plane has normal vector $\langle 2,-3,1\rangle$ and passes through the point $(100,100,100)$. Is the origin a point on this plane?

Q[8]: hint answer solution
A plane has normal vector $\langle 2, -3, a\rangle$ for some constant a and passes through the points $(2, 22, 222)$ and $(3, 33, 333)$. What is a?

Q[9]: hint answer solution
Let P be the plane with equation

$$7x - 8y + 3z = 907$$

Sketch the following.
(a) The intersection of P with the xy plane.
(b) The intersection of P with the xz plane.
(c) The intersection of P with the yz plane.

Q[10]: hint answer solution
Given the four planes below, decide which pairs are parallel, perpendicular, or identical.
P: $x - y + z = 1$ Q: $2x - 2y + 2z = 2$ R: $x + 2y + z = 13$ S: $x + y + z = 4$

Q[11]: hint answer solution
Describe the set of points that are equidistant from $(1, 2, 3)$ and $(5, 2, 7)$.

Q[12]: hint answer solution
For each set of three points, find the plane that contains them.

(a) $(1, 0, 1)$, $(2, 4, 6)$, $(1, 2, -1)$

(b) $(1, -2, -3)$, $(4, -4, 4)$, $(3, 2, -3)$

(c) $(1, -2, -3)$, $(5, 2, 1)$, $(-1, -4, -5)$

▸▸ Stage 3

Q[13]: hint answer solution
Describe the set of points equidistant from **a** and **b**.

Q[14](*): hint answer solution
A plane Π passes through the points $A = (1, 1, 3)$, $B = (2, 0, 2)$ and $C = (2, 1, 0)$ in $\mathbb{R}^3$.

(a) Find an equation for the plane Π.

(b) Find the point E in the plane Π such that the line L through $D = (6, 1, 2)$ and E is perpendicular to Π.

Q[15]: hint answer solution
Find the equation of the sphere which has the two planes $x + y + z = 3$, $x + y + z = 9$ as tangent planes if the center of the sphere is on the planes $2x - y = 0$, $3x - z = 0$.

1.4▴ Functions of Two Variables

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

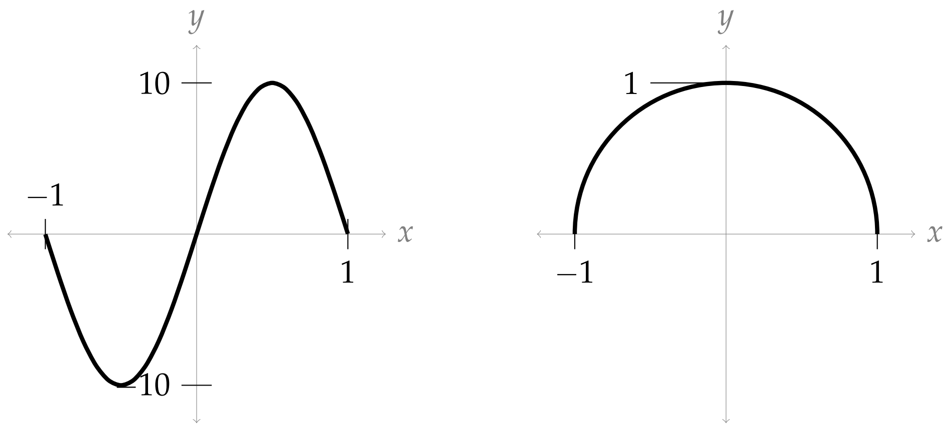
▸▸ Stage 1

Q[1]: hint answer solution

Give an example of a function that has all of $\mathbb{R}^2$ in its domain, and whose range is a single number.

Q[2]: hint answer solution

Single-variable functions $f(x)$ and $g(x)$ are sketched below. Both have domain $[-1, 1]$.



$y = f(x)$ $y = g(x)$

Based on the sketches, find the following.

(a) The range of $f(x)$,

(b) the range of $g(x)$,

(c) the domain of $f(g(x))$, and

(d) the range of $f(g(x))$.

Q[3]: hint answer solution

Is the point $(x, y) = (1, 1)$ in the domain of the implicitly defined function

$$z^2y^3 + zx^3 + xy = 1\ ?$$

▸▸ Stage 2

Q[4]: hint answer solution

Find the domain and range of the function

$$f(x, y) = \sqrt{4x^2 + y^2}$$

Q[5]: hint answer solution

Find the domain and range of the function

$$h(x,y) = \frac{x^2}{1+y^2}$$

Q[6]: hint answer solution

Find the domain and range of the function

$$k(x,y) = \arcsin\left(x^2+y^2\right)$$

▸▸ Stage 3

Q[7]: hint answer solution

Find the domain and range of the function

$$g(x,y) = \frac{1}{\ln(xy)}$$

Q[8]: hint answer solution

Find the domain and range of the two-variable function

$$f(x,y) = \frac{x^2}{x^2+1}$$

Q[9]: hint answer solution

Find the domain and range of the function

$$f(x,y) = \frac{x}{x^2+1} + \sin y$$

Q[10]: hint answer solution

If a company spends a dollars on advertisements, and sells the advertised product at p dollars each, then the number of units that will be sold is given as a function $D(a,p)$.

Give a sensible model domain and range.

Q[11]: hint answer solution

You're using the function

$$f(x,y) = \frac{1}{x^2+y^2}$$

to model some process. In your model, the only values of the range that make sense are

$$3 \leqslant f(x,y) \leqslant 5$$

What is your model domain?

Q[12]: hint answer solution

You're using the function

$$g(x,y) = 72\left[x^2 - y\right]^2 - \left[x^2 - y\right]^4$$

to model some process. In your model, the only values of the range that make sense are

$$272 \leqslant g(x,y) \leqslant 1175$$

What is the corresponding model domain?

1.5▲ Sketching Surfaces in 3d

Exercises

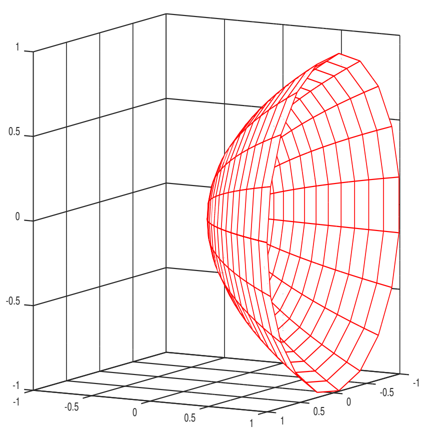
Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

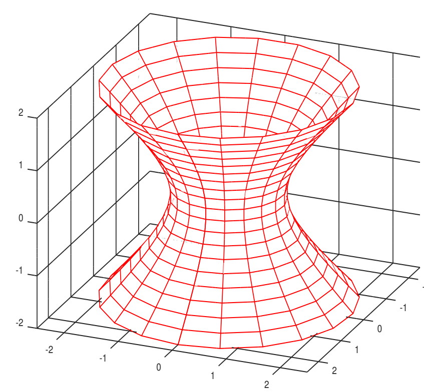
Q[1](*): hint answer solution

Match the following equations and expressions with the corresponding pictures.

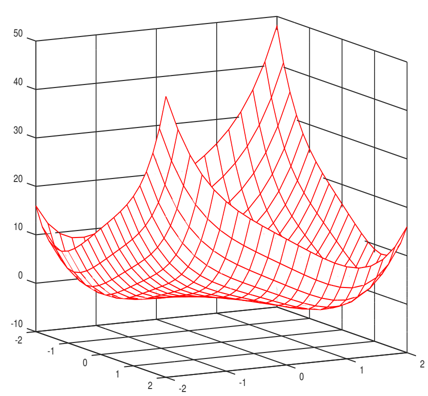
(A)



(B)



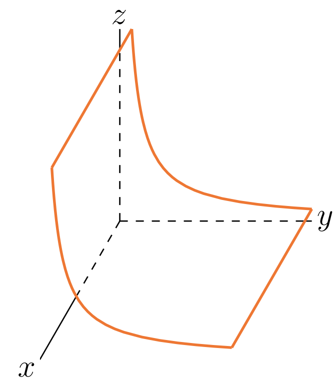
(C)



(a) $x^2 + y^2 = z^2 + 1$ (b) $y = x^2 + z^2$ (c) $z = x^4 + y^4 - 4xy$

Q[2]: hint answer solution

Sketch a few level curves for the function $f(x, y)$ whose graph $z = f(x, y)$ is sketched below.



▸▸ Stage 2

Q[3]: hint answer solution

Sketch some of the level curves of

(a) $f(x, y) = x^2 + 2y^2$

(b) $f(x, y) = xy$

(c) $f(x, y) = xe^{-y}$

Q[4](*): hint answer solution

Sketch the level curves of $f(x, y) = \frac{2y}{x^2+y^2}$.

Q[5](*): hint answer solution

A surface is given implicitly by

$$x^2 + y^2 - z^2 + 2z = 0$$

(a) Sketch several level curves z =constant.

(b) Draw a rough sketch of the surface.

Q[6](*): hint answer solution

Sketch the hyperboloid $z^2 = 4x^2 + y^2 - 1$.

Q[7]: hint answer solution

Sketch the graphs of

(a) $f(x, y) = \sin x \qquad 0 \leqslant x \leqslant 2\pi,\ 0 \leqslant y \leqslant 1$

(b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $f(x, y) = |x| + |y|$

Q[8]: hint answer solution

Sketch and describe the following surfaces.

(a) $4x^2 + y^2 = 16$

(b) $x + y + 2z = 4$

(c) $\frac{y^2}{9} + \frac{z^2}{4} = 1 + \frac{x^2}{16}$

(d) $y^2 = x^2 + z^2$

(e) $\frac{x^2}{9} + \frac{y^2}{12} + \frac{z^2}{9} = 1$

(f) $x^2 + y^2 + z^2 + 4x - by + 9z - b = 0$ where b is a constant.

(g) $\frac{x}{4} = \frac{y^2}{4} + \frac{z^2}{9}$

(h) $z = x^2$

Q[9]: hint answer solution

Sketch the level curves of the function

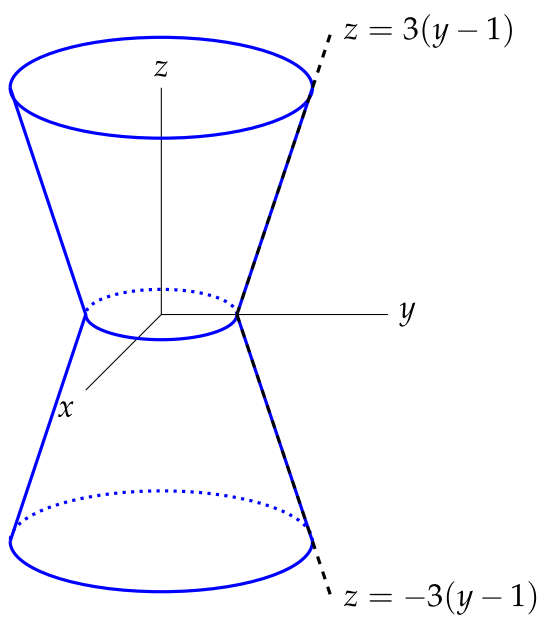
$$f(x, y) = \sin(x + y)$$

for $z = 0$, $z = 1$, and $z = 2$.

▸▸ Stage 3

Q[10]: hint answer solution

The surface below has circular level curves, centred along the z-axis. The lines given are the intersection of the surface with the right half of the yz-plane. Give an equation for the surface.



Chapter 2

PARTIAL DERIVATIVES

2.1▲ Partial Derivatives

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

You are traversing an undulating landscape. Take the z-axis to be straight up towards the sky, the positive x-axis to be due south, and the positive y-axis to be due east. Then the landscape near you is described by the equation $z = f(x, y)$, with you at the point $(0, 0, f(0, 0))$. The function $f(x, y)$ is differentiable.

Suppose $f_y(0, 0) < 0$. Is it possible that you are at a summit? Explain.

Q[2]: hint answer solution

The table below gives approximate value of $f(x,y)$ at different values of x and y. (The row gives the value of x, and the column gives the value of y.)

y ↓ \ x →	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**
1.5	8.2	9.1	10.1	11.2	12.2	13.3	14.4	15.6	16.7	17.9
1.6	8.0	9.0	9.9	10.9	12.0	13.1	14.2	15.3	16.4	17.6
1.7	7.8	8.8	9.7	10.7	11.7	12.8	13.9	15.0	16.1	17.3
1.8	7.6	8.6	9.5	10.5	11.5	12.5	13.6	14.7	15.8	17.0
1.9	7.5	8.4	9.3	10.3	11.3	12.3	13.3	14.4	15.5	16.6
2.0	7.3	8.2	9.1	10.0	11.0	12.0	13.0	14.1	15.2	16.3
2.1	7.1	8.0	8.9	9.8	10.8	11.8	12.8	13.8	14.9	16.0
2.2	7.0	7.8	8.7	9.6	10.5	11.5	12.5	13.5	14.6	15.6
2.3	6.8	7.6	8.5	9.4	10.3	11.2	12.2	13.2	14.2	15.3
2.4	6.6	7.4	8.3	9.1	10.0	11.0	11.9	12.9	13.9	15.0

Use the table to approximate the following partial derivatives.

(a) $f_y(1.5, 2.4)$

(b) $f_x(1.7, 1.7)$

(c) $f_y(1.7, 1.7)$

(d) $f_x(1.1, 2)$

▸▸ Stage 2

Q[3]: hint answer solution

Find all first partial derivatives of the following functions and evaluate them at the given point.

(a) $f(x,y,z) = x^3y^4z^5 \qquad (0,-1,-1)$

(b) $w(x,y,z) = \ln\left(1 + e^{xyz}\right) \qquad (2,0,-1)$

(c) $f(x,y) = \frac{1}{\sqrt{x^2+y^2}} \qquad (-3,4)$

Q[4]: hint answer solution

Show that the function $z(x,y) = \frac{x+y}{x-y}$ obeys

$$x\frac{\partial z}{\partial x}(x,y) + y\frac{\partial z}{\partial y}(x,y) = 0$$

Q[5](*): hint answer solution
A surface $z(x, y)$ is defined by $zy - y + x = \ln(xyz)$.
(a) Compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ in terms of x, y, z.
(b) Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x, y, z) = (-1, -2, 1/2)$.

Q[6](*): hint answer solution
Find $\frac{\partial U}{\partial T}$ and $\frac{\partial T}{\partial V}$ at $(1, 1, 2, 4)$ if (T, U, V, W) are related by

$$(TU - V)^2 \ln(W - UV) = \ln 2$$

Q[7](*): hint answer solution
Suppose that $u = x^2 + yz$, $x = \rho r \cos(\theta)$, $y = \rho r \sin(\theta)$ and $z = \rho r$. Find $\frac{\partial u}{\partial r}$ at the point $(\rho_0, r_0, \theta_0) = (2, 3, \pi/2)$.

Q[8]: hint answer solution
Use the definition of the derivative to evaluate $f_x(0, 0)$ and $f_y(0, 0)$ for

$$f(x, y) = \begin{cases} \frac{x^2 - 2y^2}{x - y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

▸▸ Stage 3

Q[9]: hint answer solution
Let f be any differentiable function of one variable. Define $z(x, y) = f(x^2 + y^2)$. Is the equation

$$y \frac{\partial z}{\partial x}(x, y) - x \frac{\partial z}{\partial y}(x, y) = 0$$

necessarily satisfied?

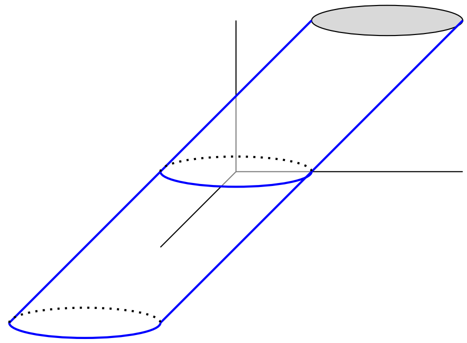
Q[10]: hint answer solution
Define the function

$$f(x, y) = \begin{cases} \frac{(x+2y)^2}{x+y} & \text{if } x + y \neq 0 \\ 0 & \text{if } x + y = 0 \end{cases}$$

(a) Evaluate, if possible, $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$.

(b) Is $f(x, y)$ continuous at $(0, 0)$?

Q[11]: hint answer solution
Consider the cylinder whose base is the radius-1 circle in the xy-plane centred at $(0, 0)$, and which slopes parallel to the line in the yz-plane given by $z = y$.



When you stand at the point $(0, -1, 0)$, what is the slope of the surface if you look in the positive y direction? The positive x direction?

Q[12](*): hint answer solution
Let

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Compute, directly from the definitions,

(a) $\frac{\partial f}{\partial x}(0,0)$

(b) $\frac{\partial f}{\partial y}(0,0)$

(c) $\frac{\mathrm{d}}{\mathrm{d}t} f(t,t)\Big|_{t=0}$

2.2▲ Higher Order Derivatives

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution
Let all of the third order partial derivatives of the function $f(x, y, z)$ exist and be continuous. Show that

$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z) = f_{zyx}(x,y,z)$$

Q[2]: hint answer solution
Find, if possible, a function $f(x, y)$ for which $f_x(x, y) = e^y$ and $f_y(x, y) = e^x$.

▸▸ Stage 2

Q[3]: hint answer solution

Find the specified partial derivatives.

(a) $f(x,y) = x^2y^3$; $f_{xx}(x,y)$, $f_{xyy}(x,y)$, $f_{yxy}(x,y)$

(b) $f(x,y) = e^{xy^2}$; $f_{xx}(x,y)$, $f_{xy}(x,y)$, $f_{xxy}(x,y)$, $f_{xyy}(x,y)$

(c) $f(u,v,w) = \dfrac{1}{u+2v+3w}$; $\dfrac{\partial^3 f}{\partial u \partial v \partial w}(u,v,w)$, $\dfrac{\partial^3 f}{\partial u \partial v \partial w}(3,2,1)$

Q[4]: hint answer solution

Find all second partial derivatives of $f(x,y) = \sqrt{x^2+5y^2}$.

Q[5]: hint answer solution

Find the specified partial derivatives.

(a) $f(x,y,z) = \arctan\left(e^{\sqrt{xy}}\right)$; $f_{xyz}(x,y,z)$

(b) $f(x,y,z) = \arctan\left(e^{\sqrt{xy}}\right) + \arctan\left(e^{\sqrt{xz}}\right) + \arctan\left(e^{\sqrt{yz}}\right)$; $f_{xyz}(x,y,z)$

(c) $f(x,y,z) = \arctan\left(e^{\sqrt{xyz}}\right)$; $f_{xx}(1,0,0)$

▸▸ Stage 3

Q[6]: hint answer solution

Let $\alpha > 0$ be a constant. Show that $u(x,y,z,t) = \dfrac{1}{t^{3/2}} e^{-(x^2+y^2+z^2)/(4\alpha t)}$ satisfies the heat equation

$$u_t = \alpha\big(u_{xx} + u_{yy} + u_{zz}\big)$$

for all $t > 0$.

Q[7]: hint answer solution

Economics consists of mathematical models that replicate how we interact with each other and resources. When modelling demand, economists prefer the following properties in utility functions. Suppose u is a utility function that depends on some variable t with $t > 0$ (for example it can signify the quantity of a good or a product). Then we may wish:

(i) $u_t > 0$ (for all values of the variables that u depends on).

(ii) $u_{tt} < 0$ (for all values of the variables that u depends on).

(iii) $u_t \to \infty$ as $t \to \infty$ (for all values of the rest of the variables that u depends on).

Determine which of the following utility functions follow or do not follow these properties by checking them for x and y with $x, y > 0$:

(a) $u(x,y) = x^{0.5}y^{0.5}$

(b) $u(x,y) = \dfrac{x^{0.5}}{y^{0.5}}$

(c) $u(x,y) = \ln(x) + \ln(y)$

(d) $u(x) = \dfrac{x^{1-a}}{1-a}$, where $a \neq 1$

Q[8]: hint answer solution

The table below gives approximate value of $f(x,y)$ at different values of x and y. (The row gives the value of y, and the column gives the value of x.)

y ↓ \ x →	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.5	8.2	9.1	10.1	11.2	12.2	13.3	14.4	15.6	16.7	17.9
1.6	8.0	9.0	9.9	10.9	12.0	13.1	14.2	15.3	16.4	17.6
1.7	7.8	8.8	9.7	10.7	11.7	12.8	13.9	15.0	16.1	17.3
1.8	7.6	8.6	9.5	10.5	11.5	12.5	13.6	14.7	15.8	17.0
1.9	7.5	8.4	9.3	10.3	11.3	12.3	13.3	14.4	15.5	16.6
2.0	7.3	8.2	9.1	10.0	11.0	12.0	13.0	14.1	15.2	16.3
2.1	7.1	8.0	8.9	9.8	10.8	11.8	12.8	13.8	14.9	16.0
2.2	7.0	7.8	8.7	9.6	10.5	11.5	12.5	13.5	14.6	15.6
2.3	6.8	7.6	8.5	9.4	10.3	11.2	12.2	13.2	14.2	15.3
2.4	6.6	7.4	8.3	9.1	10.0	11.0	11.9	12.9	13.9	15.0

Use the table to approximate $f_{xy}(1.8, 2.0)$.

2.3▲ Local Maximum and Minimum Values

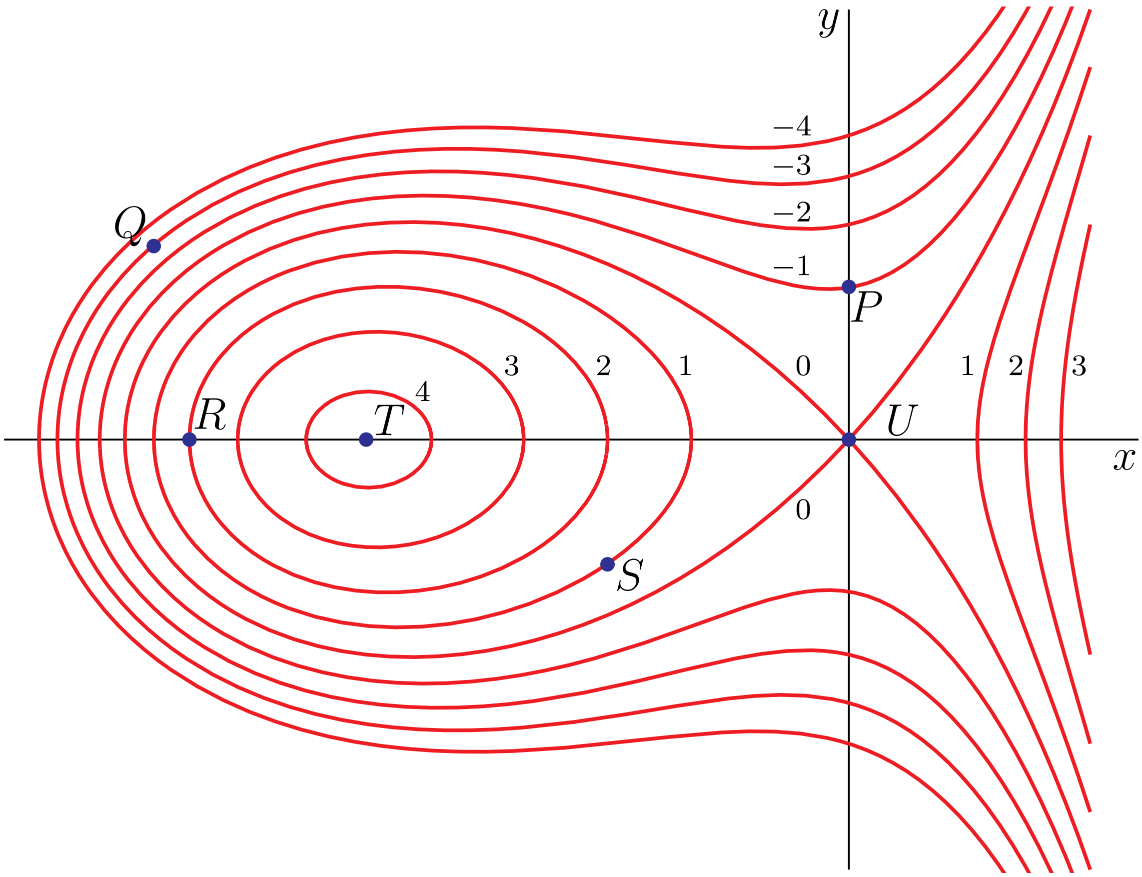
Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](∗): hint answer solution

(a) Some level curves of a function $f(x,y)$ are plotted in the xy–plane below.

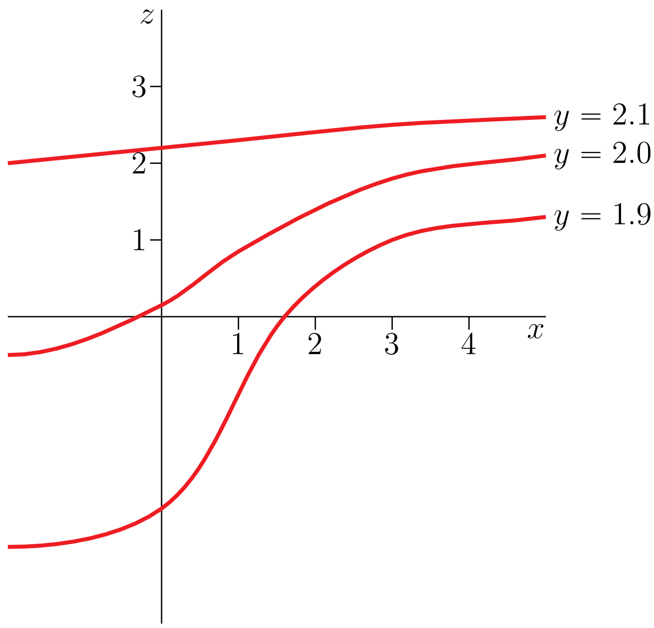


For each of the four statements below, circle the letters of **all** points in the diagram where the situation applies. For example, if the statement were "These points are on the y–axis", you would circle both P and U, but none of the other letters. You may assume that a local maximum occurs at point T.

(i) ∇f is zero	P R S T U
(ii) f has a saddle point	P R S T U
(iii) the partial derivative f_y is positive	P R S T U

(b) The diagram below shows three "y traces" of a graph $z = F(x, y)$ plotted on xz–axes. (Namely, the intersections of the surface $z = F(x, y)$ with the three planes $y = 1.9$, $y = 2$, and $y = 2.1$.) For each statement below, circle the correct word.

(i) the first order partial derivative $F_x(1, 2)$ is	positive/negative/zero (circle one)
(ii) F has a critical point at $(2, 2)$	true/false (circle one)
(iii) the second order partial derivative $F_{xy}(1, 2)$ is	positive/negative/zero (circle one)



▸▸ Stage 2

Q[2](*): hint answer solution

Let $z = f(x,y) = (y^2 - x^2)^2$.

(a) Make a reasonably accurate sketch of the level curves in the xy–plane of $z = f(x,y)$ for $z = 0, 1$ and 16. Be sure to show the scales on the coordinate axes.

(b) Verify that $(0,0)$ is a critical point for $z = f(x,y)$, and determine from part (a) or directly from the formula for $f(x,y)$ whether $(0,0)$ is a local minimum, a local maximum or a saddle point.

(c) Can you use the Second Derivative Test to determine whether the critical point $(0,0)$ is a local minimum, a local maximum or a saddle point? Give reasons for your answer.

Q[3](*): hint answer solution

Use the Second Derivative Test to find all values of the constant c for which the function $z = x^2 + cxy + y^2$ has a saddle point at $(0,0)$.

Q[4](*): hint answer solution

Find and classify all critical points of the function

$$f(x,y) = x^3 - y^3 - 2xy + 6.$$

Q[5](*): hint answer solution

Find all critical points for $f(x,y) = x(x^2 + xy + y^2 - 9)$. Also find out which of these points give local maximum values for $f(x,y)$, which give local minimum values, and which give saddle points.

Q[6]: hint answer solution

Find and classify all the critical points of $f(x,y) = x^2 + y^2 + x^2y + 4$.

Q[7](*): hint answer solution

Find all saddle points, local minima and local maxima of the function

$$f(x,y) = x^3 + x^2 - 2xy + y^2 - x.$$

Q[8](*): hint answer solution

For the surface

$$z = f(x,y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$$

Find and classify [as local maxima, local minima, or saddle points] all critical points of $f(x,y)$.

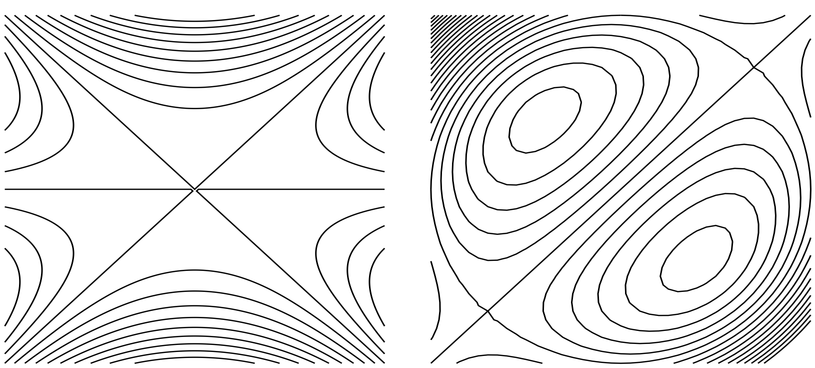
Q[9](*): hint answer solution

(a) For the function $z = f(x,y) = x^3 + 3xy + 3y^2 - 6x - 3y - 6$. Find and classify as [local maxima, local minima, or saddle points] all critical points of $f(x,y)$.

(b) The images below depict level sets $f(x,y) = c$ of the functions in the list at heights $c = 0, 0.1, 0.2, \ldots, 1.9, 2$. Label the pictures with the corresponding function and mark the critical points in each picture. (Note that in some cases, the critical points might not be drawn on the images already. In those cases you should add them to the picture.)

(i) $f(x,y) = (x^2 + y^2 - 1)(x - y) + 1$

(ii) $f(x,y) = y(x + y)(x - y) + 1$



Q[10](*): hint answer solution

Define the function

$$f(x,y) = x^3 + 3xy + 3y^2 - 6x - 3y - 6$$

Classify all critical points of $f(x,y)$ as local maxima, local minima, or saddle points.

Q[11](*): hint answer solution

Find and classify the critical points of $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 4$.

Q[12](*): hint answer solution
Find all critical points of the function $f(x,y) = x^4 + y^4 - 4xy + 2$, and for each determine whether it is a local minimum, maximum or saddle point.

Q[13](*): hint answer solution
Find all the critical points of the function

$$f(x,y) = x^4 + y^4 - 4xy$$

defined in the xy-plane. Classify each critical point as a local minimum, maximum or saddle point.

Q[14](*): hint answer solution
Find all the critical points of the function

$$f(x,y) = x^3 + xy^2 - x$$

defined in the xy-plane. Classify each critical point as a local minimum, maximum or saddle point. Explain your reasoning.

Q[15](*): hint answer solution
Find and classify all critical points of

$$f(x,y) = x^3 - 3xy^2 - 3x^2 - 3y^2$$

▸▸ Stage 3

Q[16](*): hint answer solution
Consider the function

$$f(x,y) = 3kx^2y + y^3 - 3x^2 - 3y^2 + 4$$

where $k > 0$ is a constant. Find and classify all critical points of $f(x,y)$ as local minima, local maxima, saddle points or points of indeterminate type. Carefully distinguish the cases $k < \frac{1}{2}$, $k = \frac{1}{2}$ and $k > \frac{1}{2}$.

Q[17]: hint answer solution
An experiment yields data points (x_i, y_i), $i = 1, 2, \cdots, n$. We wish to find the straight line $y = mx + b$ which "best" fits the data. The definition of "best" is "minimizes the root mean square error", i.e. minimizes $\sum_{i=1}^{n}(mx_i + b - y_i)^2$. Find m and b.

2.4▴ Absolute Minima and Maxima

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution
Suppose you want to find the maximum value of a surface $z = f(x,y)$ on the boundary of the unit circle, $x^2 + y^2 = 1$.

True or false: you should always check the points $(0, \pm 1)$ and $(\pm 1, 0)$, since these are the endpoints of the circle.

Q[2]: hint answer solution

Find the high and low points of the surface $z = \sqrt{x^2 + y^2}$ with (x, y) varying over the square $|x| \leqslant 1, |y| \leqslant 1$. Discuss the values of z_x, z_y there. Do not evaluate any derivatives in answering this question.

▸▸ Stage 2

Q[3]: hint answer solution

Find the maximum and minimum values of $f(x, y) = xy - x^3y^2$ when (x, y) runs over the square $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$.

Q[4](*): hint answer solution

Let $h(x, y) = y(4 - x^2 - y^2)$.

(a) Find and classify the critical points of $h(x, y)$ as local maxima, local minima or saddle points.

(b) Find the maximum and minimum values of $h(x, y)$ on the disk $x^2 + y^2 \leqslant 1$.

Q[5](*): hint answer solution

Find the absolute maximum and minimum values of the function
$f(x, y) = 5 + 2x - x^2 - 4y^2$ on the rectangular region

$$R = \{ (x, y) \mid -1 \leqslant x \leqslant 3, \ -1 \leqslant y \leqslant 1 \}$$

Q[6](*): hint answer solution

Find the minimum of the function $h(x, y) = -4x - 2y + 6$ on the closed bounded domain defined by $x^2 + y^2 \leqslant 1$.

Q[7](*): hint answer solution

Let $f(x, y) = xy(x + y - 3)$.

(a) Find all critical points of f, and classify each one as a local maximum, a local minimum, or saddle point.

(b) Find the location and value of the absolute maximum and minimum of f on the triangular region $x \geqslant 0, y \geqslant 0, x + y \leqslant 8$.

Q[8](*): hint answer solution

Consider the function

$$f(x, y) = 2x^3 - 6xy + y^2 + 4y$$

(a) Find and classify all of the critical points of $f(x, y)$.

(b) Find the maximum and minimum values of $f(x, y)$ in the triangle with vertices $(1, 0)$, $(0, 1)$ and $(1, 1)$.

Q[9](*): hint answer solution
Let

$$f(x,y) = xy(x+2y-6)$$

(a) Find every critical point of $f(x,y)$ and classify each one.
(b) Let D be the region in the plane between the hyperbola $xy = 4$ and the line $x + 2y - 6 = 0$. Find the maximum and minimum values of $f(x,y)$ on D.

Q[10](*): hint answer solution
A metal plate is in the form of a semi-circular disc bounded by the x-axis and the upper half of $x^2 + y^2 = 4$. The temperature at the point (x,y) is given by

$$T(x,y) = \ln\left(1 + x^2 + y^2\right) - y.$$

Find the coldest point on the plate, explaining your steps carefully. (Note: $\ln 2 \approx 0.693$, $\ln 5 \approx 1.609$)

Q[11](*): hint answer solution
Consider the function $g(x,y) = x^2 - 10y - y^2$.

(a) Find and classify all critical points of g.

(b) Find the absolute extrema of g on the bounded region given by

$$x^2 + 4y^2 \leqslant 16, \; y \leqslant 0$$

Q[12]: hint answer solution
Equal angle bends are made at equal distances from the two ends of a 100 metre long fence, so that the resulting three segment fence can be placed along an existing wall to make an enclosure of trapezoidal shape. What is the largest possible area for such an enclosure?

Q[13]: hint answer solution
Find the most economical shape of a rectangular box that has a fixed volume V and that has no top.

Q[14](*): hint answer solution
The temperature $T(x,y)$ at a point of the xy–plane is given by

$$T(x,y) = 20 - 4x^2 - y^2$$

(a) Find the maximum and minimum values of $T(x,y)$ on the disk D defined by $x^2 + y^2 \leqslant 4$.

(b) Suppose the ant is constrained to stay on the curve $y = 2 - x^2$. Where should the ant go if it wants to be as warm as possible?

▸▸ Stage 3

Q[15](*): hint answer solution
Find the largest and smallest values of x^2y^2z in the part of the plane $2x + y + z = 5$ where $x \geqslant 0$, $y \geqslant 0$ and $z \geqslant 0$. Also find all points where those extreme values occur.

Q[16](*): hint answer solution

(a) Show that the function $f(x, y) = 2x + 4y + \frac{1}{xy}$ has exactly one critical point in the first quadrant $x > 0$, $y > 0$, and find its value at that point.

(b) Use the second derivative test to classify the critical point in part (a).

(c) Explain why the inequality $2x + 4y + \frac{1}{xy} \geqslant 6$ is valid for all positive real numbers x and y.

Q[17]: hint answer solution
Let a be a constant real number. Find all points on the surface

$$z = f(x, y) = x^2 + y^2$$

that have minimum distance from the point $(0, 0, a)$.

Q[18]: hint answer solution
The Scranton branch of a well-known paper company has two sizes of paper for sale - A4 and A3.

Each ream of A4 is sold at \$6; each ream of A3 is sold at \$8. Assume that every ream produced is sold.

Suppose x is the quantity of materials that go into making A4 and y is the quantity of materials that go into making A3. Then the costs involved in turning these materials into paper are $\$1 \cdot x$ for A4 and $\$3 \cdot y$ for A3.

There are different production procedures to produce each paper size. The *production functions* below give the number of reams of paper produced out of a given amount of materials.

$$f(x) = \frac{5}{2}x^{0.8} \qquad \text{(for A4)}$$
$$g(y) = 10y^{0.6} \qquad \text{(for A3)}$$

(a) Build the (total) profit equation in terms of x and y. That is, find an equation $\Pi(x, y)$ that gives the total profit (revenue minus cost) over both paper types.

(b) Find the production quantities of both sizes of paper that maximizes profit.

(c) If the branch stops producing A4, what is the optimal production for A3 to maximize profit?

Q[19]: hint answer solution

Ayan and Pipe each have a lemonade boutique. Making each pitcher of lemonade costs \$1. If Ayan wants to sell q_A lemonades, and Pipe want to sell q_P lemonades, then each pitcher of lemonade will be sold for this price:

$$p(q_A, q_P) = 121 - 2(q_A + q_P)$$

(a) Build the profit equation in terms of q_A and q_P for Ayan. Treating q_P as a constant, find the value of q_A that maximizes Ayan's profit. (Your answer will depend on q_P.)

(b) Build the profit equation in terms of q_A and q_P for Pipe. Treating q_A as a constant, find the value of q_P that maximizes Pipe's profit. (Your answer will depend on q_A.)

(c) Guess, using your intuition, how many pitchers are Ayan and Pipe are going to produce proportional to one another so that both of them maximize their respective profit functions.

(d) Verify your answer for (c) mathematically.

(e) Calculate the profit that each seller generates under these assumptions.

(f) What would be their joint profit if they collaborate? Build a new profit equation where Ayan and Pipe are collaborating and find the optimal joint profit. Compare this to their individual profit when they are competing and decide whether it would be better for them to collaborate or compete.

(g) Is it better for thirsty consumers when the two sellers collaborate, or when they compete?

2.5▲ Lagrange Multipliers

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): hint answer solution

(a) Does the function $f(x, y) = x^2 + y^2$ have a maximum or a minimum on the curve $xy = 1$? Explain.

(b) Find all maxima and minima of $f(x, y)$ on the curve $xy = 1$.

Q[2]: hint answer solution

Give an example of a continuous surface $f(x, y)$ and a constraint function $g(x, y) = 0$ such that $f(x, y)$ has both a local max and a local min subject to the constraint, but no global max or min.

▸▸ Stage 2

Q[3](*): hint answer solution

Use the method of Lagrange multipliers to find the minimum value of $z = x^2 + y^2$ subject to $x^2y = 1$. At which point or points does the minimum occur?

Q[4](∗): hint answer solution
Use the method of Lagrange multipliers to find the maximum and minimum values of

$$f(x,y) = xy$$

subject to the constraint

$$x^2 + 2y^2 = 1.$$

Q[5](∗): hint answer solution
Find the maximum and minimum values of $f(x,y) = x^2 + y^2$ subject to the constraint $x^4 + y^4 = 1$.

Q[6]: hint answer solution
Find the absolute extrema of the function $f(x,y) = x^4 + y^4 + \frac{2}{3}y^6$ given the constraint $g(x,y) = x^2 + y^2 = 1$ using the method of Lagrange multipliers.

Q[7]: hint answer solution
Find the point(s) on the parabola $y = \frac{3}{2} - x^2$ closest to the origin using the method of Lagrange multipliers.

Q[8]: hint answer solution
What are the largest and smallest values of the product xy, for points (x,y) in the region

$$x^2 - 2xy + 5y^2 \leqslant 1 \text{ ?}$$

Q[9](∗): hint answer solution
The temperature in the plane is given by $T(x,y) = e^y\left(x^2 + y^2\right)$.

(a) (i) Give the system of equations that must be solved in order to find the warmest and coolest point on the circle $x^2 + y^2 = 100$ by the method of Lagrange multipliers.
(ii) Find the warmest and coolest points on the circle by solving that system.

(b) (i) Give the system of equations that must be solved in order to find the critical points of $T(x,y)$.
(ii) Find the critical points by solving that system.

(c) Find the coolest point on the solid disc $x^2 + y^2 \leqslant 100$.

Q[10]: hint answer solution

Use the method of Lagrange Multipliers to find the maximum and minimum values of the utility function $U = f(x,y) = 9x^{\frac{1}{3}}y^{\frac{2}{3}}$, subject to the constraint $g(x,y) = 3200x + 200y = 80,000$, where $x \geqslant 0$ and $y \geqslant 0$.

▸▸ Stage 3

Q[11](*): hint answer solution

Suppose that a and b are both greater than zero and let T be the triangle bounded by the line $ax + by = 1$ and the two axes. Use the method of Lagrange multipliers to find the smallest possible area of T if the line $ax + by = 1$ is required to pass through the point $(1,2)$.

Q[12]: hint answer solution

Find a and b so that the area πab of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passing through the point $(1,2)$ is as small as possible.

(We assume a, b are positive.)

Q[13](*): hint answer solution

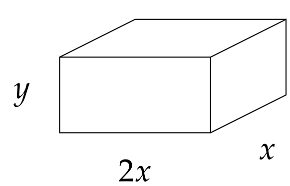
Use the method of Lagrange multipliers to find the radius of the base and the height of a right circular cylinder of maximum volume which can be fit inside the unit sphere $x^2 + y^2 + z^2 = 1$.

Q[14]: hint answer solution

A rectangular box needs the following properties:

- 72 cubic centimetre volume,
- width twice its length, and
- minimum surface area.

What are the dimensions of the box?



Use Lagrange multipliers to solve.

Q[15]: hint answer solution

Let $f(x,y)$ have continuous partial derivatives. Consider the problem of finding local minima and maxima of $f(x,y)$ on the curve $xy = 1$.

- Define $g(x,y) = xy - 1$. According to the method of Lagrange multipliers, if (x,y) is a local minimum or maximum of $f(x,y)$ on the curve $xy = 1$, then there is a real number λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = 0 \tag{E1}$$

- On the curve $xy = 1$, we have $y = \frac{1}{x}$ and $f(x,y) = f\left(x, \frac{1}{x}\right)$. Define $F(x) = f\left(x, \frac{1}{x}\right)$. If $x \neq 0$ is a local minimum or maximum of $F(x)$, we have that

$$F'(x) = 0 \tag{E2}$$

Show that (E1) is equivalent to (E2), in the sense that

there is a λ such that (x, y, λ) obeys (E1)
if and only if
$x \neq 0$ obeys (E2) and $y = 1/x$.

2.6▲ (Optional) Utility and Demand Functions

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Consumer A has a \$100 gift card to a shoe store. They bring no other cash or cards, determined to spend no money other than the gift card. In the store, they decide which combination of items will be the most satisfying, without going over their budget.

Consumer B needs to buy a new pair of running shoes. They need to be of a good-enough quality, so the consumer considers their arch support, heel padding, durability, and visual appeal. They choose the pair of shoes meeting their quality standards that costs as little as possible.

Which consumer is embodying Marshallian demand, and which is embodying Hicksian demand?

Q[2]: hint answer solution

Products X and Y have prices p_x and p_y; and the Marshallian demand functions for product X is

$$x^m(p_x, p_y, I) = \begin{cases} \frac{I}{2(p_x - p_y)} & \text{if } p_x \geqslant 2p_y \\ \\ \frac{I}{p_x} & \text{if } p_x < p_y \end{cases}$$

where I is the budget constraint.

Find the price effects of p_x and p_y on X.

Q[3]: hint answer solution
Products X and Y have prices p_x and p_y; and subject to the budget constraint $p_x x + p_y y = I$, the Marshallian demand function of X is

$$x^m(p_x, p_y, I) = \frac{p_y^2 - Ip_x}{4p_y^2 - 2p_x^2}$$

when $p_x^2 \leqslant 2p_y^2$ (and p_x, p_y, and I are in some appropriate model domain).

Is X a normal good or an inferior good?

▸▸ Stage 2

Q[4]: hint answer solution
Luiza is a microbiology student who wants to have breakfast before her 8 am class. Great Dane, the university's prime coffee shop, sells two goods: food (f) and coffee (c). Luiza knows that to maximize her utility she must look at her preferences:

$$u(f, c) = \sqrt{f} + \frac{c}{10}$$

Luiza then checks the prices of both goods and how much money she has in her pocket. Coffee (c) is 1 dollar per unit, and food (f) is 3 dollars per unit. She has 10 dollars and she is determined to use it all. Help her find:

(a) Optimal coffee consumption.

(b) Optimal food consumption.

(c) How much money would she spent on food, and how much on coffee?

Q[5]: hint answer solution
Franco has 1,000 dollars and is looking to invest on two different Italian soccer teams: Inter de Milan (m) and La Spezia (s). His utility function (representing his preferences) is as follows:

$$u(m, s) = \ln(m) + \frac{s}{16}$$

The prices for each share are \$80 for Milan and \$20 for la Spezia.

Assuming that Franco wants to spend the entirety of his budget, how many shares of each company should he buy?

Q[6]: hint answer solution
Laura is going to Whistler for the weekend. She is buying cheese (c) and strawberries (s). Cheese is 5 dollars per unit, and strawberries are 10 dollars per unit. Laura wants to spend 100 dollars and her utility function is as follows:

$$u(c, s) = 3\ln(2c) + 4\ln(s)$$

Find Laura's optimal consumption.

Q[7]: hint answer solution

Alessio plays an online multiplayer game. In this game, each character has infinite expansion packs that enhance the abilities of the character. Alessio likes playing with two characters: Keitu (k), and Nefret (n). Alessio plans to spend 84 dollars on new expansion packs. The prices per expansion pack are $4 for each of Keitu's expansion packs and $12 for each of Nefret's expansion packs. Alessio's utility function is:

$$u(k, n) = k^{0.5} n^{0.2}$$

Find how many packages Alessio will buy for each character given that Alessio wants to maximize his utility function.

▸▸ Stage 3

Q[8]: hint answer solution

Coral is going to the cinema. She always buys popcorn (p) and soda (s). Popcorn is 4.50 dollars per 100 grams, and soda is 2 dollars per 100 millilitres. Coral has 20 dollars and her utility function is as follows:

$$u(p, s) = p^{0.4}(s + p)^{0.6}$$

(a) What is Coral's optimal consumption?

(b) Coral is offered a combo: 420 gm (grams) of popcorn and 160 ml of soda for only 20 dollars.

 (i) Could Coral afford 420 gm of popcorn and 160 ml of soda at the normal price rate?

 (ii) Is this a better deal for Coral?

Q[9]: hint answer solution

The gender distribution of police officers is likely to differ across cities and is likely to depend on the hiring officials' belief and preference (HR). HR's valuation of the two genders will determine the gender distribution of the workforce, and so would any gender bias.

Please assume that the department has a fixed budget for new hires, denoted by I which is measures in CAD per hour, and let us further assume that there is no gender bias in wages 30 CAD per hour.

The utility function depicts a certain level of utility the hiring officers derive from different combinations of male and female officers. Two hiring officers Mr. Blue and Ms. Reed, both with budget I, have the following utility functions:

$$U_B(m, f) = m^{0.9} f^{0.1} \qquad \text{(for Mr. Blue)}$$
$$U_R(m, f) = m^{0.5} f^{0.5} \qquad \text{(for Ms. Reed)}$$

(a) What is the budget constraint for these officers?

(b) Sketch the level curves for Mr. Blue's and Ms. Reed's utility functions. Looking at the shape of their indifference curves, who among the two, Ms. Reed or Mr. Blue, prefers male to female police officers? Please explain and show your answer.

(c) Given the budget equation and the utility functions, use Lagrange multipliers to find the number of male and female hires by Mr. Blue and Ms. Reed in terms of I (remember both have budget I). Which officer hires a higher proportion of male police officers?

(d) What if there was a gender gap in the wage structure? Assume a bias against female officers such that the wage of a female police officer is now less than the wage of a male police officer, given everything else is the same (education, training, ability). Let's assume the wages are 35 CAD per hour for male officers, and 30 CAD per hour for female officers and budget I for both hiring managers. Explain your results.

Q[10]: hint answer solution

Recall Luiza, the microbiology student from question 4 that wants to have breakfast before her 8 am class. Great Dane, the university's prime coffee shop, sells two goods: food (f) and coffee (c). Luiza knows that to maximize her utility she must look at her preferences:

$$u(f,c) = \sqrt{f} + \frac{c}{10}$$

Luiza's budget and the price of both goods are constantly changing. Help her find:

(a) The optimal consumption of coffee for any budget or prices of the products (food and coffee).

(b) The optimal consumption of food for any budget or prices of the products.

Q[11]: hint answer solution

Recall Alessio from question 7, who plays an online multiplayer game. In this game, each character has infinite expansion packs that enhance the abilities of the character. Alessio likes playing with two characters: Keitu (k) and Nefret (n). Every month, Alessio likes to add new packages to his characters. However, every month he has a different budget and the packages vary in price.

Alessio's utility function is:

$$u(k,j,n) = k^{0.5}n^{0.2}$$

Since the package prices are variable, let p_k be the package price for Keitu, and let p_n be the package price for Nefret. You can assume in this scenario that buying fractions of packages is fine. Let I be the amount of money Alessio has to spend.

Find how many packages Alessio will buy for each of the three characters in terms of p_k, p_n, and I.

Q[12]: hint answer solution

A normal good is defined in economics as a product such that the demand increases as income increases. An inferior good is a product such that the demand decreases as income increases. You, being a student, have a limited budget. You can buy kraft dinner or chicken which we denoted their quantities by k and c respectively. You have an income of I dollars to spend on these two goods. Kraft dinner is p_k dollars per unit and chicken is p_c dollars per unit. This is your utility function:

$$u(k,c) = \ln(k-1) - 2\ln(50-c)$$

(a) What is the minimum and maximum that you can consume of each product?

(b) Find the Marshallian demand for each product. Remember that to find Marshallian demand, you must use the utility function as the objective function, and your budget as the constraint.

(c) Categorize the kraft dinner and chicken as normal or inferior goods. Note that, a normal good is mathematically defined as a good such that the Marshallian demand increases when income increases. An inferior good is a good such that the Marshallian demand decreases as income increases. Write your answer as an expression in terms of p_k and p_c. Explain this mathematically.

Q[13]: hint answer solution

Liam loves boxing. His favorite professional boxers are Lomachenko (l) and Anthony Joshua (a). He has D dollars to buy tickets for the upcoming season. Tickets for Lomachenko's matches are p_l dollars and for Anthony Joshua's are p_a dollars. Liam's utility function is as follows:

$$u(l,a) = (4l^{0.5} + 3a^{0.5})^{0.5}$$

(a) Find Liam's Marshallian demand. Remember that to find Marshallian demand, you must use the utility function as the objective function, and the Liam's budget as the constraint.

(b) Categorize Lomanchenko's and Anthony Joshua's tickets as normal or inferior goods. Note that, a normal good is mathematically defined as a good such that the Marshallian demand increases when income (here, D dollars) increases. An inferior good is a good such that the Marshallian demand decreases as income increases

(c) What happens to the Marshallian demand for Lomanchenko's tickets when p_l decreases? Right down your answer in terms of l, a, and D. Specify if the demand would increase, decrease, or not change.

(d) Find Liam's Hicksian demand. Remember that to find Hicksian demand, you must use Liam's budget as the objective function, and the utility function as the constraint by fixing $U = (4l^{0.5} + 3a^{0.5})^{0.5}$.

(e) What is the substitution effect for Anthony Joshua's demand when p_l changes? Remember, the substitution effect is mathematically defined as the change in Hicksian demand due to a change in price. In this case, how does Anthony's Hicksian demand change due to a change in p_a?

Chapter 3

INTEGRATION

3.1 Definition of the Integral

Exercises

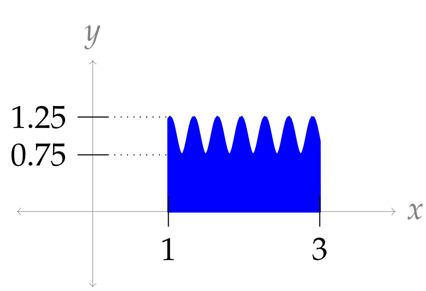
Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Stage 1

For Questions 1 through 5, we want you to develop an understanding of the model we are using to define an integral: we approximate the area under a curve by bounding it between rectangles. Later, we will learn more sophisticated methods of integration, but they are all based on this simple concept.

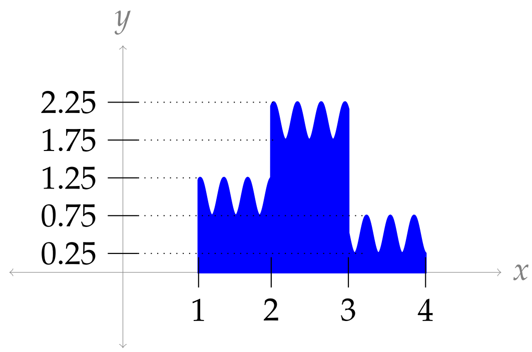
Q[1]: hint answer solution

Give a range of possible values for the shaded area in the picture below.



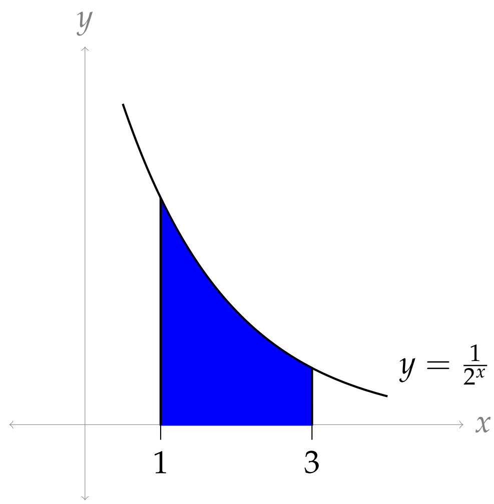
Q[2]: hint answer solution

Give a range of possible values for the shaded area in the picture below.



Q[3]: hint answer solution

Using rectangles, find a lower and upper bound for $\int_1^3 \frac{1}{2^x}\mathrm{d}x$ that differ by at most 0.2 square units.



Q[4]: hint answer solution

Let $f(x)$ be a function that is *decreasing* from $x = 0$ to $x = 5$. Which Riemann sum approximation of $\int_0^5 f(x)\mathrm{d}x$ is the largest–left, right, or midpoint?

Q[5]: hint answer solution

Give an example of a function $f(x)$, an interval $[a, b]$, and a number n such that the midpoint Riemann sum of $f(x)$ over $[a, b]$ using n intervals is *larger than* both the left and right Riemann sums of $f(x)$ over $[a, b]$ using n intervals.

In Questions 6 through 10, we practice using sigma notation. There are many ways to write a given sum in sigma notation. You can practice finding several, and deciding which looks the clearest.

Q[6]: hint answer solution

Express the following sums in sigma notation:

(a) $3 + 4 + 5 + 6 + 7$

(b) $6 + 8 + 10 + 12 + 14$

(c) $7 + 9 + 11 + 13 + 15$

(d) $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$

Q[7]: hint answer solution

Express the following sums in sigma notation:

(a) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$

(b) $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}$

(c) $-\frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81}$

(d) $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{81}$

Q[8]: hint answer solution

Express the following sums in sigma notation:

(a) $\frac{1}{3} + \frac{1}{3} + \frac{5}{27} + \frac{7}{81} + \frac{9}{243}$

(b) $\frac{1}{5} + \frac{1}{11} + \frac{1}{29} + \frac{1}{83} + \frac{1}{245}$

(c) $1000 + 200 + 30 + 4 + \frac{1}{2} + \frac{3}{50} + \frac{7}{1000}$

Q[9]: hint answer solution

Evaluate the following sums. You might want to use the formulas from Theorems 3.1.5 and 3.1.6 in the text.

(a) $\displaystyle\sum_{i=0}^{100} \left(\frac{3}{5}\right)^i$

(b) $\displaystyle\sum_{i=50}^{100} \left(\frac{3}{5}\right)^i$

(c) $\displaystyle\sum_{i=1}^{10} \left(i^2 - 3i + 5\right)$

(d) $\displaystyle\sum_{n=1}^{b} \left[\left(\frac{1}{e}\right)^n + en^3\right]$, where b is some integer greater than 1.

Q[10]: hint answer solution

Evaluate the following sums. You might want to use the formulas from Theorem 3.1.6 in the text.

(a) $\displaystyle\sum_{i=50}^{100} (i - 50) + \sum_{i=0}^{50} i$

(b) $\displaystyle\sum_{i=10}^{100} (i - 5)^3$

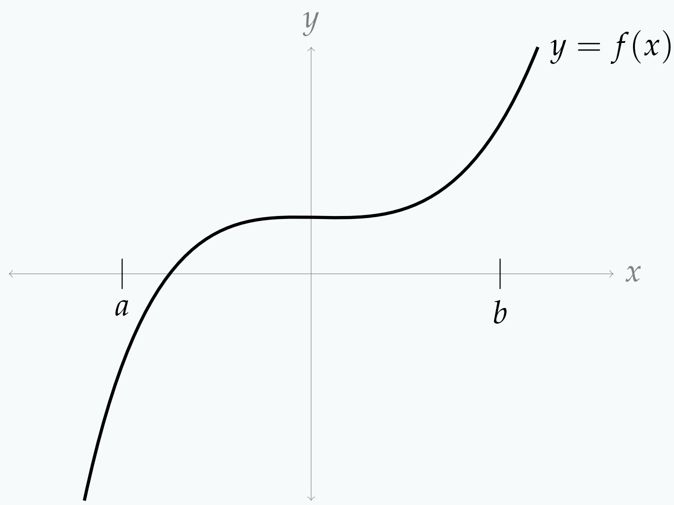
(c) $\displaystyle\sum_{n=1}^{11} (-1)^n$

(d) $\sum_{n=2}^{11} (-1)^{2n+1}$

Questions 11 through 15 are meant to give you practice interpreting the formulas in Definition 3.1.10 of the text. The formulas might look complicated at first, but if you understand what each piece means, they are easy to learn.

Q[11]: hint answer solution

In the picture below, draw in the rectangles whose (signed) area is being computed by the midpoint Riemann sum $\sum_{i=1}^{4} \frac{b-a}{4} \cdot f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{4}\right)$.



Q[12](∗): hint answer solution

$\sum_{k=1}^{4} f(1+k) \cdot 1$ is a left Riemann sum for a function $f(x)$ on the interval $[a, b]$ with n subintervals. Find the values of a, b and n.

Q[13]: hint answer solution

Draw a picture illustrating the area given by the following Riemann sum.

$$\sum_{i=1}^{3} 2 \cdot (5+2i)^2$$

Q[14]: hint answer solution

Draw a picture illustrating the area given by the following Riemann sum.

$$\sum_{i=1}^{5} \frac{\pi}{20} \cdot \tan\left(\frac{\pi(i-1)}{20}\right)$$

Q[15](∗): hint answer solution

Fill in the blanks with right, left, or midpoint; an interval; and a value of n.

$\sum_{k=0}^{3} f(1.5+k)\cdot 1$ is a ______ Riemann sum for f on the interval [___ , ___] with $n =$ ___.

Q[16]: hint answer solution

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_0^5 x\, \mathrm{d}x$$

Q[17]: hint answer solution

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_{-2}^5 x\, \mathrm{d}x$$

▸▸ Stage 2

Q[18](∗): hint answer solution

Use sigma notation to write the midpoint Riemann sum for $f(x) = x^8$ on $[5, 15]$ with $n = 50$. Do not evaluate the Riemann sum.

Q[19](∗): hint answer solution

Estimate $\int_{-1}^5 x^3\, \mathrm{d}x$ using three approximating rectangles and left hand end points.

Q[20](∗): hint answer solution

Let f be a function on the whole real line. Express $\int_{-1}^7 f(x)\, \mathrm{d}x$ as a limit of Riemann sums, using the right endpoints.

Q[21](∗): hint answer solution

The value of the following limit is equal to the area below a graph of $y = f(x)$, integrated over the interval $[0, b]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left[\sin\left(2+\frac{4i}{n}\right)\right]^2$$

Find $f(x)$ and b.

Q[22](*): hint answer solution

For a certain function $f(x)$, the following equation holds:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\sqrt{1-\frac{k^2}{n^2}}=\int_0^1 f(x)\ \mathrm{d}x$$

Find $f(x)$.

Q[23](*): hint answer solution

Express $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}e^{-i/n}\cos\left(\frac{3i}{n}\right)$ as a definite integral.

Q[24](*): hint answer solution

Let $R_n=\displaystyle\sum_{i=1}^{n}\frac{ie^{i/n}}{n^2}$. Express $\displaystyle\lim_{n\to\infty}R_n$ as a definite integral. *Do not evaluate this integral.*

Q[25](*): hint answer solution

Express $\displaystyle\lim_{n\to\infty}\left(\sum_{i=1}^{n}e^{-1-2i/n}\cdot\frac{2}{n}\right)$ as an integral in three different ways.

Questions 26 and 27 use the formula for a geometric sum, Equation 3.1.3 in the text.

Q[26]: hint answer solution

Evaluate the sum $1+r^3+r^6+r^9+\cdots+r^{3n}$.

Q[27]: hint answer solution

Evaluate the sum $r^5+r^6+r^7+\cdots+r^{100}$.

Remember that a definite integral is a signed area between a curve and the x-axis. We'll spend a lot of time learning strategies for evaluating definite integrals, but we already know lots of ways to find area of geometric shapes. In Questions 28 through 33, use your knowledge of geometry to find the signed areas described by the integrals given.

Q[28](*): hint answer solution

Evaluate $\displaystyle\int_{-1}^{2}|2x|\ \mathrm{d}x$.

Q[29]: hint answer solution

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_{-3}^{5}|t-1|\,\mathrm{d}t$$

Q[30]: hint answer solution

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_a^b x\,dx$$

where $0 \leqslant a \leqslant b$.

Q[31]: hint answer solution

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_a^b x\,dx$$

where $a \leqslant b \leqslant 0$.

Q[32]: hint answer solution

Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_0^4 \sqrt{16 - x^2}\,dx$$

Q[33](*): hint answer solution

Use elementary geometry to calculate $\int_0^3 f(x)\,dx$, where

$$f(x) = \begin{cases} x, & \text{if } x \leqslant 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Q[34](*): hint answer solution

A car's gas pedal is applied at $t = 0$ seconds and the car accelerates continuously until $t = 2$ seconds. The car's speed at half-second intervals is given in the table below. Find the best possible **upper** estimate for the distance that the car traveled during these two seconds.

t (s)	0	0.5	1.0	1.5	2
v (m/s)	0	14	22	30	40

Q[35]: hint answer solution

True or false: the answer you gave for Question 34 is definitely greater than or equal to the distance the car travelled during the two seconds in question.

Q[36]: hint answer solution

An airplane's speed at one-hour intervals is given in the table below. Approximate the distance travelled by the airplane from noon to 4pm three ways using a midpoint Riemann sum.

time	12:00 pm	1:00 pm	2:00 pm	3:00 pm	4:00 pm
speed (km/hr)	800	700	850	900	750

▸▸ Stage 3

Q[37](*): hint answer solution
(a) Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{4-\left(-2+\frac{2i}{n}\right)^2}$$

as a definite integal.
(b) Evaluate the integral of part (a).

Q[38](*): hint answer solution
Consider the integral:

$$\int_0^3 \left(7+x^3\right)\,dx. \qquad (*)$$

(a) Approximate this integral using the left Riemann sum with $n=3$ intervals.
(b) Write down the expression for the right Riemann sum with n intervals and calculate the sum. Now take the limit $n\to\infty$ in your expression for the Riemann sum, to evaluate the integral (*) exactly.

You may use the identity

$$\sum_{i=1}^{n} i^3 = \frac{n^4+2n^3+n^2}{4}$$

Q[39](*): hint answer solution
Using a limit of right–endpoint Riemann sums, evaluate $\displaystyle\int_2^4 x^2\,dx$.

You may use the formulas $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Q[40](*): hint answer solution
Find $\displaystyle\int_0^2 (x^3+x)\,dx$ using the definition of the definite integral. You may use the summation formulas $\sum_{i=1}^{n} i^3 = \frac{n^4+2n^3+n^2}{4}$ and $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$.

Q[41](*): hint answer solution

Using a limit of right–endpoint Riemann sums, evaluate $\displaystyle\int_1^4 (2x-1)\,\mathrm{d}x$. Do not use anti-differentiation, except to check your answer.* You may use the formula $\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$.

* You'll learn about this method starting in Section 3.3 of the text. You can also check this answer using geometry.

Q[42]: hint answer solution

Give a function $f(x)$ that has the following expression as a right Riemann sum when $n = 10$, $\Delta(x) = 10$ and $a = -5$:

$$\sum_{i=1}^{10} 3(7+2i)^2 \sin(4i)\,.$$

Q[43]: hint answer solution

Using the method of Example 3.1.2 in the text, evaluate

$$\int_0^1 2^x\,\mathrm{d}x$$

Q[44]: hint answer solution

(a) Using the method of Example 3.1.2 in the text, evaluate

$$\int_a^b 10^x\,\mathrm{d}x$$

(b)

Using your answer from above, make a guess for

$$\int_a^b c^x\,\mathrm{d}x$$

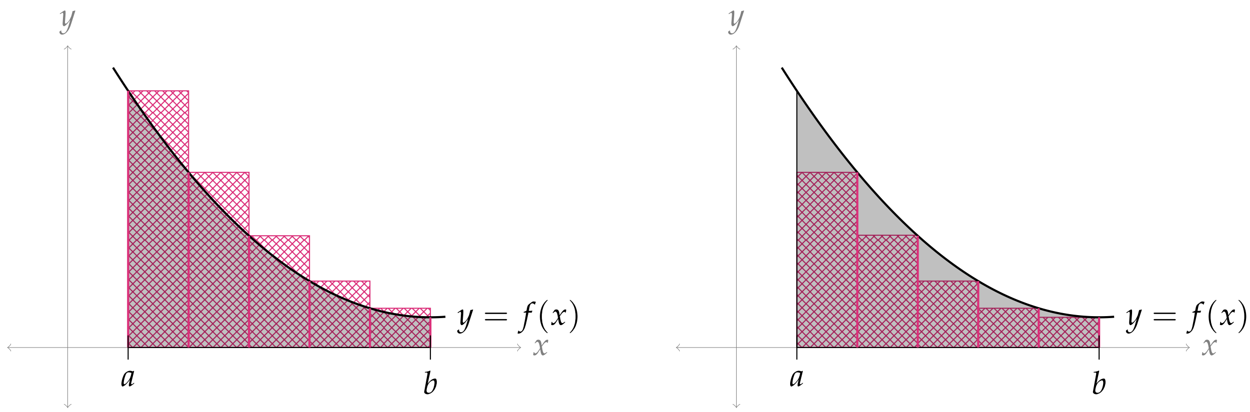
where c is a positive constant. Does this agree with Question 43?

Q[45]: hint answer solution

Evaluate $\displaystyle\int_0^a \sqrt{1-x^2}\,\mathrm{d}x$ using geometry, if $0 \leqslant a \leqslant 1$.

Q[46]: hint answer solution

Suppose $f(x)$ is a positive, decreasing function from $x = a$ to $x = b$. You give an upper and lower bound on the area under the curve $y = f(x)$ using n rectangles and a left and right Riemann sum, respectively, as in the picture below.



(a) What is the difference between the lower bound and the upper bound? (That is, if we subtract the smaller estimate from the larger estimate, what do we get?) Give your answer in terms of f, a, b, and n.

(b) If you want to approximate the area under the curve to within 0.01 square units using this method, how many rectangles should you use? That is, what should n be?

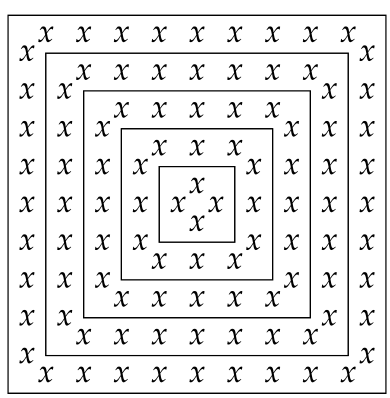
Q[47]: hint answer solution

Let $f(x)$ be a linear function, let $a < b$ be integers, and let n be a whole number. True or false: if we average the left and right Riemann sums for $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ using n rectangles, we get the same value as the midpoint Riemann sum using n rectangles.

Q[48]: hint answer solution

A square blanket is crocheted, starting from the centre and working out, with stitches placed as perimeters of ever-increasing squares (called "rounds"). The first round has one stitch on each side; the second round has three stitches on each side; the third round has five stitches on each side; and so on, with side lengths increasing by 2 every time.

In the diagram below, each x represents one stitch.



A blanket is made in this manner, with the outside having a side length of 299 stitches.

(a) How many rounds are in the blanket?

(b) How many stitches are in the blanket?

(c) During which round does the crocheter make reach the halfway point? That is, during which round are there an equal number of stitches that have and have not been made?

3.2▲ Basic properties of the definite integral

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

For each of the following properties of definite integrals, draw a picture illustrating the concept, interpreting definite integrals as areas under a curve.

For simplicity, you may assume that $a \leqslant c \leqslant b$, and that $f(x), g(x)$ give positive values.

(a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$ (Theorem 3.2.3.a in the text)

(b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$ (Theorem 3.2.3.c in the text)

(c) $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$ (Theorem 3.2.1.a in the text)

Q[2]: hint answer solution

If $\displaystyle\int_0^b \cos x\,\mathrm{d}x = \sin b$, then what is $\displaystyle\int_a^b \cos x\,\mathrm{d}x$?

Q[3](*): hint answer solution

Decide whether each of the following statements is true or false. If false, provide a counterexample. If true, provide a brief justification. (Assume that $f(x)$ and $g(x)$ are continuous functions.)

(a) $\displaystyle\int_{-3}^{-2} f(x)\mathrm{d}x = -\int_3^2 f(x)\mathrm{d}x.$

(b) If $f(x)$ is an odd function, then $\displaystyle\int_{-3}^{-2} f(x)\,\mathrm{d}x = \int_2^3 f(x)\,\mathrm{d}x.$

(c) $\displaystyle\int_0^1 f(x)\cdot g(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^1 g(x)\,\mathrm{d}x.$

Q[4]: hint answer solution

Suppose we want to make a right Riemann sum with 100 intervals to approximate $\int_5^0 f(x)\,\mathrm{d}x$, where $f(x)$ is a function that gives only positive values.

(a) What is Δx?

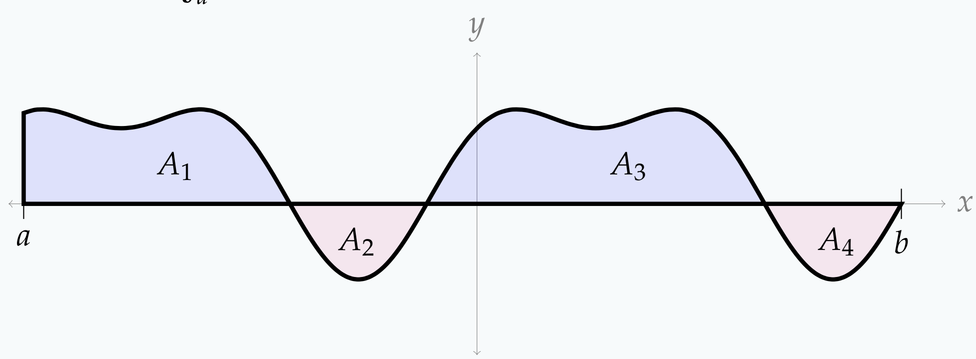
(b) Are the heights of our rectangles positive or negative?

(c) Is our Riemann sum positive or negative?

(d) Is the signed area under the curve $y = f(x)$ from $x = 0$ to $x = 5$ positive or negative?

Q[5]: hint answer solution

The function $y = g(x)$ is sketched below, with the areas of different regions given. Use the sketch to evaluate $\displaystyle\int_a^b g(x)\mathrm{d}x$.



▸▸ Stage 2

Q[6](*): hint answer solution

Suppose $\displaystyle\int_2^3 f(x)\,\mathrm{d}x = -1$ and $\displaystyle\int_2^3 g(x)\,\mathrm{d}x = 5$. Evaluate $\displaystyle\int_2^3 \big(6f(x) - 3g(x)\big)\,\mathrm{d}x$.

Q[7](*): hint answer solution

If $\displaystyle\int_0^2 f(x)\,\mathrm{d}x = 3$ and $\displaystyle\int_0^2 g(x)\,\mathrm{d}x = -4$, calculate $\displaystyle\int_0^2 \big(2f(x) + 3g(x)\big)\,\mathrm{d}x$.

Q[8](*): hint answer solution

The functions $f(x)$ and $g(x)$ obey

$$\int_0^{-1} f(x)\,\mathrm{d}x = 1 \qquad \int_0^2 f(x)\,\mathrm{d}x = 2 \qquad \int_{-1}^0 g(x)\,\mathrm{d}x = 3 \qquad \int_0^2 g(x)\,\mathrm{d}x = 4$$

Find $\int_{-1}^2 \big[3g(x) - f(x)\big]\,\mathrm{d}x$.

Q[9]: hint answer solution

In Question 45, Section 1.1, we found that

$$\int_0^a \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}$$

when $0 \leqslant a \leqslant 1$.

Using this fact, evaluate the following:

(a) $\displaystyle\int_a^0 \sqrt{1-x^2}\,\mathrm{d}x$, where $-1 \leqslant a \leqslant 0$

(b) $\displaystyle\int_a^1 \sqrt{1-x^2}\,\mathrm{d}x$, where $0 \leqslant a \leqslant 1$

Q[10](∗): hint answer solution

Evaluate $\displaystyle\int_{-1}^2 |2x|\,\mathrm{d}x$.

You may use the result from Example 3.2.5 in the text that $\int\limits_a^b x\,\mathrm{d}x = \frac{b^2-a^2}{2}$.

Q[11]: hint answer solution

Use the inequality $x^2 \leqslant x$ on the interval $[0,1]$, and Theorem 3.2.12, to give an upper bound for the value $\int_0^1 e^{x^2}\mathrm{d}x$.

Q[12]: hint answer solution

Evaluate $\displaystyle\int_{-5}^5 x|x|\,\mathrm{d}x$.

Q[13]: hint answer solution

Suppose $f(x)$ is an even function and $\displaystyle\int_{-2}^2 f(x)\mathrm{d}x = 10$. What is $\displaystyle\int_{-2}^0 f(x)\mathrm{d}x$?

▸▸ Stage 3

Q[14](∗): hint answer solution

Evaluate $\displaystyle\int_{-2}^2 \left(5+\sqrt{4-x^2}\right)\mathrm{d}x$.

Q[15]: hint answer solution

For nonnegative values of x, the following two inequalities hold:

$$\text{(a) } \sin x \leqslant x \qquad \text{(b) } \sin x \leqslant 1$$

On the interval $[0,1]$, it is additionally true that $\sin x \geqslant 0$.

Using (a), we see $\sin^2 x = \sin x \cdot \sin x \leqslant x \sin x$ on the interval $[0,1]$. Using (b), we see $\sin^2 x = \sin x \cdot \sin x \leqslant \sin x$ on $[0,1]$.

Which of the two inequalities above gives a more useful bound for the integral below?

$$\int_0^1 \sin^2 x\mathrm{d}x$$

Q[16](*): hint answer solution

Evaluate $\displaystyle\int_{-2012}^{+2012} \frac{\sin x}{\log(3+x^2)}\,dx$.

Q[17](*): hint answer solution

Evaluate $\displaystyle\int_{-2012}^{+2012} x^{1/3}\cos x\,dx$.

Q[18]: hint answer solution

Evaluate $\displaystyle\int_0^6 (x-3)^3\,dx$.

Q[19]: hint answer solution

We want to compute the area of an ellipse, $(ax)^2+(by)^2=1$ for some (let's say positive) constants a and b.

(a) Solve the equation for the upper half of the ellipse. It should have the form "$y=\cdots$"

(b) Write an integral for the area of the upper half of the ellipse. Using properties of integrals, make the integrand look like the upper half of a circle.

(c) Using geometry and your answer to part (b), find the area of the ellipse.

Q[20]: hint answer solution

Fill in the following table: the product of an (even/odd) function with an (even/odd) function is an (even/odd) function. You may assume that both functions are defined for all real numbers.

$\times$	even	odd
even		
odd		

Q[21]: hint answer solution

Suppose $f(x)$ is an odd function and $g(x)$ is an even function, both defined at $x=0$. What are the possible values of $f(0)$ and $g(0)$?

Q[22]: hint answer solution

Suppose $f(x)$ is a function defined on all real numbers that is both even and odd. What could $f(x)$ be?

Q[23]: hint answer solution

Is the derivative of an even function even or odd? Is the derivative of an odd function even or odd?

▸▸ Open-Ended Questions

Q[24]: hint answer solution

Explain how you might use Theorem 3.2.12 to check your work, after computing a particularly difficult definite integral.

Q[25]: hint answer solution

(a) Suppose a function is symmetric about the line $x = c$, for some constant c. Extend Theorem 3.2.11 to include this case.

(b) Give a definition analogous to Definition 3.2.8 to describe a function that is "symmetric but upside-down" across the line $x = c$, for some constant c. (When $c = 0$, this simply described an odd function. When $c \neq 0$, this looks like an odd function that has been moved to the left or right.) Your definition should include the constant c.

(c) Extend Theorem 3.2.11 to include the case from part (b) above.

3.3▲ The Fundamental Theorem of Calculus

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): hint answer solution

Suppose that $f(x)$ is a function and $F(x) = e^{(x^2-3)} + 1$ is an antiderivative of $f(x)$. Evaluate the definite integral $\int_1^{\sqrt{5}} f(x)\,\mathrm{d}x$.

Q[2](*): hint answer solution

For the function $f(x) = x^3 - \sin 2x$, find its antiderivative $F(x)$ that satisfies $F(0) = 1$.

Q[3](*): hint answer solution

Decide whether each of the following statements is true or false. Provide a brief justification.

(a) If $f(x)$ is continuous on $[1, \pi]$ and differentiable on $(1, \pi)$, then $\int_1^{\pi} f'(x)\,\mathrm{d}x = f(\pi) - f(1)$.

(b) $\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x = 0$.

(c) If f is continuous on $[a, b]$ then $\int_a^b xf(x)\,\mathrm{d}x = x\int_a^b f(x)\,\mathrm{d}x$.

Q[4]: hint answer solution

True or false: an antiderivative of $\frac{1}{x^2}$ is $\ln(x^2)$.

Q[5]: hint answer solution

True or false: an antiderivative of $\cos(e^x)$ is $\frac{\sin(e^x)}{e^x}$.

Q[6]: hint answer solution

Suppose $F(x) = \int_7^x \sin(t^2)\,\mathrm{d}t$. What is the instantaneous rate of change of $F(x)$ with respect to x?

Q[7]: hint answer solution

Suppose $F(x) = \int_2^x e^{1/t}\,\mathrm{d}t$. What is the slope of the tangent line to $y = F(x)$ when $x = 3$?

Q[8]: hint answer solution

Suppose $F'(x) = f(x)$. Give two different antiderivatives of $f(x)$.

Q[9]: hint answer solution

In Question 45, Section 1.1, we found that

$$\int_0^a \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}.$$

(a) Verify that $\frac{\mathrm{d}}{\mathrm{d}a}\left\{\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}\right\} = \sqrt{1-a^2}$.

(b) Find a function $F(x)$ that satisfies $F'(x) = \sqrt{1-x^2}$ and $F(0) = \pi$.

Q[10]: hint answer solution

Evaluate the following integrals using the Fundamental Theorem of Calculus Part 2, or explain why it does not apply.

(a) $\int_{-\pi}^{\pi} \cos x\,\mathrm{d}x$.

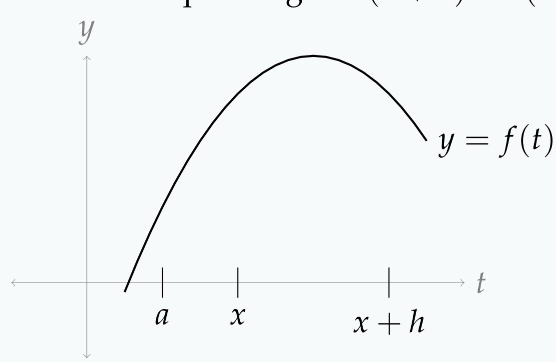
(b) $\int_{-\pi}^{\pi} \sec^2 x\,\mathrm{d}x$.

(c) $\int_{-2}^{0} \frac{1}{x+1}\,\mathrm{d}x$.

Questions 11 through 14 are meant to help reinforce key ideas in the Fundamental Theorem of Calculus and its proof.

Q[11]: hint answer solution

As in the proof of the Fundamental Theorem of Calculus, let $F(x) = \int_a^x f(t)\,\mathrm{d}t$. In the diagram below, shade the area corresponding to $F(x+h) - F(x)$.

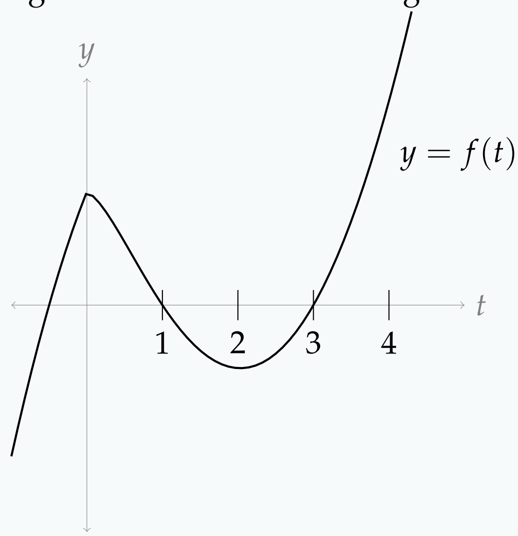


Q[12]: hint answer solution

Let $F(x) = \int_0^x f(t)dt$, where $f(t)$ is shown in the graph below, and $0 \leqslant x \leqslant 4$.

(a) Is $F(0)$ positive, negative, or zero?

(b) Where is $F(x)$ increasing and where is it decreasing?

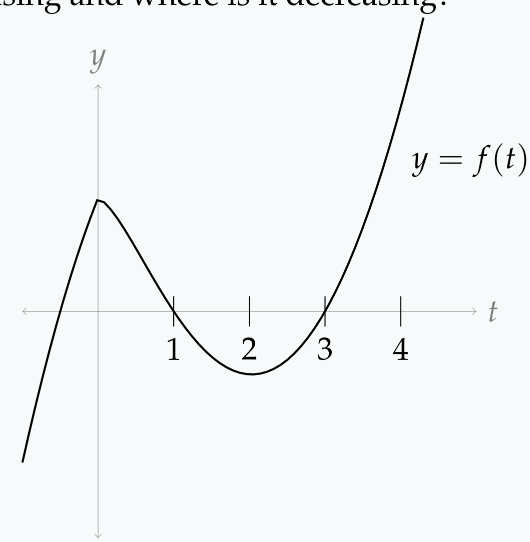


Q[13]: hint answer solution

Let $G(x) = \int_x^0 f(t)dt$, where $f(t)$ is shown in the graph below, and $0 \leqslant x \leqslant 4$.

(a) Is $G(0)$ positive, negative, or zero?

(b) Where is $G(x)$ increasing and where is it decreasing?



Q[14]: hint answer solution

Let $F(x) = \int_a^x t\, \mathrm{d}t$. Using the definition of the derivative, find $F'(x)$.

Q[15]: hint answer solution

Give a continuous function $f(x)$ so that $F(x) = \int_0^x f(t)dt$ is a constant.

So far, we have been able to guess many antiderivatives. Often, however, antiderivatives are very difficult to guess. In Questions 16 through 19, we will find some antiderivatives that might appear in a table of integrals. Coming up with the antiderivative might be quite difficult (strategies to do just that will form a large part of this semester), but verifying *that your antiderivative is correct is as simple as differentiating.*

Q[16]: hint answer solution

Evaluate and simplify $\frac{\mathrm{d}}{\mathrm{d}x}\{x\ln(ax) - x\}$, where a is some constant. What *antiderivative* does this tell you?

Q[17]: hint answer solution

Evaluate and simplify $\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\left(x^3 - 3x^2 + 6x - 6\right)\}$. What *antiderivative* does this tell you?

Q[18]: hint answer solution

Evaluate and simplify $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\ln\left|x + \sqrt{x^2 + a^2}\right|\right\}$, where a is some constant. What *antiderivative* does this tell you?

Q[19]: hint answer solution

Evaluate and simplify $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{x(a+x)} - a\ln\left(\sqrt{x} + \sqrt{a+x}\right)\right\}$, where a is some constant. What *antiderivative* does this tell you?

▸▸ Stage 2

Q[20](∗): hint answer solution

Evaluate $\displaystyle\int_0^2 \left(x^3 + \sin x\right) \mathrm{d}x$.

Q[21](∗): hint answer solution

Evaluate $\displaystyle\int_1^2 \frac{x^2+2}{x^2} \mathrm{d}x$.

Q[22]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{1+25x^2}\mathrm{d}x$.

Q[23]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{\sqrt{2-x^2}}\mathrm{d}x$.

Q[24]: hint answer solution

Evaluate $\displaystyle\int \tan^2 x \, \mathrm{d}x$.

Q[25]: hint answer solution

Evaluate $\displaystyle\int 3\sin x\cos x\,\mathrm{d}x$.

Q[26]: hint answer solution

Evaluate $\displaystyle\int \cos^2 x\,\mathrm{d}x$.

Q[27](∗): hint answer solution

If

$$F(x) = \int_0^x \ln(2+\sin t)\,\mathrm{d}t \quad \text{and} \quad G(y) = \int_y^0 \ln(2+\sin t)\,\mathrm{d}t$$

find $F'\left(\frac{\pi}{2}\right)$ and $G'\left(\frac{\pi}{2}\right)$.

Q[28](∗): hint answer solution

Let $f(x) = \displaystyle\int_1^x 100(t^2-3t+2)e^{-t^2}\,\mathrm{d}t$. Find the interval(s) on which f is increasing.

Q[29](∗): hint answer solution

If $F(x) = \displaystyle\int_0^{\cos x} \frac{1}{t^3+6}\,\mathrm{d}t$, find $F'(x)$.

Q[30](∗): hint answer solution

Compute $f'(x)$ where $f(x) = \displaystyle\int_0^{1+x^4} e^{t^2}\mathrm{d}t$.

Q[31](∗): hint answer solution

Evaluate $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_0^{\sin x}(t^6+8)\mathrm{d}t\right\}$.

Q[32](∗): hint answer solution

Let $F(x) = \displaystyle\int_0^{x^3} e^{-t}\sin\left(\frac{\pi t}{2}\right)\,\mathrm{d}t$. Calculate $F'(1)$.

Q[33](∗): hint answer solution

Find $\displaystyle\frac{\mathrm{d}}{\mathrm{d}u}\left\{\int_{\cos u}^0 \frac{\mathrm{d}t}{1+t^3}\right\}$.

Q[34](∗): hint answer solution

Find $f(x)$ if $x^2 = 1 + \displaystyle\int_1^x f(t)\,\mathrm{d}t$.

Q[35](∗): hint answer solution

If $x\sin(\pi x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t$ where f is a continuous function, find $f(4)$.

Q[36](*): hint answer solution

Consider the function $F(x) = \int_0^{x^2} e^{-t}\,dt + \int_{-x}^{0} e^{-t^2}\,dt$.

(a) Find $F'(x)$.

(b) Find the value of x for which $F(x)$ takes its minimum value.

Q[37](*): hint answer solution

If $F(x)$ is defined by $F(x) = \int_{x^4-x^3}^{x} e^{\sin t}\,dt$, find $F'(x)$.

Q[38](*): hint answer solution

Evaluate $\frac{d}{dx}\left\{ \int_{x^5}^{-x^2} \cos\left(e^t\right) dt \right\}$.

Q[39](*): hint answer solution

Differentiate $\int_x^{e^x} \sqrt{\sin t}\,dt$.

Q[40](*): hint answer solution

Evaluate $\int_1^5 f(x)\,dx$, where $f(x) = \begin{cases} 3 & \text{if } x \leqslant 3 \\ x & \text{if } x \geqslant 3 \end{cases}$.

▸▸ Stage 3

Q[41](*): hint answer solution

If $f'(1) = 2$ and $f'(2) = 3$, find $\int_1^2 f'(x) f''(x)\,dx$.

Q[42](*): hint answer solution

A car traveling at 30 m/s applies its brakes at time $t = 0$, its velocity (in m/s) decreasing according to the formula $v(t) = 30 - 10t$. How far does the car go before it stops?

Q[43](*): hint answer solution

Compute $f'(x)$ where $f(x) = \int_0^{2x-x^2} \ln\left(1 + e^t\right) dt$. Does $f(x)$ have an absolute maximum? Explain.

Q[44](*): hint answer solution

Find the minimum value of $\int_0^{x^2-2x} \frac{dt}{1+t^4}$. Express your answer as an integral.

Q[45](*): hint answer solution

Define the function $F(x) = \int_0^{x^2} \sin(\sqrt{t})\,dt$ on the interval $0 < x < 4$. On this interval, where does $F(x)$ have a maximum?

Q[46](*): hint answer solution

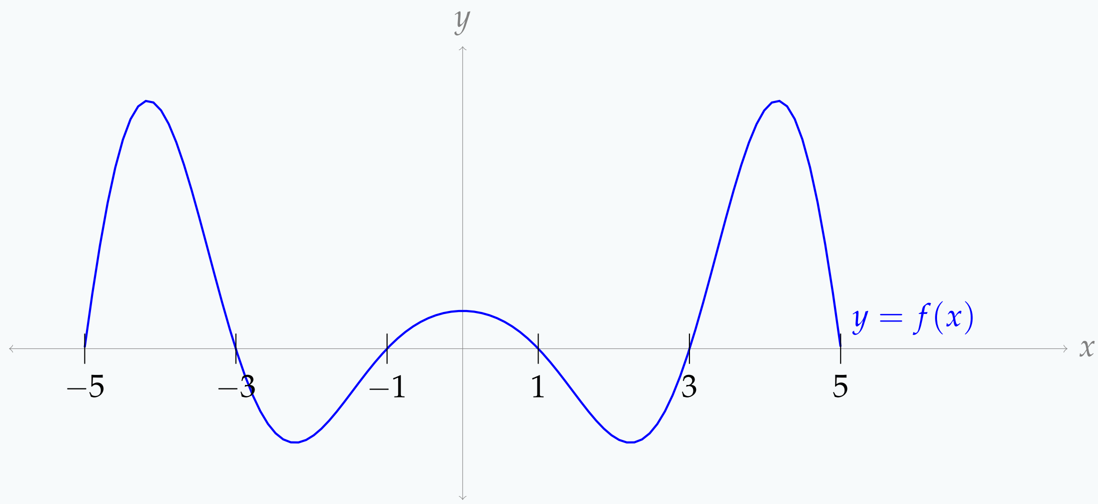
Evaluate $\lim\limits_{n\to\infty} \frac{\pi}{n} \sum\limits_{j=1}^{n} \sin\left(\frac{j\pi}{n}\right)$ by interpreting it as a limit of Riemann sums.

Q[47](∗): hint answer solution

Use Riemann sums to evaluate the limit $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{j=1}^{n} \frac{1}{1+\frac{j}{n}}$.

Q[48]: hint answer solution

Below is the graph of $y = f(t)$, $-5 \leqslant t \leqslant 5$. Define $F(x) = \int_0^x f(t)\,\mathrm{d}t$ for any x in $[-5, 5]$. Sketch $F(x)$.



Q[49](∗): hint answer solution

Define $f(x) = x^3 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t$.

(a) Find a formula for the derivative $f'(x)$. (Your formula may include an integral sign.)

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at $x = -1$.

Q[50]: hint answer solution

Two students calculate $\int f(x)\,\mathrm{d}x$ for some function $f(x)$.

- Student A calculates $\int f(x)\,\mathrm{d}x = \tan^2 x + x + C$
- Student B calculates $\int f(x)\,\mathrm{d}x = \sec^2 x + x + C$
- It is a fact that $\frac{\mathrm{d}}{\mathrm{d}x}\{\tan^2 x\} = f(x) - 1$

Who ended up with the correct answer?

Q[51]: hint answer solution

Let $F(x) = \displaystyle\int_0^x x^3 \sin(t)\, \mathrm{d}t$.

(a) Evaluate $F(3)$.

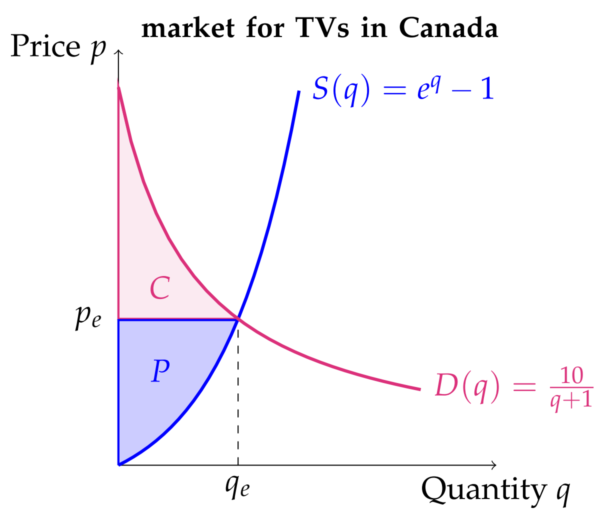
(b) What is $F'(x)$?

Q[52]: hint answer solution

Let $f(x)$ be an even function, defined everywhere, and let $F(x)$ be an antiderivative of $f(x)$. Is $F(x)$ even, odd, or not necessarily either one? (You may use your answer from Section 1.2, Question 23.)

Q[53]: hint answer solution

The following graph portrays the supply and demand functions for the Canadian TV market: $S(q) = e^q - 1$ and $D(q) = \frac{10}{q+1}$. The price where the supply function and the demand function meet is known as the equilibrium price, denoted by p_e in this example. The quantity where the supply function and the demand function meet is known as the equilibrium quantity, denoted by q_e in this example. We think of q_e as the total amount of products bought at p_e dollars.



(a) When TVs are each sold at a market price of p_e, then consumers who would have been willing to buy some of them at a higher price gain some benefit. They pay only p_e for a TV that, to them, is worth more than that.

The Consumer Surplus measures the difference between the price consumers are *willing* to pay for the TV and the price they *actually* pay for it. It is calculated as the area above the line $p = p_e$ and below the demand curve, from $q = 0$ to $q = q_e$ (area C in the figure above).

Find the Consumer Surplus in the Canadian TV market. You may leave your answer in terms of q_e and p_e.

(b) When TVs are each sold at a market price of p_e, then producers who would have been willing to sell some of them at a lower price gain some benefit.

The Producer Surplus measures the difference between the amount producers are *willing* to accept for a TV and the amount they *actually* receive. It is calculated as the area above the supply curve and below the line $p = p_e$, from $q = 0$ to $q = q_e$ (area P in the figure above).

Find the Producer Surplus in the Canadian TV market. You may leave your answer in terms of q_e and p_e.

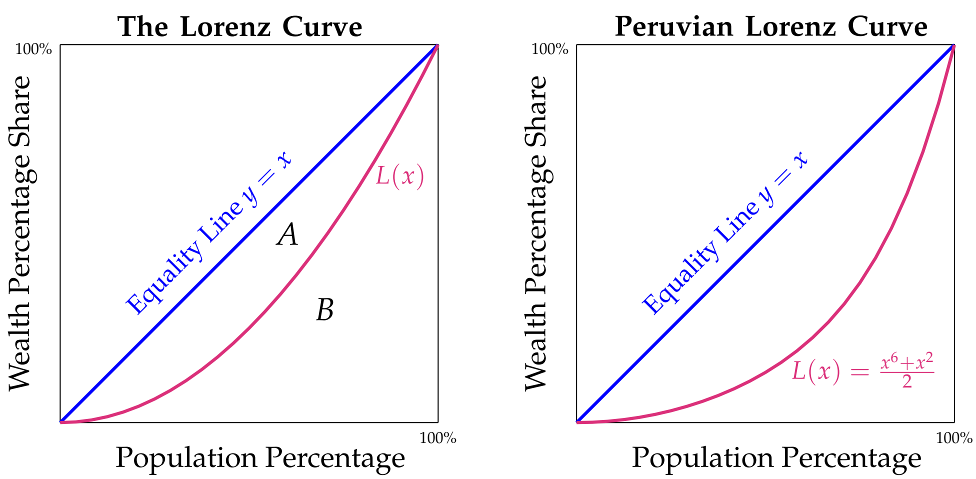
(c) The Total Surplus is the sum of the Consumer Surplus and the Producer Surplus. Find the Total Surplus in the Canadian TV Market. You may leave your answer in terms of q_e and p_e.

Q[54]: hint answer solution

In this question, we'll investigate the Gini coefficient. The Gini Coefficient helps economists measure the level of inequality in a country (or, more generally, among a population).

In the graphs below, the x-axis represents the population percentage, while the y-axis represents the cumulative percentage wealth shared. The Lorenz curve $L(x)$ represents a country's income distribution. Specifically, $L(x)$ gives the percentage of the country's wealth belonging to the poorest x percent of the population. For example, $L(0.6) = 0.2$ means that the poorest 60% of the population possess only 20% of the total wealth.

In a situation of perfect equality, where every individual owns the same amount of wealth, the distribution curve would be the straight line $y = x$.



A is the area between the equality line $y = x$ and the Lorenz curve $L(x)$. B is the area under $L(x)$. The Gini coefficient is calculated by $\dfrac{A}{A+B}$. The higher the Gini coefficient, the higher the inequality.

(a) Using integrals, define the general formula to find the Gini Coefficient. Refer to the Lorenz curve as $L(x)$. Use ratios instead of percentages (for example, 0.5 instead of 50%).

(b) If a country has perfect equality in wealth distribution, what is its Gini coefficient?

(c) Peru has a Lorenz curve of $L(x) = \dfrac{x^6 + x^2}{2}$. Find its Gini Coefficient.

Q[55]: hint answer solution

Total Cost, $\text{TC}(q)$, is the the cost of producing q of units. Total Cost is the sum of Fixed Cost (FC) and Variable Cost (VC).

$$\text{TC} = \text{FC} + \text{VC}$$

The **Fixed Cost** is defined as all expenses that do not change with quantity (for example, rent on a factory space, which is the same whether you make 1 or 1000 units). FC is a (generally nonzero) constant, and is equal to $\text{TC}(0)$, the costs incurred before before the first unit is produced.

Variable Cost, $\text{VC}(q)$ consists of expenses that depend on quantity (for example, raw materials: producing more units means using more raw materials). Variable cost is a function of the quantity of units that are made, q.

Marginal Cost is the cost of producing one extra unit of output. It includes, for example, the costs of raw materials and labour that go into making each unit. We write Marginal Cost as a function of the quantity q of units that have already been made, $\text{MC}(q)$. Thinking of MC as the rate of change of TC, we define

$$\text{MC} = \frac{\mathrm{d}}{\mathrm{d}q}\text{TC}$$

(a) If a company has $\text{MC} = \frac{1}{q+1} + q + 2$, and FC of $1,000$ dollars, find its TC function and its Total Cost to produce $2,000$ units.

(b) If a company has $\text{MC} = 40 - 10q + \frac{e^q}{10}$, and FC of $50,000$ dollars, find its TC function its Total Cost to produce 10 units.

Q[56]: hint answer solution

Marginal Revenue (MR) is the extra **Total Revenue** (TR) gained by producing one extra unit of output. We define MR as:

$$\text{MR} = \frac{d(\text{TR})}{dq}$$

where the independent variable q gives units of output. If $q = 0$, then $\text{TR} = 0$, as no products are being sold.

The **unit price** P of a product is TR divided by the amount of units sold, $P = \frac{TR}{q}$. Use this information to solve the next two questions.

(a) If a company has $\text{MR} = \cos(q) + \dfrac{q}{5} + 2$, find the TR function and the price at which each unit is sold.

(b) If a company has $\text{MR} = \dfrac{e^q}{1000} + \dfrac{1}{2\sqrt{q}}$, find the TR function and the price at which each unit is sold.

3.4▲ Substitution

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

(a) True or False: $\displaystyle\int \sin(e^x)\cdot e^x\,\mathrm{d}x = \int \sin(u)\,\mathrm{d}u\Big|_{u=e^x} = -\cos(e^x)+C$

(b) True or False: $\displaystyle\int_0^1 \sin(e^x)\cdot e^x\,\mathrm{d}x = \int_0^1 \sin(u)\,\mathrm{d}u = 1-\cos(1)$

Q[2]: hint answer solution

Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\displaystyle\int (2x+1)^2\mathrm{d}x$.

Work: We use the substitution $u = 2x+1$. Then:

$$\begin{aligned}\int (2x+1)^2\mathrm{d}x &= \int u^2\,\mathrm{d}u\\ &= \frac{1}{3}u^3 + C\\ &= \frac{1}{3}(2x+1)^3 + C\end{aligned}$$

Q[3]: hint answer solution

Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\displaystyle\int_1^\pi \frac{\cos(\ln t)}{t}\mathrm{d}t$.

Work: We use the substitution $u = \ln t$, so $\mathrm{d}u = \frac{1}{t}\mathrm{d}t$. Then:

$$\begin{aligned}\int_1^\pi \frac{\cos(\ln t)}{t}\mathrm{d}t &= \int_1^\pi \cos(u)\mathrm{d}u\\ &= \sin(\pi) - \sin(1) = -\sin(1)\,.\end{aligned}$$

Q[4]: hint answer solution

Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\displaystyle\int_0^{\pi/4} x\tan(x^2)\,\mathrm{d}x$.

Work: We begin with the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$:

$$\begin{aligned}\int_0^{\pi/4} x\tan(x^2)\,\mathrm{d}x &= \int_0^{\pi/4} \frac{1}{2}\tan(x^2)\cdot 2x\mathrm{d}x\\ &= \int_0^{\pi^2/16}\frac{1}{2}\tan u\,\mathrm{d}u\\ &= \frac{1}{2}\int_0^{\pi^2/16}\frac{\sin u}{\cos u}\mathrm{d}u\end{aligned}$$

Now we use the substitution $v = \cos u$, $\mathrm{d}v = -\sin u\,\mathrm{d}u$:

$$\begin{aligned}&= \frac{1}{2}\int_{\cos 0}^{\cos(\pi^2/16)} -\frac{1}{v}\mathrm{d}v\\ &= -\frac{1}{2}\int_1^{\cos(\pi^2/16)}\frac{1}{v}\mathrm{d}v\\ &= -\frac{1}{2}\left[\ln|v|\right]_1^{\cos(\pi^2/16)}\\ &= -\frac{1}{2}\left(\ln\left(\cos(\pi^2/16)\right) - \ln(1)\right)\\ &= -\frac{1}{2}\ln\left(\cos(\pi^2/16)\right)\end{aligned}$$

Q[5](∗): hint answer solution

What is the integral that results when the substitution $u = \sin x$ is applied to the integral $\displaystyle\int_0^{\pi/2} f(\sin x)\,\mathrm{d}x$?

Q[6]: hint answer solution

Let f and g be functions that are continuous and differentiable everywhere. Simplify

$$\int f'(g(x))g'(x)\,\mathrm{d}x - f(g(x)).$$

▸▸ Stage 2

Q[7](∗): hint answer solution

Use substitution to evaluate $\displaystyle\int_0^1 xe^{x^2}\cos(e^{x^2})\,\mathrm{d}x$.

Q[8](∗): hint answer solution

Let $f(t)$ be any function for which $\displaystyle\int_1^8 f(t)\,\mathrm{d}t = 1$. Calculate the integral $\displaystyle\int_1^2 x^2 f(x^3)\,\mathrm{d}x$.

Q[9](*): hint answer solution

Evaluate $\displaystyle\int \frac{x^2}{(x^3+1)^{101}}\,dx$.

Q[10](*): hint answer solution

Evaluate $\displaystyle\int_e^{e^4} \frac{dx}{x\ln x}$.

Q[11](*): hint answer solution

Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,dx$.

Q[12](*): hint answer solution

Evaluate $\displaystyle\int_0^{\pi/2} \cos x\cdot(1+\sin^2 x)\,dx$.

Q[13](*): hint answer solution

Evaluate $\displaystyle\int_1^3 (2x-1)e^{x^2-x}\,dx$.

Q[14](*): hint answer solution

Evaluate $\displaystyle\int \frac{(x^2-4)x}{\sqrt{4-x^2}}\,dx$.

Q[15]: hint answer solution

Evaluate $\displaystyle\int \frac{e^{\sqrt{\ln x}}}{2x\sqrt{\ln x}}\,dx$.

▸▸ Stage 3

Q[16](*): hint answer solution

Calculate $\displaystyle\int_{-2}^{2} xe^{x^2}\,dx$.

Q[17](*): hint answer solution

Calculate $\displaystyle\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sin\left(1+\frac{j^2}{n^2}\right)$.

Questions 18 through 22 can be solved by substitution, but it may not be obvious which substitution will work. In general, when evaluating integrals, it is not always immediately clear which methods are appropriate. If this happens to you, don't despair, and definitely don't give up! Just guess a method and try it. Even if it fails, you'll probably learn something that you can use to make a better guess.[1]

Q[18]: hint answer solution

Evaluate $\displaystyle\int_0^1 \frac{u^3}{u^2+1}\,\mathrm{d}u$.

Q[19]: hint answer solution

Evaluate $\displaystyle\int \tan^3\theta\,\mathrm{d}\theta$.

Q[20]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{e^x+e^{-x}}\,\mathrm{d}x$

Q[21]: hint answer solution

Evaluate $\displaystyle\int_0^1 (1-2x)\sqrt{1-x^2}\,\mathrm{d}x$

Q[22]: hint answer solution

Evaluate $\displaystyle\int \tan x\cdot\ln(\cos x)\,\mathrm{d}x$

Q[23](*): hint answer solution

Evaluate $\displaystyle\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\cos\left(\frac{j^2}{n^2}\right)$.

Q[24](*): hint answer solution

Calculate $\displaystyle\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sqrt{1+\frac{j^2}{n^2}}$.

Q[25]: hint answer solution

Using Riemann sums, prove that

$$\int_a^b 2f(2x)\mathrm{d}x = \int_{2a}^{2b} f(x)\mathrm{d}x$$

Q[26]: hint answer solution

Total Cost, $\mathrm{TC}(q)$, is the the cost of producing q of units. Total Cost is the sum of Fixed Cost (FC) and Variable Cost (VC).

$$\mathrm{TC} = \mathrm{FC} + \mathrm{VC}$$

1 This is also pretty decent life advice.

The **Fixed Cost** is defined as all expenses that do not change with quantity (for example, rent on a factory space, which is the same whether you make 1 or 1000 units). FC is a (generally nonzero) constant, and is equal to $\mathrm{TC}(0)$, the costs incurred before the first unit is produced.

Variable Cost, $\mathrm{VC}(q)$ consists of expenses that depend on quantity (for example, raw materials: producing more units means using more raw materials). Variable cost is a function of the quantity of units that are made, q.

Marginal Cost is the cost of producing one extra unit of output. It includes, for example, the costs of raw materials and labour that go into making each unit. We write Marginal Cost as a function of the quantity q of units that have already been made, $\mathrm{MC}(q)$. Thinking of MC as the rate of change of TC, we define

$$\mathrm{MC} = \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{TC}$$

Marginal Revenue (MR) is the extra **Total Revenue** (TR) gained by producing one extra unit of output. We define MR as:

$$\mathrm{MR} = \frac{d(\mathrm{TR})}{dq}$$

where the independent variable q gives units of output. If $q = 0$, then $\mathrm{TR} = 0$, as no products are being sold.

The **unit price** P of a product is TR divided by the amount of units sold, $P = \frac{TR}{q}$.

(a) A company has $\mathrm{MC} = \dfrac{6q^2 - 80}{\sqrt{2q^3 - 80q}}$ and $FC = 2,000$ dollars. Find the company's TC function.

(b) The same company has $\mathrm{MR} = \dfrac{q^3}{\sqrt{q^2+1}}$. Find the company's TR function.

(c) Find the profit formula for this company.

(d) Why might this formula not apply for some small quantities q?

3.5▲ Integration by parts

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

The method of integration by substitution comes from the ________ rule for differentiation.

The method of integration by parts comes from the ________ rule for differentiation.

Q[2]: hint answer solution

Suppose you want to evaluate an integral using integration by parts. You choose part of your integrand to be u, and part to be $\mathrm{d}v$. The part chosen as u will be: (differentiated, antidifferentiated). The part chosen as $\mathrm{d}v$ will be: (differentiated, antidifferentiated).

Q[3]: hint answer solution

Let $f(x)$ and $g(x)$ be differentiable functions. Using the quotient rule for differentiation, give an equivalent expression to $\displaystyle\int \frac{f'(x)}{g(x)}\,\mathrm{d}x$.

Q[4]: hint answer solution

Suppose we want to use integration by parts to evaluate $\displaystyle\int u(x)\cdot v'(x)\mathrm{d}x$ for some differentiable functions u and v. We need to find an antiderivative of $v'(x)$, but there are infinitely many choices. Show that every antiderivative of $v'(x)$ gives an equivalent final answer.

Q[5]: hint answer solution

Suppose you want to evaluate $\displaystyle\int f(x)\mathrm{d}x$ using integration by parts. Explain why $\mathrm{d}v = f(x)\mathrm{d}x$, $u = 1$ is generally a bad choice.

Note: compare this to Example 3.5.8 of the text, where we chose $u = f(x)$, $\mathrm{d}v = 1\mathrm{d}x$.

▸▸ Stage 2

Q[6](*): hint answer solution

Evaluate $\displaystyle\int x \ln x \,\mathrm{d}x$.

Q[7](*): hint answer solution

Evaluate $\displaystyle\int \frac{\ln x}{x^7}\,\mathrm{d}x$.

Q[8](*): hint answer solution

Evaluate $\displaystyle\int_0^{\pi} x \sin x \,\mathrm{d}x$.

Q[9](*): hint answer solution

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x \cos x \,\mathrm{d}x$.

Q[10]: hint answer solution

Evaluate $\displaystyle\int x^3 e^x \mathrm{d}x$.

Q[11]: hint answer solution

Evaluate $\displaystyle\int x \ln^3 x \, dx$.

Q[12]: hint answer solution

Evaluate $\displaystyle\int x^2 \sin x \, dx$.

Q[13]: hint answer solution

Evaluate $\displaystyle\int (3t^2 - 5t + 6) \ln t \, dt$.

Q[14]: hint answer solution

Evaluate $\displaystyle\int \sqrt{s} e^{\sqrt{s}} ds$.

Q[15]: hint answer solution

Evaluate $\displaystyle\int \ln^2 x dx$.

Q[16]: hint answer solution

Evaluate $\displaystyle\int 2x e^{x^2+1} dx$.

Q[17](*): hint answer solution

Evaluate $\displaystyle\int \arccos y \, dy$.

▸▸ Stage 3

Q[18](*): hint answer solution

Evaluate $\displaystyle\int 4y \arctan(2y) \, dy$.

Q[19]: hint answer solution

Evaluate $\displaystyle\int x^2 \arctan x \, dx$.

Q[20]: hint answer solution

Evaluate $\displaystyle\int e^{x/2} \cos(2x) dx$.

Q[21]: hint answer solution

Evaluate $\displaystyle\int \sin(\ln x) dx$.

Q[22]: hint answer solution
Evaluate $\displaystyle\int 2^{x+\log_2 x} dx$.

Q[23]: hint answer solution
Evaluate $\displaystyle\int e^{\cos x} \sin(2x) dx$.

Q[24]: hint answer solution
Evaluate $\displaystyle\int \frac{xe^{-x}}{(1-x)^2} dx$.

Q[25](*): hint answer solution
A reduction formula.

(a) Derive the reduction formula

$$\int \sin^n(x)\, dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\, dx.$$

(b) Calculate $\displaystyle\int_0^{\pi/2} \sin^8(x)\, dx$.

Q[26](*): hint answer solution
Let R be the part of the first quadrant that lies below the curve $y = \arctan x$ and between the lines $x = 0$ and $x = 1$.
Sketch the region R and determine its area.

Q[27](*): hint answer solution
Let R be the region between the curves $T(x) = \sqrt{x}e^{3x}$ and $B(x) = \sqrt{x}(1+2x)$ on the interval $0 \leqslant x \leqslant 3$. (It is true that $T(x) \geqslant B(x)$ for all $0 \leqslant x \leqslant 3$.) Compute the volume of the solid formed by rotating R about the x-axis.

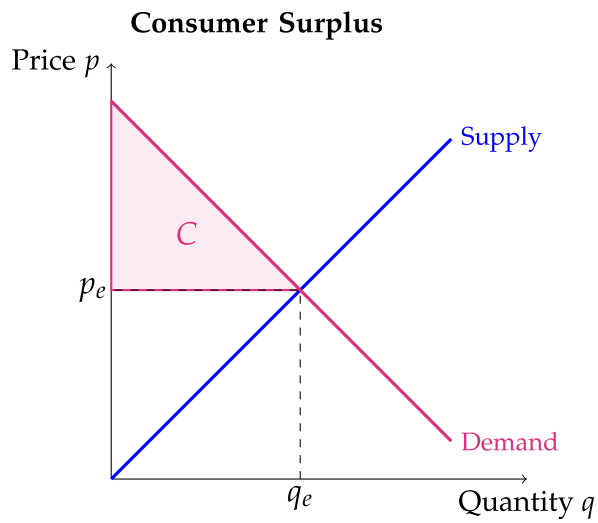
Q[28](*): hint answer solution
Let $f(0) = 1$, $f(2) = 3$ and $f'(2) = 4$. Calculate $\displaystyle\int_0^4 f''\left(\sqrt{x}\right) dx$.

Q[29]: hint answer solution
Evaluate $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\frac{2}{n}i - 1\right) e^{\frac{2}{n}i-1}$.

Q[30]: hint answer solution
The price where the supply function and the demand function meet is known as the equilibrium price, denoted by p_e in this example. The quantity where the supply function and the demand function meet is known as the equilibrium quantity, denoted by q_e in this example. We think of q_e as the total amount of products bought at p_e dollars.

When goods are each sold at a market price of p_e, then consumers who would have been willing to buy some of them at a higher price gain some benefit. They pay only p_e for a good that, to them, is worth more than that.

The Consumer Surplus measures the difference between the price consumers are *willing* to pay for a product and the price they *actually* pay for it. It is calculated as the area above the line $p = p_e$ and below the demand curve, from $q = 0$ to $q = q_e$ (area C in the figure below).



Find the Consumer Surplus for a good whose demand function is

$$p(q) = 8 - 2\ln(2q + 1)$$

and whose equilibrium quantity is $q_e = 5$.

Q[31]: hint answer solution

Total Cost, $\mathrm{TC}(q)$, is the the cost of producing q of units. Total Cost is the sum of Fixed Cost (FC) and Variable Cost (VC).

$$\mathrm{TC} = \mathrm{FC} + \mathrm{VC}$$

The **Fixed Cost** is defined as all expenses that do not change with quantity (for example, rent on a factory space, which is the same whether you make 1 or 1000 units). FC is a (generally nonzero) constant, and is equal to $\mathrm{TC}(0)$, the costs incurred before before the first unit is produced.

Variable Cost, $\mathrm{VC}(q)$ consists of expenses that depend on quantity (for example, raw materials: producing more units means using more raw materials). Variable cost is a function of the quantity of units that are made, q.

Marginal Cost is the cost of producing one extra unit of output. It includes, for example, the costs of raw materials and labour that go into making each unit. We write Marginal Cost as a function of the quantity q of units that have already been made, $\mathrm{MC}(q)$. Thinking of MC as the rate of change of TC, we define

$$\mathrm{MC} = \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{TC}$$

Suppose a company has $\mathrm{MC} = \dfrac{qe^{\frac{q}{10}}}{100} - q + 30$ and FC of $1,000$ dollars.

(a) Find its TC function.

(b) What would be the average cost per unit when producing 10 units?

3.6▲ Trigonometric Integrals

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Suppose you want to evaluate $\int_0^{\pi/4} \sin x \cos^n x \, dx$ using the substitution $u = \cos x$. Which of the following need to be true for your substitution to work?

(a) n must be even

(b) n must be odd

(c) n must be an integer

(d) n must be positive

(e) n can be any real number

Q[2]: hint answer solution

Evaluate $\int \sec^n x \tan x dx$, where n is a strictly positive integer.

Q[3]: hint answer solution

Derive the identity $\tan^2 x + 1 = \sec^2 x$ from the easier-to-remember identity $\sin^2 x + \cos^2 x = 1$.

▸▸ Stage 2

Questions 4 through 10 deal with powers of sines and cosines. Review Section 3.6.1 in the text for integration strategies.

Q[4](∗): hint answer solution

Evaluate $\int \cos^3 x \, dx$.

Q[5](∗): hint answer solution

Evaluate $\int_0^{\pi} \cos^2 x \, dx$.

Q[6](*): hint answer solution

Evaluate $\displaystyle\int \sin^{36} t\, \cos^3 t\, dt$.

Q[7]: hint answer solution

Evaluate $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\, dx$.

Q[8]: hint answer solution

Evaluate $\displaystyle\int_0^{\pi/3} \sin^4 x\, dx$.

Q[9]: hint answer solution

Evaluate $\displaystyle\int \sin^5 x\, dx$.

Q[10]: hint answer solution

Evaluate $\displaystyle\int \sin^{1.2} x \cos x\, dx$.

Questions 12 through 21 deal with powers of tangents and secants. Review Section 3.6.2 in the text for strategies.

Q[11]: hint answer solution

Evaluate $\displaystyle\int \tan x \sec^2 x dx$.

Q[12](*): hint answer solution

Evaluate $\displaystyle\int \tan^3 x \sec^5 x\, dx$.

Q[13](*): hint answer solution

Evaluate $\displaystyle\int \sec^4 x\, \tan^{46} x\, dx$.

Q[14]: hint answer solution

Evaluate $\displaystyle\int \tan^3 x \sec^{1.5} x\, dx$.

Q[15]: hint answer solution

Evaluate $\displaystyle\int \tan^3 x \sec^2 x\, dx$.

Q[16]: hint answer solution

Evaluate $\int \tan^4 x \sec^2 x \, dx$.

Q[17]: hint answer solution

Evaluate $\int \tan^3 x \sec^{-0.7} x \, dx$.

Q[18]: hint answer solution

Evaluate $\int \tan^5 x \, dx$.

Q[19]: hint answer solution

Evaluate $\int_0^{\pi/6} \tan^6 x \, dx$.

Q[20]: hint answer solution

Evaluate $\int_0^{\pi/4} \tan^8 x \sec^4 x \, dx$.

Q[21]: hint answer solution

Evaluate $\int \tan x \sqrt{\sec x} \, dx$.

Q[22]: hint answer solution

Evaluate $\int \sec^8 \theta \tan^e \theta \, d\theta$.

▸▸ Stage 3

Q[23](∗): hint answer solution

A reduction formula.

(a) Let n be a positive integer with $n \geqslant 2$. Derive the reduction formula

$$\int \tan^n(x) \, dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x) \, dx.$$

(b) Calculate $\int_0^{\pi/4} \tan^6(x) \, dx$.

Q[24]: hint answer solution

Evaluate $\int \tan^5 x \cos^2 x \, dx$.

Q[25]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{\cos^2\theta} d\theta$.

Q[26]: hint answer solution

Evaluate $\displaystyle\int \cot x \, dx$.

Q[27]: hint answer solution

Evaluate $\displaystyle\int e^x \sin(e^x)\cos(e^x) \, dx$.

Q[28]: hint answer solution

Evaluate $\displaystyle\int \sin(\cos x)\sin^3 x \, dx$.

Q[29]: hint answer solution

Evaluate $\displaystyle\int x \sin x \cos x \, dx$.

3.7▲ Trigonometric Substitution

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): hint answer solution

For each of the following integrals, choose the substitution that is most beneficial for evaluating the integral.

(a) $\displaystyle\int \frac{2x^2}{\sqrt{9x^2-16}} dx$

(b) $\displaystyle\int \frac{x^4-3}{\sqrt{1-4x^2}} dx$

(c) $\displaystyle\int (25+x^2)^{-5/2} dx$

Q[2]: hint answer solution

For each of the following integrals, choose a trigonometric substitution that will eliminate the roots.

(a) $\displaystyle\int \frac{1}{\sqrt{x^2-4x+1}} dx$

(b) $\displaystyle\int \frac{(x-1)^6}{(-x^2+2x+4)^{3/2}} dx$

(c) $\displaystyle\int \frac{1}{\sqrt{4x^2+6x+10}} dx$

(d) $\displaystyle\int \sqrt{x^2 - x}\, \mathrm{d}x$

Q[3]: hint answer solution

In each part of this question, assume θ is an angle in the interval $[0, \pi/2]$.

(a) If $\sin\theta = \dfrac{1}{20}$, what is $\cos\theta$?

(b) If $\tan\theta = 7$, what is $\csc\theta$?

(c) If $\sec\theta = \dfrac{\sqrt{x-1}}{2}$, what is $\tan\theta$?

Q[4]: hint answer solution

Simplify the following expressions.

(a) $\sin\left(\arccos\left(\frac{x}{2}\right)\right)$

(b) $\sin\left(\arctan\left(\frac{1}{\sqrt{3}}\right)\right)$

(c) $\sec\left(\arcsin\left(\sqrt{x}\right)\right)$

▸▸ Stage 2

Q[5](*): hint answer solution

Evaluate $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\, \mathrm{d}x$.

Q[6](*): hint answer solution

Evaluate $\displaystyle\int_0^4 \frac{1}{\left(4+x^2\right)^{3/2}}\, \mathrm{d}x$. Your answer may not contain inverse trigonometric functions.

Q[7](*): hint answer solution

Evaluate $\displaystyle\int_0^{5/2} \frac{\mathrm{d}x}{\sqrt{25-x^2}}$.

Q[8](*): hint answer solution

Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+25}}$. You may use that $\displaystyle\int \sec \mathrm{d}x = \ln\left|\sec x + \tan x\right| + C$.

Q[9]: hint answer solution

Evaluate $\displaystyle\int \frac{x+1}{\sqrt{2x^2+4x}}\, \mathrm{d}x$.

Q[10](*): hint answer solution

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+16}}$.

Q[11](*): hint answer solution

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-9}}$ for $x \geqslant 3$. Do not include any inverse trigonometric functions in your answer.

Q[12](*): hint answer solution

(a) Show that $\displaystyle\int_0^{\pi/4} \cos^4\theta \, d\theta = (8+3\pi)/32$.

(b) Evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{(x^2+1)^3}$.

Q[13]: hint answer solution

Evaluate $\displaystyle\int_{-\pi/12}^{\pi/12} \frac{15x^3}{(x^2+1)(9-x^2)^{5/2}}\, dx$.

Q[14](*): hint answer solution

Evaluate $\displaystyle\int \sqrt{4-x^2}\, dx$.

Q[15](*): hint answer solution

Evaluate $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\, dx$ for $x > \frac{2}{5}$.

Q[16]: hint answer solution

Evaluate $\displaystyle\int_{\sqrt{10}}^{\sqrt{17}} \frac{x^3}{\sqrt{x^2-1}}\, dx$.

Q[17](*): hint answer solution

Evaluate $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$.

Q[18]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{(2x-3)^3\sqrt{4x^2-12x+8}}\, dx$ for $x > 2$.

Q[19]: hint answer solution

Evaluate $\displaystyle\int_0^1 \frac{x^2}{(x^2+1)^{3/2}} dx$.

You may use that $\int \sec x dx = \ln|\sec x + \tan x| + C$.

Q[20]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$.

▸▸ Stage 3

Q[21]: hint answer solution

Evaluate $\displaystyle\int \frac{x^2}{\sqrt{x^2-2x+2}}\,dx$.

You may assume without proof that $\displaystyle\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$.

Q[22]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{\sqrt{3x^2+5x}}\,dx$.

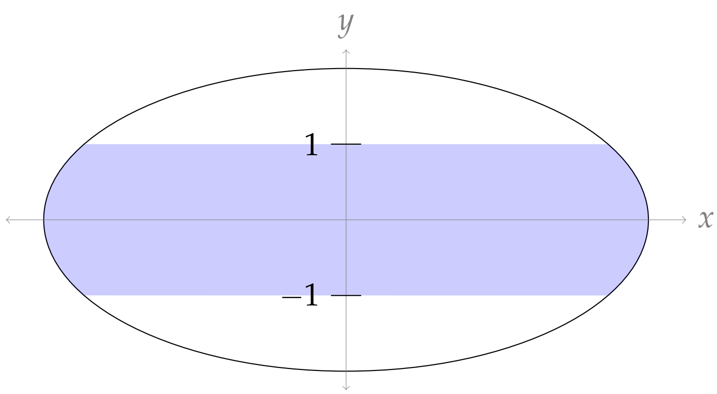
You may use that $\int \sec x\,dx = \ln|\sec x + \tan x| + C$.

Q[23]: hint answer solution

Evaluate $\displaystyle\int \frac{(1+x^2)^{3/2}}{x}dx$. You may use the fact that $\displaystyle\int \csc\theta\,d\theta = \ln|\cot\theta - \csc\theta| + C$.

Q[24]: hint answer solution

Below is the graph of the ellipse $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. Find the area of the shaded region using the ideas from this section.



Q[25]: hint answer solution

Let $f(x) = \dfrac{|x|}{\sqrt[4]{1-x^2}}$, and let R be the region between $f(x)$ and the x-axis over the interval $[-\frac{1}{2}, \frac{1}{2}]$.

(a) Find the area of R.

(b) Find the volume of the solid formed by rotating R about the x-axis.

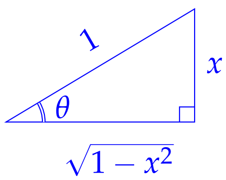
Q[26]: hint answer solution

Evaluate $\displaystyle\int \sqrt{1+e^x}\,dx$. You may use the antiderivative $\displaystyle\int \csc\theta\,d\theta = \ln|\cot\theta - \csc\theta| + C$.

Q[27]: hint answer solution

Consider the following work.

$$\begin{aligned}
\int \frac{1}{1-x^2}\,dx &= \int \frac{1}{1-\sin^2\theta}\cos\theta\,d\theta && \text{using } x=\sin\theta,\quad dx=\cos\theta\,d\theta\\
&= \int \frac{\cos\theta}{\cos^2\theta}\,d\theta\\
&= \int \sec\theta\,d\theta\\
&= \ln|\sec\theta+\tan\theta|+C && \text{Example A.8.1 in the text}\\
&= \ln\left|\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}\right|+C\\
&= \ln\left|\frac{1+x}{\sqrt{1-x^2}}\right|+C
\end{aligned}$$



(a) Differentiate $\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right|$.

(b) True or false: $\displaystyle\int_2^3 \frac{1}{1-x^2}\,dx = \left[\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right|\right]_{x=2}^{x=3}$

(c) Was the work in the question correct? Explain.

Q[28]: hint answer solution

(a) Suppose we are evaluating an integral that contains the term $\sqrt{a^2-x^2}$, where a is a positive constant, and we use the substitution $x = a\sin u$ (with inverse $u = \arcsin(x/a)$), so that

$$\sqrt{a^2-x^2} = \sqrt{a^2\cos^2 u} = |a\cos u|$$

Under what circumstances is $|a\cos u| \neq a\cos u$?

(b) Suppose we are evaluating an integral that contains the term $\sqrt{a^2+x^2}$, where a is a positive constant, and we use the substitution $x = a\tan u$ (with inverse $u = \arctan(x/a)$), so that

$$\sqrt{a^2+x^2} = \sqrt{a^2\sec^2 u} = |a\sec u|$$

Under what circumstances is $|a\sec u| \neq a\sec u$?

(c) Suppose we are evaluating an integral that contains the term $\sqrt{x^2-a^2}$, where a is a positive constant, and we use the substitution $x = a\sec u$ (with inverse $u = \operatorname{arcsec}(x/a) = \arccos(a/x)$), so that

$$\sqrt{x^2-a^2} = \sqrt{a^2\tan^2 u} = |a\tan u|$$

Under what circumstances is $|a\tan u| \neq a\tan u$?

3.8▲ Partial Fractions

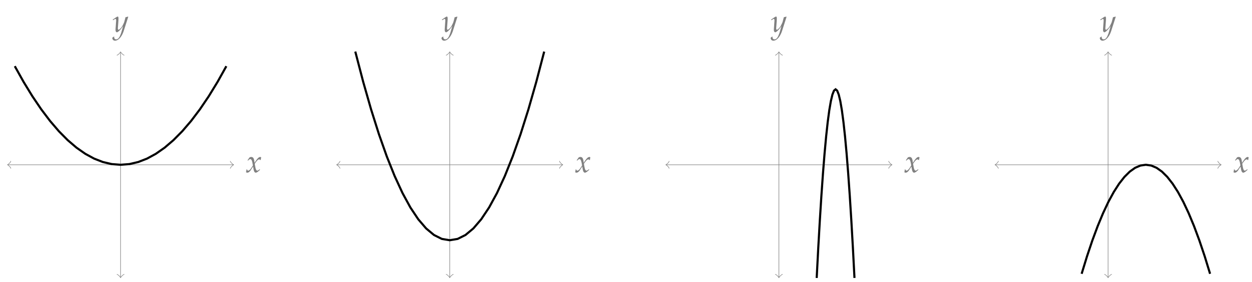
Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Below are the graphs of four different quadratic functions. For each quadratic function, decide whether it is: (i) the product of two distinct linear factors, or (ii) the product of a repeated linear factor (and possibly a constant).



(a) (b) (c) (d)

Q[2](∗): hint answer solution

Write out the **general form** of the partial-fractions decomposition of $\dfrac{x^3+3}{(x^2-1)^2(x+1)}$.

You need not determine the values of any of the coefficients.

Q[3](∗): hint answer solution

Find the coefficient of $\dfrac{1}{x-1}$ in the partial fraction decomposition of $\dfrac{3x^3-2x^2+11}{x^2(x-1)(x+3)}$.

Q[4]: hint answer solution

Re-write the following rational functions as the sum of a polynomial and a rational function whose numerator has a strictly smaller degree than its denominator. (Remember our method of partial fraction decomposition of a rational function only works when the degree of the numerator is strictly smaller than the degree of the denominator.)

(a) $\dfrac{x^3+2x+2}{x^2+1}$

(b) $\dfrac{15x^4+6x^3+34x^2+4x+20}{5x^2+2x+8}$

(c) $\dfrac{2x^5+9x^3+12x^2+10x+30}{2x^2+5}$

Q[5]: hint answer solution

Factor the following polynomials into linear factors.

(a) $5x^3 - 3x^2 - 10x + 6$

(b) $x^4 - 11x^2 + 30$

Q[6]: hint answer solution

Factor the following polynomials into linear factors.

(a) $x^4 - 3x^3 - 15x^2 + 15x + 50$

(b) $2x^4 + 12x^3 - x^2 - 52x + 15$

Q[7]: hint answer solution

Here is a fact:

> Suppose we have a rational function with a repeated linear factor $(ax + b)^n$ in the denominator, and the degree of the numerator is strictly less than the degree of the denominator. In the partial fraction decomposition, we can replace the terms
>
> $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n} \tag{1}$$
>
> with the single term
>
> $$\frac{B_1 + B_2x + B_3x^2 + \cdots + B_nx^{n-1}}{(ax+b)^n} \tag{2}$$
>
> and still be guaranteed to find a solution.

Why do we use the sum in (1), rather than the single term in (2), in partial fraction decomposition?

▸▸ Stage 2

Q[8](∗): hint answer solution

Evaluate $\displaystyle\int_1^2 \frac{\mathrm{d}x}{x + x^2}$.

Q[9](∗): hint answer solution

Evaluate $\displaystyle\int \frac{x - 13}{x^2 - x - 6}\mathrm{d}x$.

Q[10](∗): hint answer solution

Evaluate $\displaystyle\int \frac{5x + 1}{x^2 + 5x + 6}\mathrm{d}x$.

Q[11]: hint answer solution

Evaluate $\displaystyle\int \frac{5x^2-3x-1}{x^2-1}\,dx$.

Q[12]: hint answer solution

Evaluate $\displaystyle\int \frac{x^2+2x-1}{x^4-2x^3+x^2}\,dx$.

Q[13]: hint answer solution

Evaluate $\displaystyle\int \frac{9x^2+36x+35}{(3x+5)^3}dx$

Q[14]: hint answer solution

Evaluate $\displaystyle\int \frac{8x^2-27x+29}{(x-4)(2x-1)^2}dx$

Q[15]: hint answer solution

Evaluate $\displaystyle\int \frac{13x^3-20x^2+x-6}{(x^2-1)^2}dx$

Q[16]: hint answer solution

Evaluate $\displaystyle\int_0^1 \frac{10x^2-68x+102}{(x-5)(x-3)^2}dx$

Q[17]: hint answer solution

Evaluate $\displaystyle\int \frac{3x^2-4x-10}{2x^3-x^2-8x+4}\,dx$.

▸▸ Stage 3

Q[18](*): hint answer solution

Calculate $\displaystyle\int \frac{1}{x^4+x^2}\,dx$.

In Questions 19 and 20, we use partial fraction to find the antiderivatives of two important functions: cosecant, and cosecant cubed.

Q[19]: hint answer solution

Using the method of Example 3.8.4 in the text, integrate $\displaystyle\int \csc x\,dx$.

Q[20]: hint answer solution

Using the method of Example 3.8.5 in the text, integrate $\displaystyle\int \csc^3 x\,dx$.

In Questions 21 through 23, we use substitution to turn a non-rational integrand into a rational integrand, then evaluate the resulting integral using partial fraction. Till now, the partial fraction problems you've seen have all looked largely the same, but keep in mind that a partial fraction decomposition can be a small step in a larger problem.

Q[21]: hint answer solution

Evaluate $\displaystyle\int \frac{\cos\theta}{3\sin\theta + \cos^2\theta - 3}\,d\theta$.

Q[22]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{e^{2t} - e^t - 2}\,dt$.

Q[23]: hint answer solution

Evaluate $\displaystyle\int \sqrt{1+e^x}\,dx$ using partial fraction decomposition.

Q[24]: hint answer solution

Find the area of the finite region bounded by the curves $y = \dfrac{4}{3+x^2}$, $y = \dfrac{2}{x(x+1)}$, $x = \dfrac{1}{4}$, and $x = 3$.

Q[25]: hint answer solution

Let $F(x) = \displaystyle\int_1^x \frac{1}{t^2-9}\,dt$.

(a) Give a formula for $F(x)$ that does not involve an integral.

(b) Find $F'(x)$.

Questions 26 through 29 include fractional powers of x. To evaluate, first make the appropriate substitution.

Q[26]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{10x^{6/5} + 5x}\,dx$ using the substitution $u^5 = x$.

Q[27]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{x^{3/2} - x^{4/3}}\,dx$ using the substitution $u^6 = x$.

Q[28]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{x + 3x^{2/3} + 2x^{1/3}}\,dx$.

Q[29]: hint answer solution

Evaluate $\displaystyle\int \frac{\sqrt{x-5}}{x-7}\,dx$ using the substitution $u = \sqrt{x-5}$.

3.9▲ Numerical Integration

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

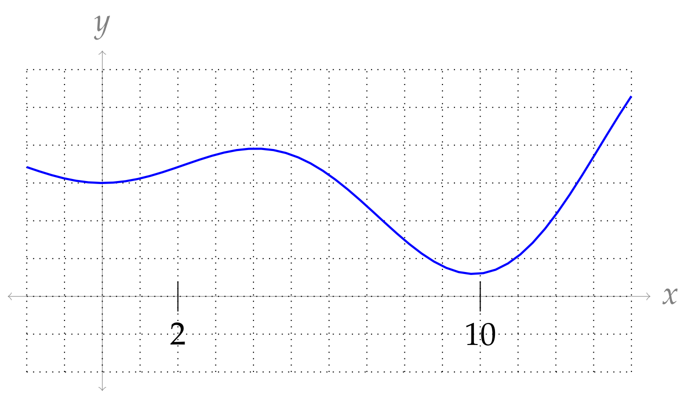
▸▸ Stage 1

Q[1]: hint answer solution

Suppose we approximate an object to have volume 1.5m^3, when its exact volume is 1.387m^3. Give the relative error, absolute error, and percent error of our approximation.

Q[2]: hint answer solution

Consider approximating $\displaystyle\int_2^{10} f(x)\,\mathrm{d}x$, where $f(x)$ is the function in the graph below.



(a) Draw the rectangles associated with the midpoint rule approximation and $n = 4$.

(b) Draw the trapezoids associated with the trapezoidal rule approximation and $n = 4$.

You don't have to give an approximation.

Q[3]: hint answer solution

Let $f(x) = -\dfrac{1}{12}x^4 + \dfrac{7}{6}x^3 - 3x^2$.

(a) Find a reasonable value M such that $|f''(x)| \leqslant M$ for all $1 \leqslant x \leqslant 6$.

(b) Find a reasonable value L such that $|f^{(4)}(x)| \leqslant L$ for all $1 \leqslant x \leqslant 6$.

Q[4]: hint answer solution

Let $f(x) = x\sin x + 2\cos x$. Find a reasonable value M such that $|f''(x)| \leqslant M$ for all $-3 \leqslant x \leqslant 2$.

Q[5]: hint answer solution

Consider the quantity $A = \int_{-\pi}^{\pi} \cos x \, dx$.

(a) Find the upper bound on the error using Simpson's rule with $n = 4$ to approximate A using Theorem 3.9.12 in the text.
(b) Find the Simpson's rule approximation of A using $n = 4$.
(c) What is the (actual) absolute error in the Simpson's rule approximation of A with $n = 4$?

Q[6]: hint answer solution

Give a function $f(x)$ such that:

- $f''(x) \leqslant 3$ for every x in $[0, 1]$, and
- the error using the trapezoidal rule approximating $\int_0^1 f(x) \, dx$ with $n = 2$ intervals is exactly $\frac{1}{16}$.

Q[7]: hint answer solution

Suppose my mother is under 100 years old, and I am under 200 years old.[2] Who is older?

Q[8]: hint answer solution

(a) True or False: for fixed positive constants M, n, a, and b, with $b > a$,

$$\frac{M}{24} \frac{(b-a)^3}{n^2} \leqslant \frac{M}{12} \frac{(b-a)^3}{n^2}$$

(b) True or False: for a function $f(x)$ and fixed constants n, a, and b, with $b > a$, the n-interval midpoint approximation of $\int_a^b f(x) \, dx$ is more accurate than the n-interval trapezoidal approximation.

Q[9](*): hint answer solution

Decide whether the following statement is true or false. If false, provide a counterexample. If true, provide a brief justification.

When $f(x)$ is positive and concave up, any trapezoidal rule approximation for $\int_a^b f(x) \, dx$ will be an upper estimate for $\int_a^b f(x) \, dx$.

Q[10]: hint answer solution

Give a polynomial $f(x)$ with the property that the Simpson's rule approximation of $\int_a^b f(x) \, dx$ is exact for all a, b, and n.

2 We're going somewhere with this.

▸▸ Stage 2

Questions 11 and 12 ask you to approximate a given integral using the formulas in Equations 3.9.2, 3.9.6, and 3.9.9 in the text.

Q[11]: hint answer solution

Write out all three approximations of $\displaystyle\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x$ with $n = 6$. (That is: midpoint, trapezoidal, and Simpson's.) You do not need to simplify your answers.

Q[12](*): hint answer solution

Find the midpoint rule approximation to $\displaystyle\int_0^{\pi} \sin x\,\mathrm{d}x$ with $n = 3$.

Questions 13 though 17 ask you to approximate a quantity based on observed data.

Q[13](*): hint answer solution

The solid V is 40 cm high and the horizontal cross sections are circular disks. The table below gives the diameters of the cross sections in centimetres at 10 cm intervals. Use the trapezoidal rule to estimate the volume of V.

height	0	10	20	30	40
diameter	24	16	10	6	4

Q[14](*): hint answer solution

A 6 metre long cedar log has cross sections that are approximately circular. The diameters of the log, measured at one metre intervals, are given below:

metres from left end of log	0	1	2	3	4	5	6
diameter in metres	1.2	1	0.8	0.8	1	1	1.2

Use Simpson's Rule to estimate the volume of the log.

Q[15](*): hint answer solution

The circumference of an 8 metre high tree at different heights above the ground is given in the table below. Assume that all horizontal cross–sections of the tree are circular disks.

height (metres)	0	2	4	6	8
circumference (metres)	1.2	1.1	1.3	0.9	0.2

Use Simpson's rule to approximate the volume of the tree.

Q[16](∗): hint answer solution

By measuring the areas enclosed by contours on a topographic map, a geologist determines the cross sectional areas A in m^2 of a 60 m high hill. The table below gives the cross sectional area $A(h)$ at various heights h. The volume of the hill is $V = \int_0^{60} A(h)\,\mathrm{d}h$.

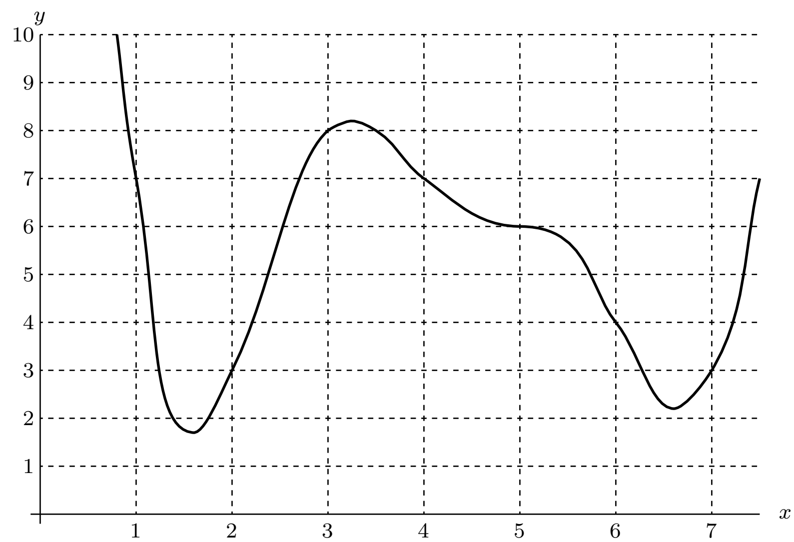
h	0	10	20	30	40	50	60
A	10,200	9,200	8,000	7,100	4,500	2,400	100

(a) If the geologist uses the Trapezoidal Rule to estimate the volume of the hill, what will be their estimate, to the nearest $1{,}000\mathrm{m}^3$?

(b) What will be the geologist's estimate of the volume of the hill if they use Simpson's Rule instead of the Trapezoidal Rule?

Q[17](∗): hint answer solution

The graph below applies to both parts (a) and (b).



(a) Use the Trapezoidal Rule, with $n = 4$, to estimate the area under the graph between $x = 2$ and $x = 6$. Simplify your answer completely.

(b) Use Simpson's Rule, with $n = 4$, to estimate the area under the graph between $x = 2$ and $x = 6$.

In Questions 18 through 24, we practice finding error bounds for our approximations.

Q[18](∗): hint answer solution

The integral $\displaystyle\int_{-1}^{1} \sin(x^2)\,\mathrm{d}x$ is estimated using the Midpoint Rule with 1000 intervals.

Show that the absolute error in this approximation is at most $2 \cdot 10^{-6}$.

You may use the fact that when approximating $\int_a^b f(x)\,\mathrm{d}x$ with the Midpoint Rule using n points, the absolute value of the error is at most $M(b-a)^3/24n^2$ when $|f''(x)| \leqslant M$ for all $x \in [a,b]$.

Q[19](∗): hint answer solution

The total error using the midpoint rule with n subintervals to approximate the integral of $f(x)$ over $[a, b]$ is bounded by $\dfrac{M(b-a)^3}{(24n^2)}$, if $|f''(x)| \leqslant M$ for all $a \leqslant x \leqslant b$.

Using this bound, if the integral $\displaystyle\int_{-2}^{1} 2x^4\,\mathrm{d}x$ is approximated using the midpoint rule with 60 subintervals, what is the largest possible error between the approximation M_{60} and the true value of the integral?

Q[20](*): hint answer solution

Both parts of this question concern the integral $I = \displaystyle\int_0^2 (x-3)^5\,\mathrm{d}x$.

(a) Write down the Simpson's Rule approximation to I with $n = 6$. Leave your answer in calculator-ready form.

(b) Which method of approximating I results in a smaller error bound: the Midpoint Rule with $n = 100$ intervals, or Simpson's Rule with $n = 10$ intervals? You may use the formulas

$$|E_M| \leqslant \frac{M(b-a)^3}{24n^2} \qquad \text{and} \qquad |E_S| \leqslant \frac{L(b-a)^5}{180n^4},$$

where M is an upper bound for $|f''(x)|$ and L is an upper bound for $|f^{(4)}(x)|$, and E_M and E_S are the absolute errors arising from the midpoint rule and Simpson's rule, respectively.

Q[21](*): hint answer solution

Find a bound for the error in approximating $\displaystyle\int_1^5 \frac{1}{x}\,\mathrm{d}x$ using Simpson's rule with $n = 4$. Do not write down the Simpson's rule approximation S_4.

In general the error in approximating $\int_a^b f(x)\ \mathrm{d}x$ using Simpson's rule with n steps is bounded by $\dfrac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \dfrac{b-a}{n}$ and $L \geqslant |f^{(4)}(x)|$ for all $a \leqslant x \leqslant b$.

Q[22](*): hint answer solution

Find a bound for the error in approximating

$$\int_0^1 \left(e^{-2x} + 3x^3\right)\mathrm{d}x$$

using Simpson's rule with $n = 6$. Do not write down the Simpson's rule approximation S_n.

In general, the error in approximating $\int_a^b f(x)\ \mathrm{d}x$ using Simpson's rule with n steps is bounded by $\dfrac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \dfrac{b-a}{n}$ and $L \geqslant |f^{(4)}(x)|$ for all $a \leqslant x \leqslant b$.

Q[23](*): hint answer solution

Let $I = \displaystyle\int_1^2 (1/x)\,\mathrm{d}x$.

(a) Write down the trapezoidal approximation T_4 for I. *You do not need to simplify your answer.*

(b) Write down the Simpson's approximation S_4 for I. *You do not need to simplify your answer.*

(c) Without computing I, find an upper bound for $|I - S_4|$. You may use the fact that if $\left|f^{(4)}(x)\right| \leqslant L$ on the interval $[a, b]$, then the error in using S_n to approximate $\int_a^b f(x)\,\mathrm{d}x$ has absolute value less than or equal to $L(b-a)^5/180n^4$.

Since $\dfrac{1}{x^5}$ is a decreasing function when $x > 0$, look for its maximum value when x is as small as possible.

Q[24](*): hint answer solution

A function $s(x)$ satisfies $s(0) = 1.00664$, $s(2) = 1.00543$, $s(4) = 1.00435$, $s(6) = 1.00331$, $s(8) = 1.00233$. Also, it is known to satisfy $\left|s^{(k)}(x)\right| \leqslant \dfrac{k}{1000}$ for $0 \leqslant x \leqslant 8$ and all positive integers k.

(a) Find the best Trapezoidal Rule and Simpson's Rule approximations that you can for $I = \displaystyle\int_0^8 s(x)\,\mathrm{d}x$.

(b) Determine the maximum possible sizes of errors in the approximations you gave in part (a). Recall that if a function $f(x)$ satisfies $\left|f^{(k)}(x)\right| \leqslant K_k$ on $[a, b]$, then

$$\left|\int_a^b f(x)\,\mathrm{d}x - T_n\right| \leqslant \frac{K_2(b-a)^3}{12n^2} \quad \text{and} \quad \left|\int_a^b f(x)\,\mathrm{d}x - S_n\right| \leqslant \frac{K_4(b-a)^5}{180n^4}$$

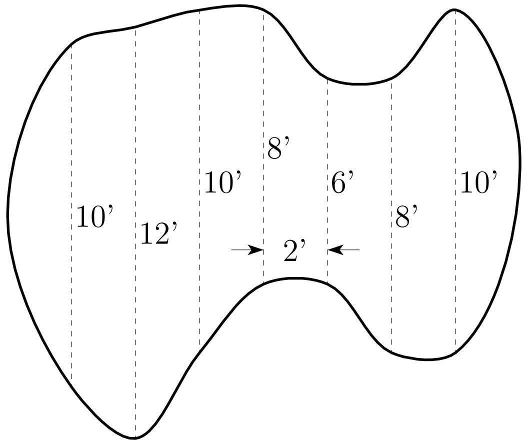
Q[25](*): hint answer solution

Consider the trapezoidal rule for making numerical approximations to $\displaystyle\int_a^b f(x)\,\mathrm{d}x$. The error for the trapezoidal rule satisfies $|E_T| \leqslant \dfrac{M(b-a)^3}{12n^2}$, where $|f''(x)| \leqslant M$ for $a \leqslant x \leqslant b$. If $-2 < f''(x) < 0$ for $1 \leqslant x \leqslant 4$, find a value of n to guarantee the trapezoidal rule will give an approximation for $\displaystyle\int_1^4 f(x)\,\mathrm{d}x$ with absolute error, $|E_T|$, less than 0.001.

▸▸ Stage 3

Q[26](*): hint answer solution

A swimming pool has the shape shown in the figure below. The vertical cross–sections of the pool are semi–circular disks. The distances in feet across the pool are given in the figure at 2–foot intervals along the sixteen–foot length of the pool. Use Simpson's Rule to estimate the volume of the pool.



Q[27](*): hint answer solution

A piece of wire 1m long with radius 1mm is made in such a way that the density varies in its cross–section, but is radially symmetric (that is, the local density $g(r)$ in kg/m^3 depends only on the distance r in mm from the centre of the wire). Take as given that the total mass W of the wire in kg is given by

$$W = 2\pi 10^{-6} \int_0^1 rg(r)\,\mathrm{d}r$$

Data from the manufacturer is given below:

r	0	1/4	1/2	3/4	1
$g(r)$	8051	8100	8144	8170	8190

(a) Find the best Trapezoidal Rule approximation that you can for W based on the data in the table.

(b) Suppose that it is known that $|g'(r)| < 200$ and $|g''(r)| < 150$ for all values of r. Determine the maximum possible size of the error in the approximation you gave in part (a). Recall that if a function $f(x)$ satisfies $|f''(x)| \leqslant M$ on $[a, b]$, then

$$|I - T_n| \leqslant \frac{M(b-a)^3}{12n^2}$$

where $I = \int_a^b f(x)\,\mathrm{d}x$ and T_n is the Trapezoidal Rule approximation to I using n subintervals.

Q[28](*): hint answer solution

Simpson's rule can be used to approximate $\ln 2$, since $\ln 2 = \displaystyle\int_1^2 \frac{1}{x}\,\mathrm{d}x$.

(a) Use Simpson's rule with 6 subintervals to approximate $\ln 2$.

(b) How many subintervals are required in order to guarantee that the absolute error is less than 0.00001?

Note that if E_n is the error using n subintervals, then $|E_n| \leqslant \dfrac{L(b-a)^5}{180n^4}$ where L is the maximum absolute value of the fourth derivative of the function being integrated and a and b are the end points of the interval.

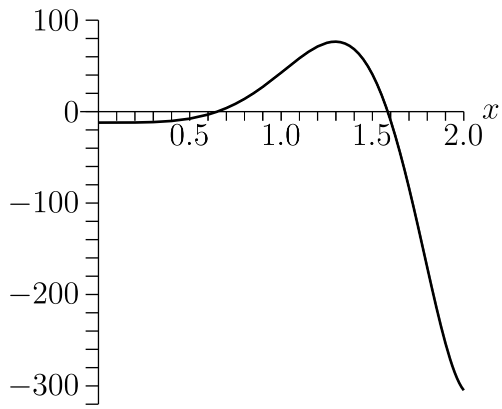
Q[29](*): hint answer solution

Let $I = \int_0^2 \cos(x^2)\,\mathrm{d}x$ and let S_n be the Simpson's rule approximation to I using n subintervals.

(a) Estimate the maximum absolute error in using S_8 to approximate I.

(b) How large should n be in order to ensure that $|I - S_n| \leqslant 0.0001$?

Note: The graph of $f''''(x)$, where $f(x) = \cos(x^2)$, is shown below. The absolute error in the Simpson's rule approximation is bounded by $\frac{L(b-a)^5}{180n^4}$ when $|f''''(x)| \leqslant L$ on the interval $[a, b]$.



Q[30](*): hint answer solution

Define a function $f(x)$ and an integral I by

$$f(x) = \int_0^{x^2} \sin(\sqrt{t})\,\mathrm{d}t, \qquad I = \int_0^1 f(t)\,\mathrm{d}t$$

Estimate how many subdivisions are needed to calculate I to five decimal places of accuracy using the trapezoidal rule.

Note that if E_n is the error using n subintervals, then $|E_n| \leqslant \frac{M(b-a)^3}{12n^2}$, where M is the maximum absolute value of the second derivative of the function being integrated and a and b are the limits of integration.

Q[31]: hint answer solution

Let $f(x)$ be a function[3] with $f''(x) = \frac{x^2}{x+1}$.

(a) Show that $|f''(x)| \leqslant 1$ whenever x is in the interval $[0, 1]$.

(b) Find the maximum value of $|f''(x)|$ over the interval $[0, 1]$.

3 For example, $f(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + (1+x)\ln|x+1|$ will do, but you don't need to know what $f(x)$ is for this problem.

(c) Assuming $M = 1$, how many intervals should you use to approximate $\int_0^1 f(x)\,\mathrm{d}x$ to within 10^{-5}?

(d) Using the value of M you found in (b), how many intervals should you use to approximate $\int_0^1 f(x)\,\mathrm{d}x$ to within 10^{-5}?

Q[32]: hint answer solution
Approximate the function $\ln x$ with a rational function by approximating the integral $\int_1^x \frac{1}{t}\,\mathrm{d}t$ using Simpson's rule. Your rational function $f(x)$ should approximate $\ln x$ with an error of not more than 0.1 for any x in the interval $[1, 3]$.

Q[33]: hint answer solution
Using an approximation of the area under the curve $\frac{1}{x^2+1}$, show that the constant $\arctan 2$ is in the interval $\left[\frac{\pi}{4} + 0.321, \frac{\pi}{4} + 0.323\right]$.

You may assume use without proof that $\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{\frac{1}{1+x^2}\right\} = \frac{24(5x^4 - 10x^2 + 1)}{(x^2+1)^5}$. You may use a calculator, but only to add, subtract, multiply, and divide.

3.10▲ Improper Integrals

Exercises

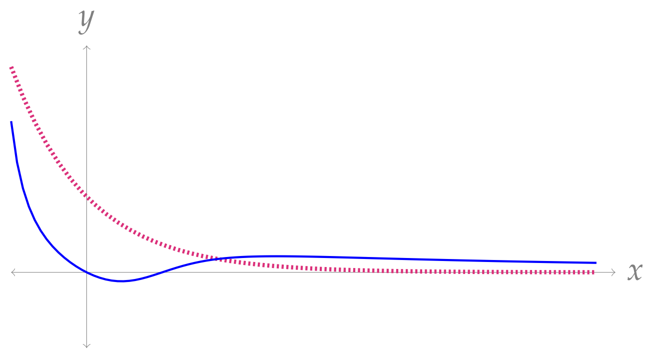
Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution
For which values of b is the integral $\int_0^b \frac{1}{x^2-1}\,\mathrm{d}x$ improper?

Q[2]: hint answer solution
For which values of b is the integral $\int_0^b \frac{1}{x^2+1}\,\mathrm{d}x$ improper?

Q[3]: hint answer solution
Below are the graphs $y = f(x)$ and $y = g(x)$. Suppose $\int_0^\infty f(x)\,\mathrm{d}x$ converges, and $\int_0^\infty g(x)\,\mathrm{d}x$ diverges. Assuming the graphs continue on as shown as $x \to \infty$, which graph is $f(x)$, and which is $g(x)$?



Note: one function is shown as a red dotted line, and the other is shown as a blue solid line. Both functions are continuous – the dots are only there to distinguish the two functions, not to imply discontinuities.

Q[4](*): hint answer solution

Decide whether the following statement is true or false. If false, provide a counterexample. If true, provide a brief justification. (Assume that $f(x)$ and $g(x)$ are continuous functions.)

$$\text{If } \int_1^\infty f(x)\,\mathrm{d}x \text{ converges and } g(x) \geqslant f(x) \geqslant 0 \text{ for all } x\text{, then } \int_1^\infty g(x)\,\mathrm{d}x \text{ converges.}$$

Q[5]: hint answer solution

Let $f(x) = e^{-x}$ and $g(x) = \dfrac{1}{x+1}$. Note $\int_0^\infty f(x)\,\mathrm{d}x$ converges while $\int_0^\infty g(x)\,\mathrm{d}x$ diverges.

For each of the functions $h(x)$ described below, decide whether $\int_0^\infty h(x)\,\mathrm{d}x$ converges or diverges, or whether there isn't enough information to decide. Justify your decision.

(a) $h(x)$, continuous and defined for all $x \geqslant 0$, $h(x) \leqslant f(x)$.

(b) $h(x)$, continuous and defined for all $x \geqslant 0$, $f(x) \leqslant h(x) \leqslant g(x)$.

(c) $h(x)$, continuous and defined for all $x \geqslant 0$, $-2f(x) \leqslant h(x) \leqslant f(x)$.

▸▸ Stage 2

Q[6](*): hint answer solution

Evaluate the integral $\displaystyle\int_0^1 \frac{x^4}{x^5-1}\,\mathrm{d}x$ or state that it diverges.

Q[7](*): hint answer solution

Determine whether the integral $\displaystyle\int_{-2}^2 \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x$ is convergent or divergent. If it is convergent, find its value.

Q[8](∗): hint answer solution

Does the improper integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{4x^2-x}}\,dx$ converge? Justify your answer.

Q[9](∗): hint answer solution

Does the integral $\displaystyle\int_0^\infty \frac{dx}{x^2+\sqrt{x}}$ converge or diverge? Justify your claim.

Q[10]: hint answer solution

Does the integral $\displaystyle\int_{-\infty}^\infty \cos x\,dx$ converge or diverge? If it converges, evaluate it.

Q[11]: hint answer solution

Does the integral $\displaystyle\int_{-\infty}^\infty \sin x\,dx$ converge or diverge? If it converges, evaluate it.

Q[12]: hint answer solution

Evaluate $\displaystyle\int_{10}^\infty \frac{x^4-5x^3+2x-7}{x^5+3x+8}\,dx$, or state that it diverges.

Q[13]: hint answer solution

Evaluate $\displaystyle\int_0^{10} \frac{x-1}{x^2-11x+10}\,dx$, or state that it diverges.

Q[14](∗): hint answer solution

Determine (with justification!) which of the following applies to the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{x^2+1}dx$:

(i) $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{x^2+1}dx$ diverges

(ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{x^2+1}dx$ converges but $\displaystyle\int_{-\infty}^{+\infty} \left|\frac{x}{x^2+1}\right| dx$ diverges

(iii) $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{x^2+1}dx$ converges, as does $\displaystyle\int_{-\infty}^{+\infty} \left|\frac{x}{x^2+1}\right| dx$

Remark: these options, respectively, are that the integral diverges, converges conditionally, and converges absolutely. You'll see this terminology used for series in Section 5.4 of the text.

Q[15](∗): hint answer solution

Decide whether $I = \displaystyle\int_0^\infty \frac{|\sin x|}{x^{3/2}+x^{1/2}}\,dx$ converges or diverges. Justify.

▸▸ Stage 3

Q[16](*): hint answer solution

What is the largest value of q for which the integral $\displaystyle\int_1^\infty \frac{1}{x^{5q}}\,dx$ diverges?

Q[17]: hint answer solution

For which values of p does the integral $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^p}\,dx$ converge?

Q[18]: hint answer solution

Does the integral $\displaystyle\int_{-5}^{5}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)dx$ converge or diverge?

Q[19]: hint answer solution

Evaluate $\displaystyle\int_0^\infty e^{-x}\sin x\,dx$, or state that it diverges.

Q[20](*): hint answer solution

Is the integral $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^2}\,dx$ convergent or divergent? Explain why.

Q[21]: hint answer solution

Does the integral $\displaystyle\int_0^\infty \frac{x}{e^x+\sqrt{x}}\,dx$ converge or diverge?

Q[22](*): hint answer solution

Let $M_{n,t}$ be the Midpoint Rule approximation for $\displaystyle\int_0^t \frac{e^{-x}}{1+x}\,dx$ with n equal subintervals.

Find a value of t and a value of n such that $M_{n,t}$ differs from $\int_0^\infty \frac{e^{-x}}{1+x}\,dx$ by at most 10^{-4}. Recall that the error E_n introduced when the Midpoint Rule is used with n subintervals obeys

$$|E_n| \leqslant \frac{M(b-a)^3}{24n^2}$$

where M is the maximum absolute value of the second derivative of the integrand and a and b are the end points of the interval of integration.

Q[23]: hint answer solution

Suppose $f(x)$ is continuous for all real numbers, and $\displaystyle\int_1^\infty f(x)\,dx$ converges.

(a) If $f(x)$ is odd, does $\displaystyle\int_{-\infty}^{-1} f(x)\,dx$ converge or diverge, or is there not enough information to decide?

(b) If $f(x)$ is even, does $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ converge or diverge, or is there not enough information to decide?

Q[24]: hint answer solution

True or false: There is some real number x, with $x \geqslant 1$, such that $\displaystyle\int_0^x \frac{1}{e^t}\,\mathrm{d}t = 1$.

3.11▲ Overview of Integration Techniques

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Match the integration method to a common kind of integrand it's used to antidifferentiate.

(A)	u-substitution	(I)	a function multiplied by its derivative
(B)	trigonometric substitution	(II)	a polynomial function times an exponential function
(C)	integration by parts	(III)	a rational function
(D)	partial fractions	(IV)	the square root of a quadratic function

▸▸ Stage 2

Q[2]: hint answer solution

Evaluate $\displaystyle\int_0^{\pi/2} \sin^4 x \cos^5 x\,\mathrm{d}x$.

Q[3]: hint answer solution

Evaluate $\displaystyle\int \sqrt{3-5x^2}\,\mathrm{d}x$.

Q[4]: hint answer solution

Evaluate $\displaystyle\int_0^\infty \frac{x-1}{e^x}\,\mathrm{d}x$.

Q[5]: hint answer solution

Evaluate $\displaystyle\int \frac{-2}{3x^2+4x+1}\mathrm{d}x$.

Q[6]: hint answer solution

Evaluate $\displaystyle\int_1^2 x^2 \ln x\,\mathrm{d}x$.

Q[7](∗): hint answer solution

Evaluate $\displaystyle\int \frac{x}{x^2-3}\,\mathrm{d}x$.

Q[8](∗): hint answer solution

Evaluate the following integrals.

(a) $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}\,\mathrm{d}x$

(b) $\displaystyle\int_0^{\pi/2} \cos^3 x\ \sin^2 x \, dx$

(c) $\displaystyle\int_1^{e} x^3 \ln x \, dx$

Q[9](*): hint answer solution
Evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/2} x \sin x \, dx$

(b) $\displaystyle\int_0^{\pi/2} \cos^5 x \, dx$

Q[10](*): hint answer solution
Evaluate the following integrals.

(a) $\displaystyle\int_0^{2} x e^x \, dx$

(b) $\displaystyle\int_0^{1} \frac{1}{\sqrt{1+x^2}} \, dx$

(c) $\displaystyle\int_3^{5} \frac{4x}{(x^2-1)(x^2+1)} \, dx$

Q[11](*): hint answer solution
Calculate the following integrals.

(a) $\displaystyle\int_0^{3} \sqrt{9-x^2} \, dx$

(b) $\displaystyle\int_0^{1} \ln(1+x^2) \, dx$

(c) $\displaystyle\int_3^{\infty} \frac{x}{(x-1)^2(x-2)} \, dx$

Q[12]: hint answer solution
Evaluate $\displaystyle\int \frac{\sin^4\theta - 5\sin^3\theta + 4\sin^2\theta + 10\sin\theta}{\sin^2\theta - 5\sin\theta + 6} \cos\theta \, d\theta.$

Q[13](*): hint answer solution
Evaluate the following integrals. *Show your work.*

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2(2x)\cos^3(2x) \, dx$

(b) $\displaystyle\int \left(9+x^2\right)^{-\frac{3}{2}} \, dx$

(c) $\displaystyle\int \frac{dx}{(x-1)(x^2+1)}$

(d) $\displaystyle\int x \arctan x \, \mathrm{d}x$

Q[14](*): hint answer solution

Evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/4} \sin^5(2x)\,\cos(2x)\,\mathrm{d}x$

(b) $\displaystyle\int \sqrt{4-x^2}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{x+1}{x^2(x-1)}\,\mathrm{d}x$

Q[15](*): hint answer solution

Calculate the following integrals.

(a) $\displaystyle\int_0^{\infty} e^{-x}\sin(2x)\,\mathrm{d}x$

(b) $\displaystyle\int_0^{\sqrt{2}} \frac{1}{(2+x^2)^{3/2}}\,\mathrm{d}x$

(c) $\displaystyle\int_0^1 x\ln(1+x^2)\,\mathrm{d}x$

(d) $\displaystyle\int_3^{\infty} \frac{1}{(x-1)^2(x-2)}\,\mathrm{d}x$

Q[16](*): hint answer solution

Evaluate the following integrals.

(a) $\displaystyle\int x\,\ln x\,\mathrm{d}x$

(b) $\displaystyle\int \frac{(x-1)\,\mathrm{d}x}{x^2+4x+5}$

(c) $\displaystyle\int \frac{\mathrm{d}x}{x^2-4x+3}$

(d) $\displaystyle\int \frac{x^2\,\mathrm{d}x}{1+x^6}$

Q[17](*): hint answer solution

Evaluate the following integrals.

(a) $\displaystyle\int_0^1 \arctan x\,\mathrm{d}x.$

(b) $\displaystyle\int \frac{2x-1}{x^2-2x+5}\,\mathrm{d}x.$

Q[18](*): hint answer solution

(a) Evaluate $\displaystyle\int \frac{x^2}{(x^3+1)^{101}}\,dx$.

(b) Evaluate $\displaystyle\int \cos^3 x\ \sin^4 x\,dx$.

Q[19]: hint answer solution

Evaluate $\displaystyle\int_{\pi/2}^{\pi} \frac{\cos x}{\sqrt{\sin x}}\,dx$.

Q[20](*): hint answer solution

Evaluate the following integrals.

(a) $\displaystyle\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx$

(b) $\displaystyle\int_2^4 \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\,dx$

Q[21](*): hint answer solution

Evaluate these integrals.

(a) $\displaystyle\int \frac{\sin^3 x}{\cos^3 x}\,dx$

(b) $\displaystyle\int_{-2}^{2} \frac{x^4}{x^{10}+16}\,dx$

Q[22]: hint answer solution

Evaluate $\displaystyle\int x\sqrt{x-1}\,dx$.

Q[23]: hint answer solution

Evaluate $\displaystyle\int \frac{\sqrt{x^2-2}}{x^2}dx$.

You may use that $\int \sec x dx = \ln|\sec x + \tan x| + C$.

Q[24]: hint answer solution

Evaluate $\displaystyle\int_0^{\pi/4} \sec^4 x \tan^5 x\,dx$.

Q[25]: hint answer solution

Evaluate $\displaystyle\int \frac{3x^2+4x+6}{(x+1)^3}\,dx$.

Q[26]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{x^2+x+1}\,dx$.

Q[27]: hint answer solution

Evaluate $\displaystyle\int \sin x \cos x \tan x\,dx$.

Q[28]: hint answer solution

Evaluate $\displaystyle\int \frac{1}{x^3+1}\,\mathrm{d}x$.

Q[29]: hint answer solution

Evaluate $\displaystyle\int (3x)^2 \arcsin x\,\mathrm{d}x$.

▸▸ Stage 3

Q[30]: hint answer solution

Evaluate $\displaystyle\int_0^{\pi/2} \sqrt{\cos t + 1}\,\mathrm{d}t$.

Q[31]: hint answer solution

Evaluate $\displaystyle\int_1^{e} \frac{\ln\sqrt{x}}{x}\,\mathrm{d}x$.

Q[32]: hint answer solution

Evaluate $\displaystyle\int_{0.1}^{0.2} \frac{\tan x}{\ln(\cos x)}\,\mathrm{d}x$.

Q[33](*): hint answer solution

Evaluate these integrals.

(a) $\displaystyle\int \sin(\ln x)\,\mathrm{d}x$

(b) $\displaystyle\int_0^1 \frac{1}{x^2-5x+6}\,\mathrm{d}x$

Q[34](*): hint answer solution

Evaluate (with justification).

(a) $\displaystyle\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{4x+8}{(x-2)(x^2+4)}\,\mathrm{d}x$

(c) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{e^x+e^{-x}}\,\mathrm{d}x$

Q[35]: hint answer solution

Evaluate $\displaystyle\int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x$.

Q[36]: hint answer solution

Evaluate $\displaystyle\int_0^1 e^{2x}e^{e^x}\,\mathrm{d}x$.

Q[37]: hint answer solution

Evaluate $\displaystyle\int \frac{xe^x}{(x+1)^2}\,\mathrm{d}x$.

Q[38]: hint answer solution

Evaluate $\displaystyle\int \frac{x \sin x}{\cos^2 x}\, dx$.

You may use that $\int \sec x dx = \ln|\sec x + \tan x| + C$.

Q[39]: hint answer solution

Evaluate $\displaystyle\int x(x+a)^n\, dx$, where a and n are constants.

Q[40]: hint answer solution

Evaluate $\displaystyle\int \arctan(x^2)\, dx$.

3.12▲ Differential Equations

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Below are pairs of functions $y = f(x)$ and differential equations. For each pair, decide whether the function is a solution of the differential equation.

	function	differential equation
(a)	$y = 5(e^x - 3x^2 - 6x - 6)$	$\frac{dy}{dx} = y + 15x^2$
(b)	$y = \frac{-2}{x^2+1}$	$y'(x) = xy^2$
(c)	$y = x^{3/2} + x$	$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = y$

Q[2]: hint answer solution

Following Definition 3.12.3 in the text, a separable differential equation has the form

$$g(y(x))\frac{dy}{dx}(x) = f(x).$$

Show that each of the following equations can be written in this form, identifying $f(x)$ and $g(y)$.

(a) $3y\frac{dy}{dx} = x \sin y$

(b) $\frac{dy}{dx} = e^{x+y}$

(c) $\frac{dy}{dx} + 1 = x$

(d) $\left(\frac{dy}{dx}\right)^2 - 2x\frac{dy}{dx} + x^2 = 0$

Q[3]: hint answer solution

Suppose we have the following functions:

- y is a differentiable function of x
- f is a function of x, with $\int f(x)\,\mathrm{d}x = F(x)$
- g is a nonzero function of y, with $\int g(y)\,\mathrm{d}y = G(y) = G(y(x))$.

In the work below, we set up a solution to the separable differential equation

$$g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

without using the mnemonic in the text.

By deleting some portion of our work, we can create the solution as it would look using the mnemonic. What portion can be deleted?

Remark: the purpose of this exercise is to illuminate what, exactly, the mnemonic is a shortcut for. Despite its peculiar look, it agrees with what we already know about integration.

$$g(y(x)) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

If these functions of x are the same, then they have the same antiderivative with respect to x.

$$\int g(y(x)) \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int f(x)\,\mathrm{d}x$$

The left integral is in the correct form for a change of variables to y. To make this easier to see, we'll use a u-substitution, since it's a little more familiar than a y-substitution. If $u = y$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}$, so $\mathrm{d}u = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x$.

$$\int g(u)\,\mathrm{d}u = \int f(x)\,\mathrm{d}x$$

Since u was just the same as y, again for cosmetic reasons, we can swap it back. (Formally, you could have skipped the step above–we just included it to be extra clear that we're not using any integration techniques we haven't seen before.)

$$\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

We're given the antiderivatives in question.

$$G(y) + C_1 = F(x) + C_2$$
$$G(y) = F(x) + (C_2 - C_1)$$

where C_1 and C_2 are arbitrary constants. Then also $C_2 - C_1$ is an arbitrary constant, so we might as well call it C.

$$G(y) = F(x) + C$$

Q[4]: hint answer solution

Suppose $y = f(x)$ is a solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$.

True or false: $f(x) + C$ is also a solution, for any constant C.

Q[5]: hint answer solution

Suppose a function $y = f(x)$ satisfies $|y| = Cx$, for some constant $C > 0$.

(a) What is the largest possible domain of $f(x)$, given the information at hand?

(b) Give an example of function $y = f(x)$ with the following properties, or show that none exists:

- $|y| = Cx$,
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ exists for all $x > 0$, and
- $y > 0$ for some values of x, and $y < 0$ for others.

Q[6]: hint answer solution

Express the following sentence* as a differential equation. You don't have to solve the equation.

> About 0.3 percent of the total quantity of morphine in the bloodstream is eliminated every minute.

* The sentence is paraphrased from the Pharmakokinetics website of Université de Lausanne, "Elimination Kinetics," at https://sepia.unil.ch/pharmacology/index.php?id=94 . The half-life of morphine is given on the same website at https://sepia.unil.ch/pharmacology/index.php?id=85 . Accessed 12 August 2017.

Q[7]: hint answer solution

Suppose a particular change is occurring in a language, from an old form to a new form.* Let $p(t)$ be the proportion (measured as a number between 0, meaning none, and 1, meaning all) of the time that speakers use the new form. Piotrowski's law† predicts the following.

> Use of the new form over time spreads at a rate that is proportional to the product of the proportion of the new form and the proportion of the old form.

Express this as a differential equation. You do not need to solve the differential equation.

* An example is the change in German from "wollt" to "wollst" for the second-person conjugation of the verb "wollen." This example is provided by the site Laws in Quantitative Linguistics, "Change in Language" http://lql.uni-trier.de/index.php/Change_in_language accessed 18 August 2017.

† Piotrowski's law is paraphrased from the page **Piotrowski-Gesetz** on Glottopedia, http://www.glottopedia.org/index.php/Piotrowski-Gesetz, accessed 18 August 2017. According to this source, the law was based on work by the married couple R. G. Piotrowski and A. A. Piotrowskaja, later generalized by G. Altmann.

Q[8]: hint answer solution

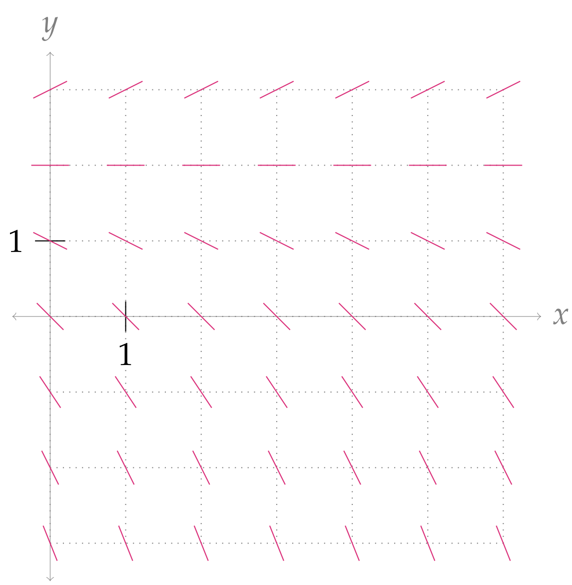
Consider the differential equation $y' = \frac{y}{2} - 1$.

(a) When $y = 0$, what is y'?

(b) When $y = 2$, what is y'?

(c) When $y = 3$, what is y'?

(d) On the axes below, interpret the marks we have made, and use them to sketch a possible solution to the differential equation.



▸▸ Stage 2

Q[9]: hint answer solution

Find the function $y(t)$ satisfying

$$\frac{dy}{dt} = 5y - 7$$

and $y(0) = -3$.

Q[10]: hint answer solution

Find the function $y(t)$ satisfying

$$\frac{dy}{dt} = 1 + 2y$$

and $y(0) = 0$.

Q[11]: hint answer solution

Find the function $y(t)$ satisfying

$$\frac{dy}{dt} = 2y + 3$$

and $y(1) = 1$.

Q[12]: hint answer solution

What is the steady-state solution to the differential equation

$$\frac{dy}{dt} = 3y - 7\ ?$$

Q[13]: hint answer solution

What are the steady-state solutions to the differential equation

$$\frac{dy}{dt} = y(y^2 - 1)\ ?$$

Q[14](*): hint answer solution

Find the solution to the separable initial value problem:

$$\frac{dy}{dx} = \frac{2x}{e^y}, \qquad\qquad y(0) = \ln 2$$

Express your solution explicitly as $y = y(x)$.

Q[15](*): hint answer solution

Find the solution $y(x)$ of $\dfrac{dy}{dx} = \dfrac{xy}{x^2+1}$, $\quad y(0) = 3$.

Q[16](*): hint answer solution

Solve the differential equation $y'(t) = e^{\frac{y}{3}} \cos t$. You should express the solution $y(t)$ in terms of t explicitly.

Q[17](*): hint answer solution

Solve the differential equation

$$\frac{dy}{dx} = xe^{x^2 - \ln(y^2)}$$

Q[18](*): hint answer solution

Let $y = y(x)$. Find the general solution of the differential equation $y' = xe^y$.

Q[19](*): hint answer solution

Find the solution to the differential equation $\dfrac{yy'}{e^x - 2x} = \dfrac{1}{y}$ that satisfies $y(0) = 3$. Solve completely for y as a function of x.

Q[20](*): hint answer solution

Find the function $y = f(x)$ that satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -xy^3 \qquad \text{and} \qquad f(0) = -\frac{1}{4}$$

Q[21](*): hint answer solution

Find the function $y = y(x)$ that satisfies $y(1) = 4$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{15x^2 + 4x + 3}{y}$$

Q[22](*): hint answer solution

Find the solution $y(x)$ of $y' = x^3 y$ with $y(0) = 1$.

Q[23](*): hint answer solution

Solve the initial value problem

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2 \qquad y(1) = -1$$

Q[24](*): hint answer solution

A function $f(x)$ is always positive, has $f(0) = e$ and satisfies $f'(x) = x\,f(x)$ for all x. Find this function.

Q[25](*): hint answer solution

Solve the following initial value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(x^2 + x)y} \qquad y(1) = 2$$

Q[26](*): hint answer solution

Find the solution of the differential equation $\dfrac{1 + \sqrt{y^2 - 4}}{\tan x} y' = \dfrac{\sec x}{y}$ that satisfies $y(0) = 2$. You don't have to solve for y in terms of x.

Q[27](*): hint answer solution

The fish population in a lake is attacked by a disease at time $t = 0$, with the result that the size $P(t)$ of the population at time $t \geqslant 0$ satisfies

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -k\sqrt{P}$$

where k is a positive constant. If there were initially 90,000 fish in the lake and 40,000 were left after 6 weeks, when will the fish population be reduced to 10,000?

Q[28](∗): hint answer solution

An object of mass m is projected straight upward at time $t = 0$ with initial speed v_0. While it is going up, the only forces acting on it are gravity (assumed constant) and a drag force proportional to the square of the object's speed $v(t)$. It follows that the differential equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -(mg + kv^2)$$

where g and k are positive constants. At what time does the object reach its highest point?

Q[29](∗): hint answer solution

A motor boat is traveling with a velocity of 40 ft/sec when its motor shuts off at time $t = 0$. Thereafter, its deceleration due to water resistance is given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -k\,v^2$$

where k is a positive constant. After 10 seconds, the boat's velocity is 20 ft/sec.

(a) What is the value of k?

(b) When will the boat's velocity be 5 ft/sec?

Q[30](∗): hint answer solution

Consider the initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = k(3-x)(2-x)$, $x(0) = 1$, where k is a positive constant. (This kind of problem occurs in the analysis of certain chemical reactions.)

(a) Solve the initial value problem. That is, find x as a function of t.

(b) What value will $x(t)$ approach as t approaches $+\infty$.

Q[31](∗): hint answer solution

The quantity $P = P(t)$, which is a function of time t, satisfies the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 4P - P^2$$

and the initial condition $P(0) = 2$.

(a) Solve this equation for $P(t)$.

(b) What is P when $t = 0.5$? What is the limiting value of P as t becomes large?

Q[32](∗): hint answer solution

An object moving in a fluid has an initial velocity v of 400 m/min. The velocity is decreasing at a rate proportional to the square of the velocity. After 1 minute the velocity is 200 m/min.

(a) Give a differential equation for the velocity $v = v(t)$ where t is time.

(b) Solve this differential equation.

(c) When will the object be moving at 50 m/min?

Q[33](*): hint answer solution
Let $f(r, \theta) = r^m \cos(m\theta)$ be a function of r and θ, where m is a positive integer.

(a) Find the second order partial derivatives f_{rr}, $f_{r\theta}$, $f_{\theta\theta}$ and evaluate their respective values at $(r, \theta) = (1, 0)$.

(b) Determine the value of the real number λ so that $f(r, \theta)$ satisfies the equation

$$f_{rr} + \frac{\lambda}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0$$

▸▸ Stage 3

Q[34](*): hint answer solution
An investor places some money in a mutual fund where the interest is compounded continuously and where the interest rate fluctuates between 4% and 8%. Assume that the amount of money $B = B(t)$ in the account in dollars after t years satisfies the differential equation

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \big(0.06 + 0.02 \sin t\big) B$$

(a) Solve this differential equation for B as a function of t.
(b) If the initial investment is \$1000, what will the balance be at the end of two years?

Q[35](*): hint answer solution
A certain continuous function $y = y(x)$ satisfies the integral equation

$$y(x) = 3 + \int_0^x \big(y(t)^2 - 3y(t) + 2\big) \sin t \, dt \qquad (*)$$

for all x in some open interval containing 0. Find $y(x)$ and the largest interval for which $(*)$ holds.

Q[36](*): hint answer solution
Consider the equation

$$f(x) = 3 + \int_0^x \big(f(t) - 1\big)\big(f(t) - 2\big) \, \mathrm{d}t$$

(a) What is $f(0)$?

(b) Find the differential equation satisfied by $f(x)$.

(c) Solve the initial value problem determined in (a) and (b).

Q[37]: hint answer solution
Suppose $f(t)$ is a continuous, differentiable function and the root mean square of $f(t)$ on $[a, x]$ is equal to the average of $f(t)$ on $[a, x]$ for all x. That is,

$$\frac{1}{x-a} \int_a^x f(t) \, \mathrm{d}(t) = \sqrt{\frac{1}{x-a} \int_a^x f^2(t) \, \mathrm{d}t} \qquad (*)$$

You may assume $x > a$.

(a) Guess a function $f(t)$ for which the average of $f(t)$ is the same as the root mean square of $f(t)$ on any interval.

(b) Differentiate both sides of the given equation.

(c) Simplify your answer from (b) by using Equation ($*$) to replace all terms containing $\int_a^x f^2(t)\,\mathrm{d}t$ with terms containing $\int_a^x f(t)\,\mathrm{d}t$.

(d) Let $Y(x) = \int_a^x f(t)\,\mathrm{d}t$, so the equation from (c) becomes a differential equation. Find all functions that satisfy it.

(e) What is $f(t)$?

Q[38]: hint answer solution

Find the function $y(x)$ such that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2}{y^3} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

and if $x = -\frac{1}{16}\ln 3$, then $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$.

You do not need to solve for y explicitly.

Q[39]: hint answer solution

In the paper *Mathematics of Marital Conflict: Qualitative Dynamic Mathematical Modeling of Marital Interaction*[*] the authors investigate the interactions between opposite-gender married couples.

Qualitative aspects of interactions are coded as correlating positively or negatively with marital success, as investigated in previous research, leading to a score that changes as the interaction progresses. Denote the score as $W(t)$ for wives and $H(t)$ for husbands. We'll call these functions *demeanours*.

The proposed model was that the change in a person's demeanour was dependent on three things:

1. their emotional volatility (how easily they changed emotional states);
2. influence from their partner; and
3. a tendency to revert to a fixed neutral state over time in the absence of external influence.

With this framework, they propose the following model:

$$\begin{cases} \frac{\mathrm{d}W}{\mathrm{d}t} &= rW + I(H) + a \\ \frac{\mathrm{d}H}{\mathrm{d}t} &= sH + J(W) + b \end{cases}$$

where

- r resp. s is a measure of emotional volatility, related (but not equal) to how readily an individual's demeanour changes;
- I resp. J are influence functions describing how the individual's demeanour is affected by their partner's; and
- a resp. b are constants related (but not necessarily equal) to the neutral demeanours each individual reverts to over time.

(a) Suppose we are in an uninfluenced state, where $I(H) = I(W) = 0$. (For example, the partners are in different locations.) Find when W resp. H are constant. (This corresponds to the "personal neutral state" we expect each person to stay at, in the absence of external influence.)

(b) Suppose the wife is in an uninfluenced state, where $I(H) = I(W) = 0$. If her demeanour is higher than the personal neutral state you found in part (a), it will decrease; if her demeanour is lower than the personal neutral state, it will increase. What does that tell you about the signs of a and r?

(c) We don't know how to solve systems of differential equations like this, so let's make some simplifying assumptions. Suppose that the husband and wife are perfectly matched, with $W(t) = H(t)$, and that $I(H) = cH$ for some constant c. Find $W(t)$ in terms of r, a, and c.

* Cook et.al. Journal of Family Psychology 1995, Vol. 9, No. 2, 110-130. Accessed online Sept 21, 2020 at https://pdfs.semanticscholar.org/b654/6eb6e02b0e6008d0966c9c09ab6ee7a31ee1.pdf?_ga=2.104641033.825755858.1600723639-1520202967.1600723639 Some notation from the article has been changed to fit our own conventions.

Q[40]: hint answer solution

In Example 3.12.18 from the textbook, we approximated the sum $\sum_{t=1}^{300} e^{t/400}$ by interpreting it as a right-hand Riemann sum. Unfortunately, when we talked about numerical approximations in Section 3.9, we didn't give a way of bounding the error in that particular type of approximation.

If we're going to be using an approximation, it's best if we can bound the error. That way we can define a range where the actual value definitely lies. In this problem, we will find a range containing the actual value of the number

$$\frac{1875}{e^{3/4}-1}\left(300 \cdot e^{3/4} - \sum_{t=1}^{300} e^{t/400}\right)$$

(a) Equation 3.9.6 gives the trapezoid rule as

$$\int_a^b f(x)\,\mathrm{d}x \approx \left[\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)\right]\Delta x$$

where

$$\Delta x = \tfrac{b-a}{n}, \quad x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \cdots, \quad x_{n-1} = b - \Delta x, \quad x_n = b$$

Write the trapezpoid rule approximation for the integral $\int_0^{300} e^{t/400}\mathrm{d}t$ using $n = 300$.

(b) Theorem 3.9.12 says that the total error introduced by the trapezoidal rule is bounded by $\frac{M}{12}\frac{(b-a)^3}{n^2}$ when approximating $\displaystyle\int_a^b f(x)\mathrm{d}x$, where $|f''(x)| \leqslant M$ for all x in $[a, b]$.

Use this formula to bound the error on the approximation from (a).

(c) How does your answer from (a) differ from the sum $\sum_{t=1}^{300} e^{t/400}$?

(d) Use your work above to give an interval (as small as possible) that $\frac{1875}{e^{3/4}-1}\left(300 \cdot e^{3/4} - \sum_{t=1}^{300} e^{t/400}\right)$ definitely lies in.

(e) The text gave the approximation $\frac{1875}{e^{3/4}-1}\left(300 \cdot e^{3/4} - \sum_{t=1}^{300} e^{t/400}\right) \approx 316081.01$. Using your answer from (d), give a range for the absolute and relative error of that approximation.

Q[41]: hint answer solution

A loan is made for P_0 dollars. Every month, a portion of the loan is repaid, along with r percent interest on the remaining portion of the loan. Your loan is paid off over a term of N months.

Let $P(t)$ be the amount of your loan remaining after t months. Your answers below will depend on the constants given.

(a) In terms of P, what is the interest owed at time t?

(b) Interpret $-P'(t)$ in terms of this model.

(c) You want your monthly payments to be a constant value, say some constant C. Set up a differential equation for P.

(d) Solve your differential equation for P in terms of the constants given.

(e) What is C?

(f) Suppose your job gives you cost-of-living raises of 0.1% each month. Instead of paying the constant rate of C dollars every month, you want to pay a rate that increases in line with your salary. Say you want to pay $C_0 \cdot 1.001^t$ dollars in month t. Set up (but do not solve) a differential equation for $P(t)$ in this new scenario, assuming everything else is the same as before.

Chapter 4

PROBABILITY

4.1 ▲ Introduction

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Let X be a random variable. Write, in our shorthand notation, "the probability that X takes the value 5 is 10%."

Q[2]: hint answer solution

Suppose the sample space of a random variable is $\{1, 2, 3\}$. Is the variable discrete?

Q[3]: hint answer solution

Suppose X is a discrete random variable taking whole-number values from 1 to 10, and Y is a continuous random variable taking real values in the interval $[1, 10]$.

Let A be the event that a variable is less than 5, and let B be the event that a variable is greater than 4.5.

For which variable are these events disjoint?

Q[4]: hint answer solution

Suppose X is a random variable and $Pr(X \leqslant 5) = \frac{99}{100}$. What is $Pr(X > 5)$?

Q[5]: hint answer solution
Suppose X is a random variable and $Pr(X = x) = \frac{78}{93}$ for some number x. What is $Pr(X \neq x)$?

▸▸ Stage 2

Q[6]: hint answer solution
Let X be the random variable corresponding to throwing a fair six-sided dice. What is $Pr(X \text{ even OR } X = 3)$?

▸▸ Stage 3

Q[7]: hint answer solution
Let Y be a random variable with sample space $\mathcal{S} = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$, where each value in $\mathcal{S}$ is equally likely. What is the probability that Y is divisible by 3 or 10?

(Recall we use "or" to mean "and/or.")

4.2▴ Probability Mass Function (PMF)

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution
What does the abbreviation PMF stand for?

Q[2]: hint answer solution
Is it discrete random variables or continuous random variables that have PMFs?

Q[3]: hint answer solution
A random variable X has PMF given below.

x	$Pr(X = x)$
5	$Pr(X = 5) = 0.7$
6	$Pr(X = 6) = 0.1$
7	$Pr(X = 7) = 0.1$
8	$Pr(X = 8) = 0.1$

(a) What is sample space?

(b) Are all values equally likely?

▸▸ Stage 2

Q[4]: hint answer solution

Z is a random variable with sample space $\mathcal{S} = \{3.1, 3.2, 3.3, 3.4, 3.5\}$. Complete its PMF below.

x	$Pr(X = x)$
3.1	0.10
3.2	0.15
3.3	0.35
3.4	0.2
3.5	$\boxed{?}$

Q[5]: hint answer solution

Given the PMF below of a variable X, which value in the sample space is the most likely?

x	$Pr(X = x)$
3.7	$1/15$
8.1	$1/5$
8.2	$1/3$
9	$1/5$
10	$1/5$

▸▸ Stage 3

Q[6]: hint answer solution

A random variable X has PMF as below.

$$f(x) = \begin{cases} \frac{1}{2^x} & x = 1, 2, 3, 4, 5, \ldots \\ 0 & \text{else} \end{cases}$$

(Note: you'll see in Example 5.2.4 that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$.)

What is the sample space of X? What is $Pr(X \leqslant 3)$?

4.3▴ Cumulative Distribution Function (PDF)

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution
Suppose T is the continuous random variable resulting from choosing a day at uniformly random and finding the temperature in Vancouver at noon. 9 out of 10 days had a temperature of more than 0 degrees; 1 out of 10 days had a temperature of higher than 30 degrees; and 4 out of 10 days had a temperature of at most 20 degrees.
Let $F(x)$ be the CDF of T. What are $F(0)$, $F(20)$, and $F(30)$?

Q[2]: hint answer solution
Suppose X is a continuous random variable, uniformly distributed on its sample space, the interval $[a, b]$. Sketch a dot diagram for X.

Q[3]: hint answer solution
A continuous random variable X has sample space $S = [0, 5]$. X is most likely to be in the middle of the interval, and less likely to be near the endpoints of the interval. Sketch a dot diagram for X.

Q[4]: hint answer solution
Let W be a random variable with CDF

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leqslant x \leqslant 1 \\ 1 & 1 < x \end{cases}$$

Is W a continuous random variable?

Q[5]: hint answer solution
Let X be a random variable with CDF $F(x)$. For each statement below, decide whether it is always true, or whether it might be false in some cases. If the statement might be false, give an example.
(a) $\lim\limits_{x\to\infty} F(x) = 1$
(b) $F(1,000,000) \approx 1$
(c) $F(1) > F(-1)$
(d) $F(1) \geqslant 0$

Q[6]: hint answer solution
Suppose $F(x)$ is the CDF of a random variable, and $F(10) = 1$. What is $F(11)$?

▸▸ Stage 2

Q[7]: hint answer solution
Let X be a discrete random variable that takes the value -1 half the time, and the value 1 the other half the time. Find the CDF for X.

Q[8]: hint answer solution

Let D be the outcome of a roll of a fair 6-sided dice. Determine the CDF of D, and decide whether D is continuous.

Q[9]: hint answer solution

Consider a random variable Z with the following PMF:

x	$Pr(Z = x)$
-4	$\frac{1}{2}$
-2	$\frac{1}{3}$
-1	$\frac{1}{6}$

Give the CDF of Z.

Q[10]: hint answer solution

Consider the function

$$F(x) = \begin{cases} \frac{Ax^3}{x^3+B} & x \geqslant 0 \\ 0 & x \leqslant 0 \end{cases}$$

For what values of A and B is $F(X)$ a CDF?

▸▸ Stage 3

Q[11]: hint answer solution

For what values of A, B, C, and D is the function

$$F(x) = \begin{cases} A + \frac{Bx}{x+1} & x \geqslant 0 \\ C + \frac{Dx}{1-x} & x < 0 \end{cases}$$

the CDF of a continuous random variable?

Q[12]: hint answer solution

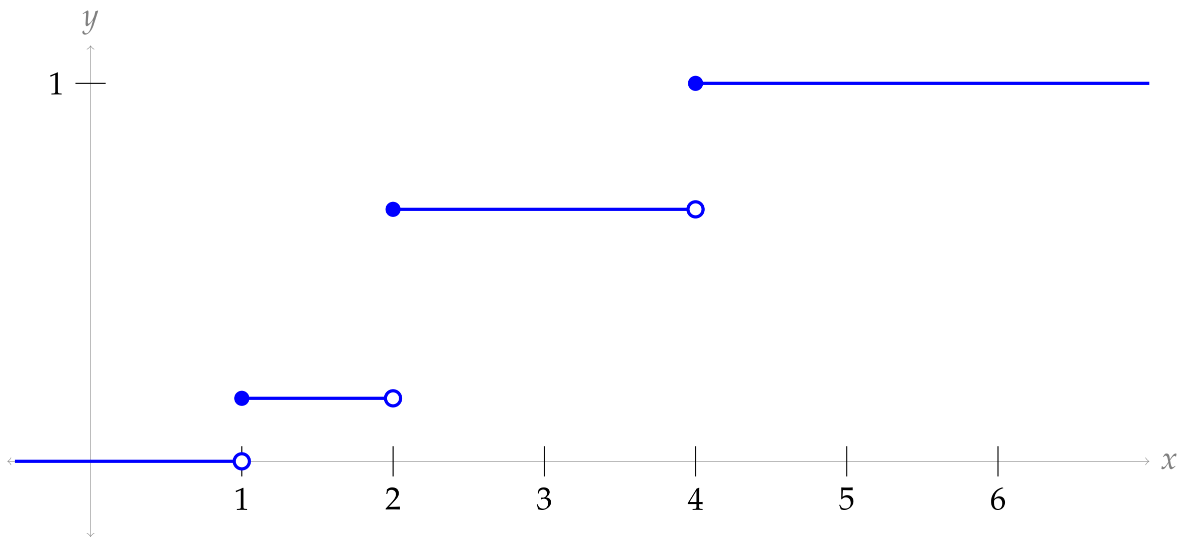
A discrete random variable W has the following CDF:

$$F(x) = \begin{cases} 0 & x < 5 \\ \frac{1}{4} & 5 \leqslant x < 6 \\ \frac{1}{3} & 6 \leqslant x < 8 \\ \frac{1}{2} & 8 \leqslant x < 12 \\ 1 & 12 \leqslant x \end{cases}$$

Find its PMF.

Q[13]: hint answer solution

Below is a sketch of $F(x)$, the CDF of a discrete random variable Y.



Find the sample space of Y, and order its values from most likely to least likely.

4.4▲ Probability Density Function (PDF)

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Draw a dot diagram for each PDF below.

(a) $f(x) = 4x(x-1)(x-2), 0 \leqslant x \leqslant 1$

(b) $f(x) = \begin{cases} \frac{1}{2} & 0 < x < 1 \\ \frac{1}{3} & 1 < x < 2 \\ \frac{1}{6} & 2 < x < 3 \end{cases}$

(c) $f(x) = x^2, 1 \leqslant x \leqslant \sqrt[3]{2}$

Q[2]: hint answer solution

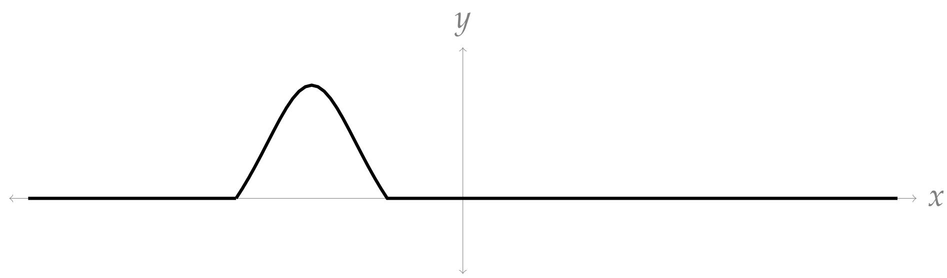
For each function property below, decide whether the property holds for all PDFs, all CDFs, both, or neither.

(a) limit at negative infinity is 1
(b) never negative
(c) nondecreasing
(d) never more than 1
(e) area under the curve gives a probability
(f) value of function gives a probability
(g) area under the curve from $-\infty$ to ∞ is 1

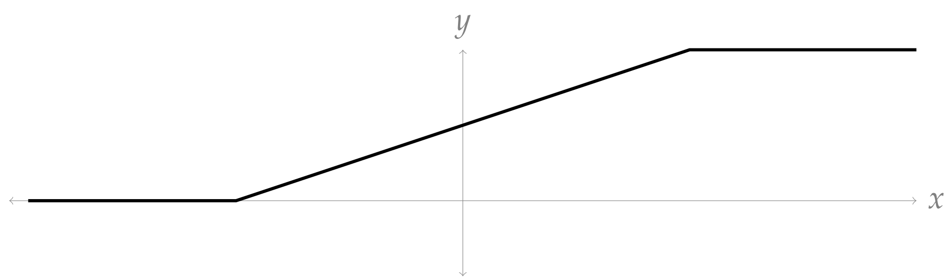
Q[3]: hint answer solution

Each sketch below is of a PDF or a CDF of a continuous random variable. Identify each as either PDF or CDF.

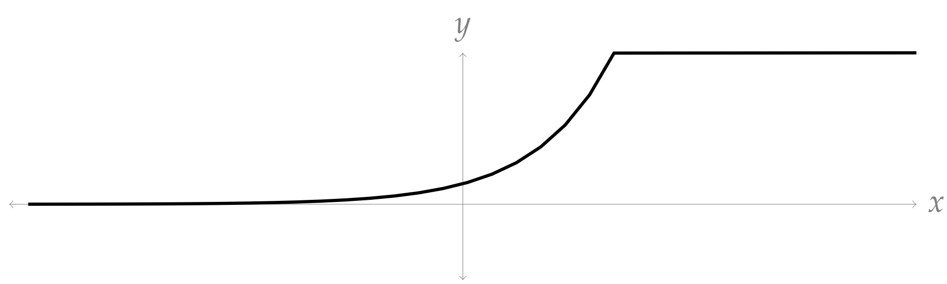
(a)



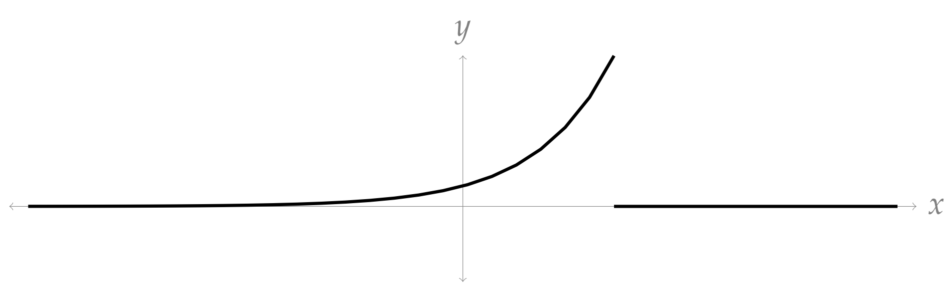
(b)



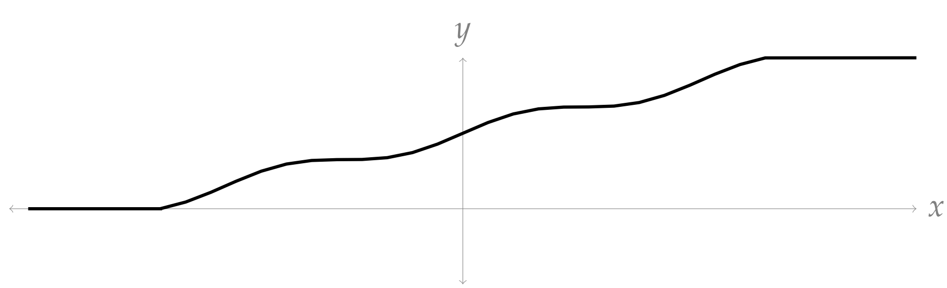
(c)



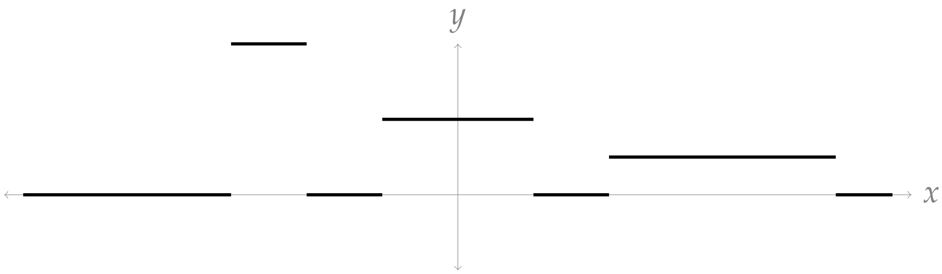
(d)



(e)



(f)

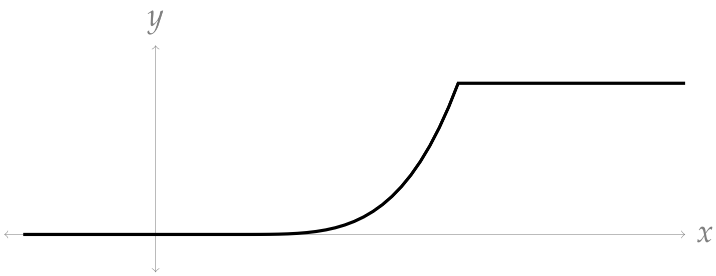


▸▸ Stage 2

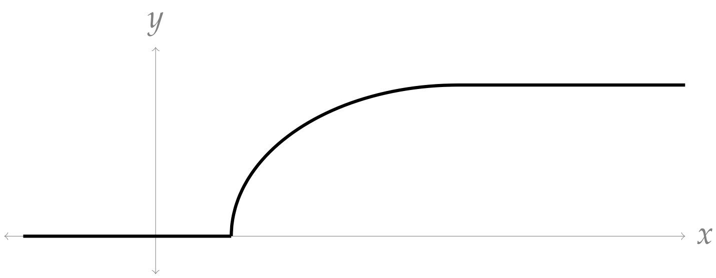
Q[4]: hint answer solution

Each function sketched below is the CDF of a random variable. If the random variable is continuous, sketch its PDF.

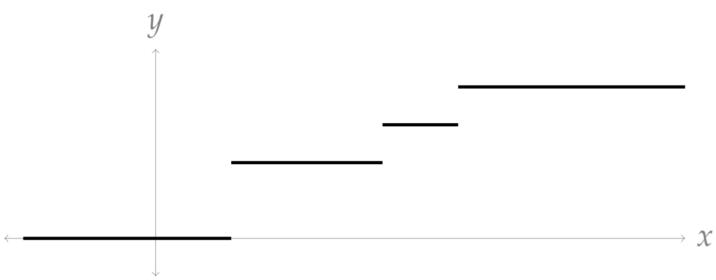
(a)



(b)



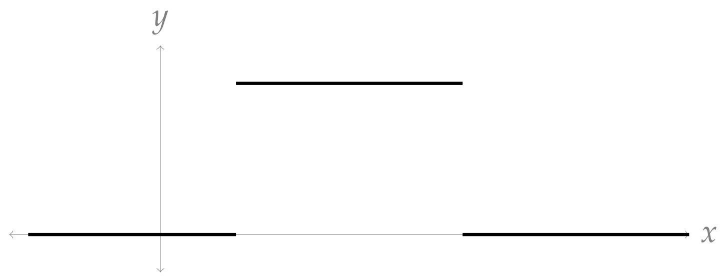
(c)



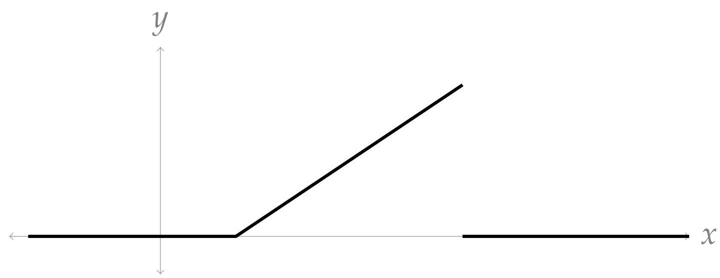
Q[5]: hint answer solution

Each function sketched below is the PDF of a continuous random variable. Sketch the CDF for each variable.

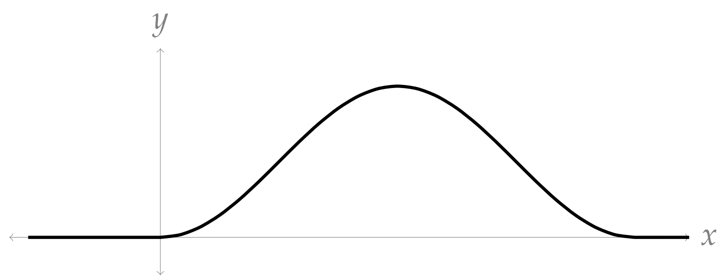
(a)



(b)



(c)



Q[6]: hint answer solution

Let W be a continuous random variable with PDF

$$f(x) = \frac{10/\pi}{1+100x^2}$$

Find $Pr(4 \leqslant W \leqslant 17)$.

Q[7]: hint answer solution

Let Q be a continuous random variable with PDF

$$f(x) = \begin{cases} \frac{1}{10} & 0 < x < 3 \\ \frac{1}{5} & 4 < x < 6 \\ \frac{3}{10} & 7 < x < 8 \end{cases}$$

Find $Pr(Q \geqslant 4.5)$.

Q[8]: hint answer solution

Let M be a continuous random variable with PDF

$$f(x) = \frac{x}{50}, \quad 0 \leqslant x < 10$$

What is the probability that M is in the interval $(0, 1)$ OR $(9, 10)$?

Q[9]: hint answer solution

Let M be a continuous random variable with PDF

$$f(x) = \frac{x}{50}, \quad 0 \leqslant x < 10$$

Find the CDF of M.

Q[10]: hint answer solution

Let Q be a continuous random variable with PDF

$$f(x) = \begin{cases} \frac{1}{10} & 0 < x < 3 \\ \frac{1}{5} & 4 < x < 6 \\ \frac{3}{10} & 7 < x < 8 \end{cases}$$

Find the CDF of Q.

Q[11]: hint answer solution

Suppose a random variable X has cumulative distribution function (CDF)

$$F(x) = \begin{cases} e^x & x \leqslant 0 \\ 1 & x > 0 \end{cases}$$

(a) Is X a continuous random variable?

(b) Find the PMF of X if it is a discrete random variable, or find the PDF of X if it is a continuous random variable.

Q[12]: hint answer solution

Suppose a random variable X has cumulative distribution function (CDF)

$$F(x) = \begin{cases} \frac{x}{x+1} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

(a) Is X a continuous random variable?

(b) Find the PMF of X if it is a discrete random variable, or find the PDF of X if it is a continuous random variable.

Q[13]: hint answer solution

Suppose the function

$$f(x) = \begin{cases} e^x & -1 \leqslant x \leqslant b \\ 0 & \text{else} \end{cases}$$

is a PDF. What is b?

Q[14]: hint answer solution

Suppose the function

$$f(x) = \frac{A}{x^2+4}$$

is a PDF. What is A?

Q[15]: hint answer solution

Let X be a continuous random variable with PDF

$$f(x) = \begin{cases} 2x^3 & 0 < x < 1 \\ 2(x-2)^3 & 2 < x < 3 \end{cases}$$

Find the CDF of X.

▸▸ Stage 3

Q[16]: hint answer solution

Let X be a random variable with PDF

$$f(x) = \begin{cases} |x| & -1 \leqslant x \leqslant 1 \\ 0 & \text{else} \end{cases}$$

Find the CDF of X.

Q[17]: hint answer solution

A silviculturist is studying the growth of yellow cedar across Pacific Spirit Park. They define the random variable M to be the mass of a tree, measured in 1000s of kg, at its hundreth birthday, where the tree is chosen uniformly at random from all trees in the park.

We can think of M as a continuous random variable, because the mass of a tree exists on the continuum $(0, \infty)$.

Suppose the PDF of M is

$$f(x) = cx^2(200-x)$$

over the sample space $\mathcal{S} = [0, 200]$, where c is a constant.

(a) Find c.

(b) Actually finding the mass of a tree is tough – not least because much of it is buried underground. Suppose the measurements of the silviculturist are only accurate to within 5,000 kg, so in practice every measurement gets rounded to the nearest 10,000 kg. Let M_1 be the discrete random variable where M_1 is the rounded mass of our randomly chosen tree. (So, for example, if $M = 12$, then $M_1 = 10$.)

What is the sample space of M_1?

(c) Evaluate $Pr(M_1 = m)$ if m is in the sample space of M_1.
Hint 1: $b^3 - a^3 = (b-a)(b^2 + ba + a^2)$.

(d) Using a graphing program[1], compare the graphs of $f(x)$ (the PDF of M) and the PMF of M_1.

Q[18]: hint answer solution
A continuous random variable X has PDF

$$f(x) = \sqrt{\frac{2}{\pi}} e^{-2x^2}$$

(There is a special name for this type of distribution: it's the normal distribution with expected value 0 and standard deviation $\frac{1}{2}$.)

The function $f(x)$ is a tough one to antidifferentiate, so it's hard to use $f(x)$ to evaluate probabilities exactly. Use Simpson's Rule with 4 intervals to approximates $Pr(0 \leqslant X \leqslant 1)$. Your final answer doesn't have to be simplified, but it should be calculator-ready.

▸▸ Open-Ended Questions

Q[19]: hint answer solution
PMFs (for discrete variables) and PDFs (for continuous variables) both serve to describe how our random variables tend to behave. In the case of a continuous random variable, the area under the curve of a PDF tells us the probability of a variable being in that range. Can you use the area under the curve of a PMF in a similar manner? Why or why not?

Q[20]: hint answer solution
No function can be both a CDF and a PDF. Explain why.

4.5▴ Expected Value

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

Problems 5, 6, 7, and 8 in this section are adapted from Ch 4.2 of Introductory Statistics by Barbara Illowsky and Susan Dean, published on OpenStax under Creative Commons Attribution License 4.0.

▸▸ Stage 1

Q[1]: hint answer solution

1 or not – actually you can do these by hand

Let X be a random variable. True or false: After one event, the most likely value of X to take is $\mathbb{E}(X)$.

Q[2]: hint answer solution

Suppose we've rolled a 6-sided dice a large number of times – one million rolls. We averaged the results of the million rolls, and got 4.16. Do you think the dice was fair? That is, do you think each number 1 through 6 was equally likely?

Q[3]: hint answer solution

Suppose X is a random variable whose sample space is the single number $\mathcal{S} = \{a\}$. What is $\mathbb{E}(X)$?

Q[4]: hint answer solution

Suppose Z is a continuous random variable whose sample space is the interval $[-1, 1]$, and $\mathbb{E}(Z) = 0$. True or false: Z must be uniformly distributed on $[-1, 1]$.

▸▸ Stage 2

Q[5]: hint answer solution

A soccer team plays zero, one, or two days a week. The probability that they play zero days is 0.2, the probability that they play one day is 0.5, and the probability that they play two days is 0.3. Find the expected value of the number of days per week the men's soccer team plays soccer.

Q[6]: hint answer solution

A hospital researcher is interested in the number of times the average post-op patient will ring the nurse during a 12-hour shift. For a random sample of 50 patients, the following information was obtained, where X was the number of times the patient rang the nurse.

x	$Pr(X = x)$
0	$Pr(X = 0) = \frac{4}{50}$
1	$Pr(X = 1) = \frac{8}{50}$
2	$Pr(X = 2) = \frac{16}{50}$
3	$Pr(X = 3) = \frac{14}{50}$
4	$Pr(X = 4) = \frac{6}{50}$
5	$Pr(X = 5) = \frac{2}{50}$

What is the expected value of X?

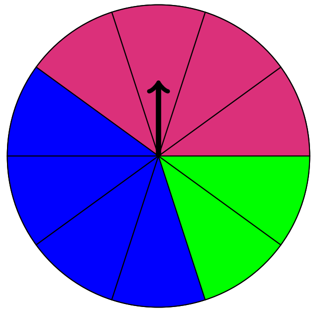
Q[7]: hint answer solution

You are playing a game of chance in which four cards are drawn from a standard deck of 52 cards. You guess the suit of each card before it is drawn. The cards are replaced in the deck on each draw. That means every time you play, the probability of you winning is $\frac{1}{4}$.

You pay \$1 to play. If you guess the right suit every time, you get your dollar back plus \$4. What is your expected profit of playing the game over the long term?

Q[8]: hint answer solution

A spinning contraption can land on red, blue, or green.



You play a game by spinning the spinner once. It lands on red $\frac{2}{5}$ of the time, on blue $\frac{2}{5}$ of the time, and on green $\frac{1}{5}$ of the time. If you land on red, you pay \$10. If you land on blue, you win \$1. If you land on green, you win \$7.

If you play the game a lot of times, how much money do you expect to win?

Q[9]: hint answer solution
Let M be the continuous random variable with PDF

$$f(x) = \frac{x}{5000}, \quad 0 \leqslant x \leqslant 100$$

Compute $\mathbb{E}(M)$, and check your answer as in Section 4.5.3.

Q[10]: hint answer solution
Let N be the continuous random variable with PDF

$$f(x) = \frac{2/\pi}{x^2+1}, \quad -1 \leqslant x \leqslant 1$$

Compute $\mathbb{E}(N)$, and check your answer as in Section 4.5.3.

Q[11]: hint answer solution
Let P be the continuous random variable with PDF

$$f(x) = \begin{cases} \frac{2}{3}x(1-x) & 0 \leqslant x \leqslant 1 \\ \frac{2}{3}(x-2) & 2 \leqslant x \leqslant 3 \end{cases}$$

Compute $\mathbb{E}(P)$, and check your answer as in Section 4.5.3.

Q[12]: hint answer solution
Let Q be the continuous random variable with PDF

$$f(x) = \left(\frac{e}{e-2}\right) \cdot \frac{\ln x}{x^2}, \quad 1 \leqslant x \leqslant e$$

Compute $\mathbb{E}(Q)$, and check your answer as in Section 4.5.3.

Q[13]: hint answer solution
Suppose Y is a random variable that is uniformly distributed on the interval $[a, b]$. What is $\mathbb{E}(Y)$?

▸▸ Stage 3

Q[14]: hint answer solution

Let X_p be the continuous random variable with PDF

$$f_p(x) = \frac{a_p}{x^p}, \quad 1 < x$$

where p is some positive number and a_p is some constant (possibly different constants for different values of p).

(a) For which values of p does $f_p(x)$ actually have the characteristics of a PDF?

(b) For those values of p from a, what is a_p?

(c) Which values of p from a give finite expected values for $\mathbb{E}(X_p)$?

Q[15]: hint answer solution

Let A be the continuous random variable with PDF

$$f(x) = xe^x, \quad 0 \leqslant x \leqslant 1$$

Find $\mathbb{E}(A)$.

Q[16]: hint answer solution

Let B be the continuous random variable with PDF

$$f(x) = \frac{2}{\ln^2 2}\left(\frac{\ln x}{x}\right), \quad 1 \leqslant x \leqslant 2$$

Find $\mathbb{E}(B)$.

Q[17]: hint answer solution

Let C be the continuous random variable with PDF

$$f(x) = \frac{x}{x^2-1}, \quad 2 \leqslant x \leqslant b$$

where $b = \sqrt{3e^2+1}$. Find $\mathbb{E}(C)$. You may leave your answer in terms of b.

Q[18]: hint answer solution

Let D be the continuous random variable with PDF

$$f(x) = \frac{4}{4-\pi}\tan^2 x, \quad 0 \leqslant x \leqslant \frac{\pi}{4}$$

Find $\mathbb{E}(D)$.

Q[19]: hint answer solution

A random variable X has PDF

$$f(x) = ce^{bx^2}, \quad -\infty < x < \infty$$

for a positive constant c and a negative constant b. (This is a type of "normal distribution.")

Find the expected value of X.

Q[20]: hint answer solution
Sonic is an insurance company in Vancouver. Fred has recently bought insurance from Sonic for his whale-watching boat. The premium insurance is \$ 1,000 per year. If Fred crashes his boat, Sonic will pay Fred \$ 15,000. If Fred crashes his boat again during the same year, the insurance company will not cover it again. The probability that Fred crashes his boat at least once during the first year is one in a thousand.

Calculate Sonic's expected profit (how much money Fred paid, minus the expected sum Sonic would pay for a crash) from Fred's insurance purchase for the first year.

Q[21]: hint answer solution
Andrea is about to finish her master's degree at UBC. She applied to jobs offering four different monthly salaries. She estimates her odds of being offered each job, and comes up with the table below describing the probability that, one year from graduation, she is earning a given monthly salary.

x	$Pr(X = x)$
0	$\frac{15}{100}$
2,500	$\frac{20}{100}$
3,000	$\frac{30}{100}$
3,500	$\frac{20}{100}$
4,000	$\frac{15}{100}$

(a) What is the probability that she is unemployed?

(b) Calculate her expected monthly salary.

Q[22]: hint answer solution
Riley is planning to invest in two assets. **Asset A** pays 20% after one year (investing \$1 would return \$1.20) with 100% certainty. **Asset B** pays 300% after one year (investing \$1 would return \$4) with a probability of 0.2, and 0% after one year (investing \$1 would return \$1) with a probability of 0.8.

(a) What would be Riley's expected monetary gain if she spends x dollars in A and y dollars in B?

(b) Riley is somewhat risk-averse. She wants to invest \$300 with a guarantee that, at the end of the year, her return is at least \$350. Subject to that constraint, how much money should she allocate to each asset to maximize her expected return?

4.6▲ Variance and Standard Deviation

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Match each term to its unofficial, intuitive description.

Terms:

A. expected value
B. variance
C. standard deviation
D. probability mass function
E. probability distribution function
F. cumulative distribution function

Descriptions:

1. description of how likely each outcome is
2. $Pr(X \leqslant x)$
3. weights giving relative likelihood of X being in a particular region
4. long-term average
5. usual difference from the average
6. average squared difference from the average

Q[2]: hint answer solution

Given an example of a random variable X that takes on values from $[-100, 100]$ (maybe not all of them) where $\sigma(X) = 100$, or explain why no such X exists.

Q[3]: hint answer solution

Suppose X is a continuous random variable, uniformly distributed on its sample space, the interval $[a, b]$.

(a) Find the variance and standard deviation of X.

(b) On a number line from a to b, indicate which values are within one standard deviation of $\mathbb{E}(X)$.

Q[4]: hint answer solution

Suppose X is a discrete random variable, and its sample space is the single number $\mathcal{S} = \{a\}$. What are the variance and standard deviation of X?

Q[5]: hint answer solution

Suppose X is a continuous random variable with sample space $[-a, a]$, and its PDF has even symmetry. What is $\mathbb{E}(X)$?

▸▸ Stage 2

Q[6]: hint answer solution

A continuous random variable X has PDF

$$f(x) = \sin x, \quad 0 \leqslant x \leqslant \frac{\pi}{2}$$

Find the following.

(a) CDF of X

(b) $\mathrm{Var}(X)$

(c) $\sigma(X)$

Q[7]: hint answer solution

A continuous random variable W has PDF

$$f(x) = 1 - |x|, \quad -1 \leqslant x \leqslant 1$$

Find the expectation, variance, and standard deviation of W.

Q[8]: hint answer solution

A random variable X has CDF

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{3}(x+1)^3 & -1 \leqslant x < 0 \\ \frac{2}{3}(x-1)^3 + 1 & 0 \leqslant x < 1 \\ 1 & x \geqslant 1 \end{cases}$$

(a) Is X continuous?

(b) Find the expectation, variance, and standard deviation of X.

Q[9]: hint answer solution

A discrete random variable T has the following PMF.

x	$Pr(T = x)$
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{4}$

Find the expected value, variance, and standard deviation of T.

Q[10]: hint answer solution

A discrete random variable S has the following PMF.

x	$Pr(S = x)$
-5	$\frac{1}{9}$
-4	$\frac{2}{9}$
2	$\frac{1}{9}$
7	$\frac{5}{9}$

Find the expected value, variance, and standard deviation of S.

Q[11]: hint answer solution

A random variable U has CDF

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x < 2 \\ \frac{2}{3} & x < 3 \\ \frac{3}{4} & x < 4 \\ 1 & x \geqslant 4 \end{cases}$$

Find the expected value, variance, and standard deviation of U.

▸▸ Stage 3

Q[12]: hint answer solution

Let Z be a continuous random variable with PDF

$$f(x) = a\frac{\sin^3 x}{x}, \quad 0 < x \leqslant \pi$$

for a suitable constant a.

Find the variance and standard deviation of Z (in terms of a).

▸▸ Open-Ended Questions

Q[13]: hint answer solution

When we motivated variance, we calculated "distance from the average" as

$$|E(X) - X|$$

and then took the average of these values. What undesirable thing happens if we don't include the absolute value, and calculate "distance from the average" as $(E(X) - X)$?

Chapter 5

SEQUENCES AND SERIES

5.1 ▲ Sequences

Exercises

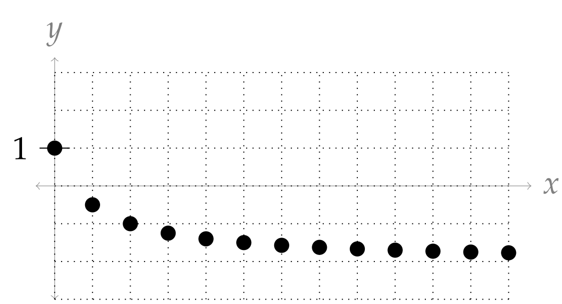
Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

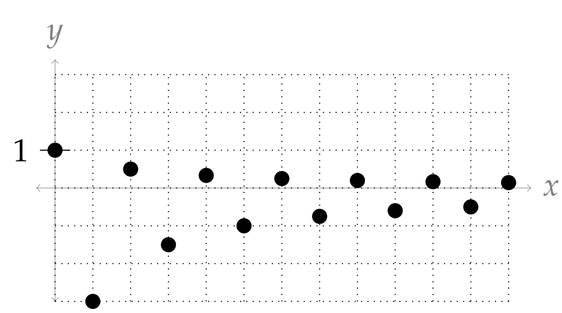
Q[1]: hint answer solution

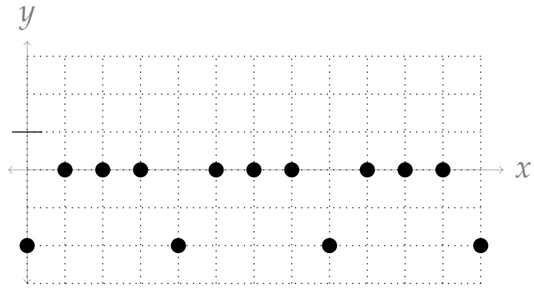
Assuming the sequences continue as shown, estimate the limit of each sequence from its graph.

(a)



(b)



(c) 

Q[2]: hint answer solution

Suppose a_n and b_n are sequences, and $a_n = b_n$ for all $n \geqslant 100$, but $a_n \neq b_n$ for $n < 100$. True or false: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

Q[3]: hint answer solution

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$, be sequences with $\lim\limits_{n\to\infty} a_n = A$, $\lim\limits_{n\to\infty} b_n = B$, and $\lim\limits_{n\to\infty} c_n = C$. Assume A, B, and C are nonzero real numbers.

Evaluate the limits of the following sequences.

(a) $\dfrac{a_n - b_n}{c_n}$

(b) $\dfrac{c_n}{n}$

(c) $\dfrac{a_{2n+5}}{b_n}$

Q[4]: hint answer solution

Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 1000$ for all $n \leqslant 1000$,
- $a_{n+1} < a_n$ for all n, and
- $\lim\limits_{n\to\infty} a_n = -2$

Q[5]: hint answer solution

Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 0$ for all even n,
- $a_n < 0$ for all odd n,
- $\lim\limits_{n\to\infty} a_n$ does not exist.

Q[6]: hint answer solution

Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 0$ for all even n,
- $a_n < 0$ for all odd n,
- $\lim\limits_{n\to\infty} a_n$ exists.

Q[7]: hint answer solution

The limits of the sequences below can be evaluated using the Squeeze Theorem. For each sequence, choose an upper bounding sequence and lower bounding sequence that will work with the Squeeze Theorem. You do not have to evaluate the limits.

(a) $a_n = \dfrac{\sin n}{n}$

(b) $b_n = \dfrac{n^2}{e^n(7 + \sin n - 5\cos n)}$

(c) $c_n = (-n)^{-n}$

Q[8]: hint answer solution

Below is a list of sequences, and a list of functions.

(a) Match each sequence a_n to any and all functions $f(x)$ such that $f(n) = a_n$ for all whole numbers n.

(b) Match each sequence a_n to any and all functions $f(x)$ such that $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x)$.

$$a_n = 1 + \frac{1}{n} \qquad\qquad f(x) = \cos(\pi x)$$

$$b_n = 1 + \frac{1}{|n|} \qquad\qquad g(x) = \frac{\cos(\pi x)}{x}$$

$$c_n = e^{-n} \qquad\qquad h(x) = \begin{cases} \frac{x+1}{x} & x \text{ is a whole number} \\ 1 & \text{else} \end{cases}$$

$$d_n = (-1)^n \qquad\qquad i(x) = \begin{cases} \frac{x+1}{x} & x \text{ is a whole number} \\ 0 & \text{else} \end{cases}$$

$$e_n = \frac{(-1)^n}{n} \qquad\qquad j(x) = \frac{1}{e^x}$$

Q[9]: hint answer solution

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by $a_n = \cos n$.

(a) Give three different whole numbers n that are within 0.1 of an odd integer multiple of π, and find the corresponding values of a_n.

(b) Give three different whole numbers n such that a_n is close to 0. Justify your answers.

Remark: this demonstrates intuitively, though not rigorously, why $\lim\limits_{n\to\infty} \cos n$ is undefined. We consistently find terms in the sequence that are close to -1, and also consistently find terms in the sequence that are close to 1. Contrast this to a sequence like $\{\cos(2\pi n)\}$, whose terms are always 1, and whose limit therefore is 1. It is possible to turn the ideas of this question into a rigorous proof that $\lim\limits_{n\to\infty} \cos n$ is undefined. See the solution.

Q[10]: hint answer solution

Below are a number of recurrence relations defining different sequences. Write out the first five terms of each sequences.

(a) $a_0 = 4$, $a_{n+1} = 10a_n - 6$

(b) $b_0 = 1$, $b_{n+1} = \frac{b_n}{2}$

(c) $c_0 = 0$, $c_{n+1} = \frac{c_n}{2}$

(d) $d_0 = 1$, $d_1 = -1$, $d_{n+2} = d_n - d_{n+1}$

Q[11]: hint answer solution

Below are a number of explicit definitions for different sequences. Write out the first five terms of each sequences, starting with $n = 0$.

(a) $a_n = 1$

(b) $b_n = n + 1$

(c) $c_n = \tan(\pi n)$

(d) $d_n = (-1)^n$

▸▸ Stage 2

Q[12]: hint answer solution

Below are a number of recurrence relations defining different sequences. Give an explicit definition for each sequence.

(a) $a_0 = 2$, $a_{n+1} = (a_n)^2$

(b) $b_0 = 5$, $b_{n+1} = -b_n$

(c) $c_0 = 8$, $c_{n+1} = \frac{c_n}{2} + 4$

Q[13]: hint answer solution

Below are a number of lists representing the first terms of a sequence. Give an explicit definitions for each sequence.

(a) $\{0, 1, 4, 9, 16, \ldots\}$

(b) $\{1, -2, 4, -8, 16, -32, \ldots\}$

(c) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

(d) $\{1.5, 2, 2.5, 3, 3.5, 4, \ldots\}$

Q[14]: hint answer solution

Determine the limits of the following sequences.

(a) $a_n = \dfrac{3n^2 - 2n + 5}{4n + 3}$

(b) $b_n = \dfrac{3n^2 - 2n + 5}{4n^2 + 3}$

(c) $c_n = \dfrac{3n^2 - 2n + 5}{4n^3 + 3}$

Q[15]: hint answer solution

Determine the limit of the sequence $a_n = \dfrac{4n^3 - 21}{n^e + \frac{1}{n}}$.

Q[16]: hint answer solution

Determine the limit of the sequence $b_n = \dfrac{\sqrt[4]{n} + 1}{\sqrt{9n + 3}}$.

Q[17]: hint answer solution

Determine the limit of the sequence $c_n = \dfrac{\sin n}{n}$.

Q[18]: hint answer solution

Determine the limit of the sequence $a_n = \dfrac{n^{\sin n}}{n^2}$.

Q[19]: hint answer solution

Determine the limit of the sequence $d_n = e^{-1/n}$.

Q[20]: hint answer solution

Determine the limit of the sequence $a_n = \dfrac{1 + 3\sin(n^2) - 2\sin n}{n}$.

Q[21]: hint answer solution

Determine the limit of the sequence $b_n = \dfrac{e^n}{2^n + n^2}$.

Q[22](*): hint answer solution

Find the limit, if it exists, of the sequence $\{a_k\}$, where

$$a_k = \frac{k! \sin^3 k}{(k+1)!}$$

Q[23](*): hint answer solution

Consider the sequence $\left\{(-1)^n \sin\left(\frac{1}{n}\right)\right\}$. State whether this sequence converges or diverges, and if it converges give its limit.

Q[24](*): hint answer solution

Evaluate $\displaystyle\lim_{n\to\infty} \left[\frac{6n^2 + 5n}{n^2 + 1} + 3\cos(1/n^2)\right]$.

▸▸ Stage 3

Q[25]: hint answer solution

Below are explicit definitions of several sequences. Give an explicit definition of a *different* sequence has the same first three terms (starting at $n = 0$).

(a) $a_n = \frac{9}{2}n^2 - \frac{3}{2}n + 1$

(b) $c_n = n^3 - 3n^2 + 5n + 3$

(c) $e_n = n(n-1)(n-2)$

Q[26]: hint answer solution

For which initial values a_0 does the recurrence relation

$$a_{n+1} = 5a_n - a_n^2$$

generate a sequence that has only negative values?

Q[27](*): hint answer solution

Find the limit of the sequence $\left\{\ln\left(\sin\frac{1}{n}\right) + \ln(2n)\right\}$.

Q[28]: hint answer solution

Evaluate $\lim\limits_{n\to\infty}\left[\sqrt{n^2+5n} - \sqrt{n^2-5n}\right]$.

Q[29]: hint answer solution

Evaluate $\lim\limits_{n\to\infty}\left[\sqrt{n^2+5n} - \sqrt{2n^2-5}\right]$.

Q[30]: hint answer solution

Evaluate the limit of the sequence $\left\{n\left[\left(2+\frac{1}{n}\right)^{100} - 2^{100}\right]\right\}_{n=1}^{\infty}$.

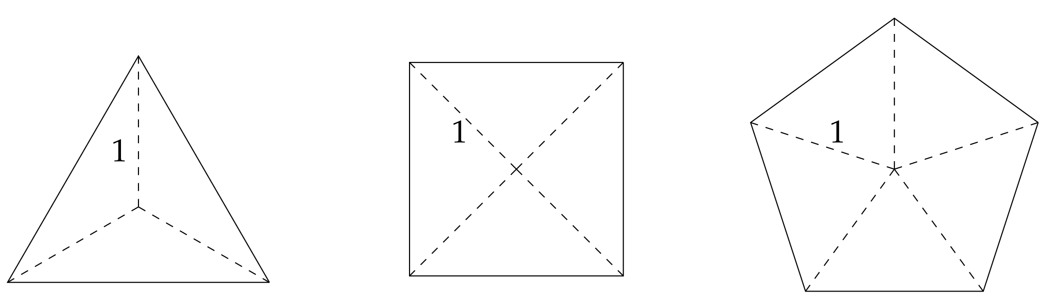
Q[31]: hint answer solution

Write a sequence $\{a_n\}_{n=1}^{\infty}$ whose limit is $f'(a)$ for a function $f(x)$ that is differentiable at the point a.

Your answer will depend on f and a.

Q[32]: hint answer solution

Let $\{A_n\}_{n=3}^{\infty}$ be the area of a regular polygon with n sides, with the distance from the centroid of the polygon to each corner equal to 1.



$A(3) = \frac{3\sqrt{3}}{4}$ $A(4) = 2$ $A(5) = 2.5\sin(0.4\pi)$

(a) By dividing the polygon into n triangles, give a formula for A_n.

(b) What is $\lim\limits_{n\to\infty} A_n$?

Q[33]: hint answer solution

The book *The Mathematics of Behaviour* by Earl Hunt[1] has these two paragraphs towards the end:

> I have been told that an author loses half the readers with every equation. If that is so, and if every person on the globe started to try to read this book, I have a reader left! My thanks and congratulations, lonely reader.
>
> Actually, I am not that pessimistic. It seems more likely to me that you lose half your readers on the first, one-third on the second, and so on. Some hardy souls will persevere. They may even expect mathematics. And we are at the end.

Assuming there are 7 billion people in the world, how many equations were there in the text? Assuming the attrition mentioned in the second paragraph, how many of those 7 billion people are left reading the book?

Q[34]: hint answer solution

Suppose we define a sequence $\{f_n\}$, which depends on some constant x, as the following:

$$f_n(x) = \begin{cases} 1 & n \leqslant x < n+1 \\ 0 & \text{else} \end{cases}$$

For a fixed constant $x \geqslant 1$, $\{f_n\}$ is the sequence $\{0, 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0, 0, \ldots\}$. The sole nonzero element comes in position k, where k is what we get when we round x *down* to a whole number. If $x < 1$, then the sequence consists of all zeroes.

Since we can plug in different values of x, we can think of $f_n(x)$ as a function of sequences: a different x gives you a different sequence. On the other hand, if we imagine fixing n, then $f_n(x)$ is just a function, where $f_n(x)$ gives the nth term in the sequence corresponding to x.

(a) Sketch the curve $y = f_2(x)$.

(b) Sketch the curve $y = f_3(x)$.

(c) Define $A_n = \int_0^\infty f_n(x)\,\mathrm{d}x$. Give a simple description of the sequence $\{A_n\}_{n=1}^\infty$.

(d) Evaluate $\lim\limits_{n\to\infty} A_n$.

(e) Evaluate $\lim\limits_{n\to\infty} f_n(x)$ for a constant x, and call the result $g(x)$.

(f) Evaluate $\displaystyle\int_0^\infty g(x)\,\mathrm{d}x$.

Q[35]: hint answer solution

Determine the limit of the sequence $b_n = \left(1 + \dfrac{3}{n} + \dfrac{5}{n^2}\right)^n$.

1 Hunt, E. (2006). L'ENVOI. In The Mathematics of Behavior (pp. 325-327). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511618222.014, accessed online at https://www.cambridge.org/core/books/mathematics-of-behavior/lenvoi/01BDE784FC609A87928803587997DD77 on September 25, 2020

Q[36]: hint answer solution

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers satisfies the recursion relation $a_{n+1} = \dfrac{a_n + 8}{3}$ for $n \geqslant 2$.

(a) Suppose $a_1 = 4$. What is $\lim_{n\to\infty} a_n$?

(b) Find x if $x = \dfrac{x+8}{3}$.

(c) Suppose $a_1 = 1$. Show that $\lim_{n\to\infty} a_n = L$, where L is the solution to equation above.

Q[37]: hint answer solution

Zipf's Law applied to word frequency can be phrased as follows:

> The most-used word in a language is used n times as frequently as the n-th most word used in a language.

(a) Suppose the sequence $\{w_1, w_2, w_3, \ldots\}$ is a list of all words in a language, where w_n is the word that is the nth most frequently used. Let f_n be the frequency of word w_n. Is $\{f_1, f_2, f_3, \ldots\}$ an increasing sequence or a decreasing sequence?

(b) Give a general formula for f_n, treating f_1 as a constant.

(c) Suppose in a language, w_1 (the most frequently used word) has frequency 6%. If the language follows Zipf's Law, then what frequency does w_3 have?

(d) Suppose $f_6 = 0.3\%$ for a language following Zipf's law. What is f_{10}?

(e) The word "the" is the most-used word in contemporary American English. In a collection of about 450 million words, "the" appeared 22,038,615 times. The second-most used word is "be," followed by "and." About how many usages of these words do you expect in the same collection of 450 million words?

Sources: *Zipf's word frequency law in natural language: A critical review and future directions*, Steven T. Piantadosi. Psychon Bull Rev. 2014 Oct; 21(5): 1112–1130. Accessed online 11 October 2017

Word Frequency Data, `https://www.wordfrequency.info/free.asp?s=y` Accessed online 11 October 2017

Q[38]: hint answer solution

Binghao has a savings account. Every year, the bank pays him whatever was on the account at the start of the year, P, plus a percentage rate r of P.

(So, for example, if the percentage rate is 10% and Binghao invests 1 dollar, at the end of the year the bank will give him his dollar back plus 10% of 1 dollar. This will give a total of 1.10 dollars in the account at the end of one year.)

Below is a table of expected return per year, assuming that Binghao does not withdraw money from his account:

Year	Money	Simplify
0	P	P
1	$P+Pr$	$P(1+r)$
2	$P(1+r)+rP(1+r)$	I
3	II	III

(a) Complete the empty spaces in the table: I, II, and III

(b) If Binghao puts 100 dollars into his account at 10% rate, leaving in both the initial 100 dollars and the interest, how much money would he have in the account after two years?

(c) Determine the formula for this sequence in terms of time t measured in year, percentage rate r, and initial money (deposit) P.

(d) How much money does Binghao need to invest to have 300 dollars in 2 years? (For your interest, in finance this amount is called the "present value.")

5.1.1 ▸▸ (Optional) Geometric and harmonic sequences in musical scales

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Suppose we compose a 4-note ode to calculus using these four frequencies:

$$100, 110, 150, 200$$

A student from physics would like to sing the same song, but higher. The first note will have frequency 150 Hz, rather than 100 Hz, but the intervals will be the same. (This is called transposing a song.)

What frequencies should they use for the remaining three notes?

Q[2]: hint answer solution

Is the scale below even-tempered? How many intervals is an octave divided into?

$$100\text{ Hz}, 120\text{ Hz}, 140\text{ Hz}, 160\text{ Hz}, 180\text{ Hz}, 200\text{ Hz}$$

▸▸ Stage 2

Q[3]: hint answer solution

Some orchestras use 444 Hz to tune, rather than 440 Hz. Determine the note frequencies of an even-tempered scale that divides the octave from 444 to 888 into 12 intervals.

Q[4]: hint answer solution

Suppose you want to divide the octave from 100 Hz to 200 Hz into ten intervals to make an even-tempered scale. What are the frequencies of the notes in that scale?

Q[5]: hint answer solution
An instrument is only able to play the following[2] eight notes:

$$440,\ 495,\ 556.875,\ 586.\overline{66},\ 660,\ 742.5,\ 835.3125,\ 880$$

1. A song uses these notes: 440, 495, 556.875. Can the same instrument play this song if it is transposed so that the lowest note is $586.\overline{66}$ Hz, rather than 440 Hz?
2. A song uses these notes: 440, 495, 556.875. Can the same instrument play this song if it is transposed so that the lowest note is 495 Hz, rather than 440 Hz?

▸▸ Stage 3

Q[6]: hint answer solution
An instrument is only able to play notes from this (even-tempered) scale:

$$\left\{a_n = 100 \cdot 2^{\frac{n}{12}}\right\}_{n=0}^{\infty}$$

(To make things easy on the math side – though this would be hard on the instrumentalist – let's assume the instrument can play *every* note in this scale.)

A song, named *Ode to Question 6*, uses these notes in particular:

$$a_0, a_2, a_5, a_7$$

A singer wants to sing along with *Ode to Question 6*, but finds it too low. So, they want to replace the note a_0 in the song (100 Hz) with some higher note on the instrument, a_k. As long as all the intervals are the same, the song will sound the same – only higher.

For which values of a_k can the song still be played on the instrument?

Q[7]: hint answer solution
An instrument is only able to play notes from this (harmonic) scale:

$$\left\{b_n = 100n\right\}_{n=1}^{\infty}$$

(Again, let's assume the instrument can play *every* note in this scale.)

A song, named *Something Clever*, uses these notes in particular:

$$b_{10}, b_{11}, b_{12}$$

A singer wants to sing along with *Something Clever*, but finds it too high. So, they want to replace the highest note in the song (1200 Hz) with some lower note on the instrument, b_k. As long as all the intervals are the same, the song will sound the same – only lower.

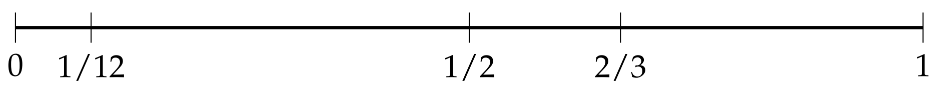
For which values of b_k can the song still be played on the instrument?

2 This isn't an entirely contrived example: the ratios follow Pythagorean tuning. See https://pages.mtu.edu/~suits/pythagorean.html.

Q[8]: hint answer solution
In Question 3 and 4, the common ratio between consecutive notes was an integer root of 2 (12th root of 2 and 10th root of 2, respectively). Why did the number 2 appear this way in both?

Q[9]: hint answer solution
On a guitar, different notes are played by the same string when a player pinches one end of the string against the instrument, effectively shortening the length of the string. Let's assume the frequency played by that string is inversely proportional to its length. (This is a reasonable assumption as long as the string is not very short or very thick.)

Suppose a string is shortened by pinching it at the different locations marked below. The portion that vibrates is the part of the string to the *right* of the pinched location. The sting at its longest (from position 0 to position 1) plays a tone of 330 Hz. What are the other notes?



5.2▴ Series

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution
Write out the first five partial sums corresponding to the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.
You don't need to simplify the terms.

Q[2]: hint answer solution
Every student who comes to class brings their instructor cookies, and leaves them on the instructor's desk. Let C_k be the total number of cookies on the instructor's desk after the kth student comes.

If $C_{11} = 20$, and $C_{10} = 17$, how many cookies did the 11th student bring to class?

Q[3]: hint answer solution

Suppose the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is $\{S_N\} = \left\{\frac{N}{N+1}\right\}$.

(a) What is $\{a_n\}$?
(b) What is $\lim_{n\to\infty} a_n$?
(c) Evaluate $\sum_{n=1}^{\infty} a_n$.

Q[4]: hint answer solution

Suppose the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is $\{S_N\} = \left\{(-1)^N + \frac{1}{N}\right\}$.

What is $\{a_n\}$?

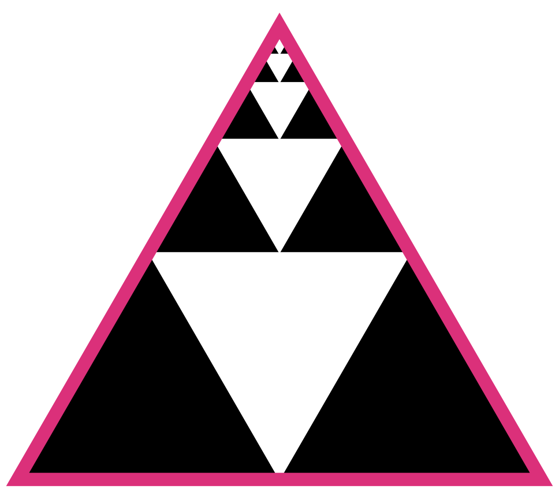
Q[5]: hint answer solution

Let $f(N)$ be a formula for the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. (That is, $f(N) = S_N$.) If $f'(N) < 0$ for all $N > 1$, what does that say about a_n?

Questions 6 through 8 invite you to explore geometric sums in a geometric way. This is complementary to than the algebraic method discussed in the text.

Q[6]: hint answer solution

Suppose the triangle outlined in red in the picture below has area one.

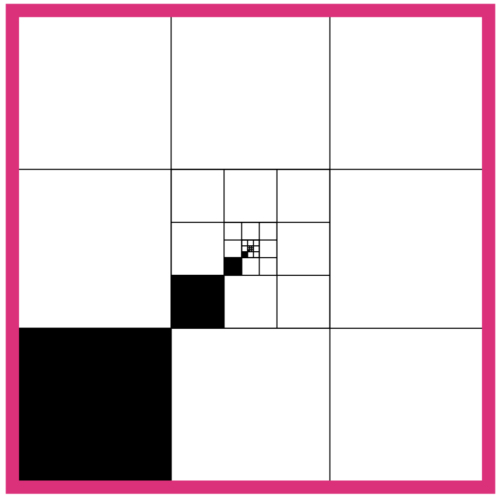


(a) Express the combined area of the black triangles as a series, assuming the pattern continues forever.

(b) Evaluate the series using the picture (*not* the formula from your book).

Q[7]: hint answer solution

Suppose the square outlined in red in the picture below has area one.



(a) Express the combined area of the black square as a series, assuming the pattern continues forever.

(b) Evaluate the series using the picture (*not* the formula from your book).

Q[8]: hint answer solution

In the style of Questions 6 and 7, draw a picture that represents $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ as an area.

Q[9]: hint answer solution

Evaluate $\displaystyle\sum_{n=0}^{100} \frac{1}{5^n}$.

Q[10]: hint answer solution

Every student who comes to class brings their instructor cookies, and leaves them on the instructor's desk. Let C_k be the total number of cookies on the instructor's desk after the kth student comes.

If $C_{20} = 53$, and $C_{10} = 17$, what does $C_{20} - C_{10} = 36$ represent?

Q[11]: hint answer solution

Evaluate $\displaystyle\sum_{n=50}^{100} \frac{1}{5^n}$. (Note the starting index.)

Q[12]: hint answer solution

(a) Starting on day $d = 1$, every day you give your friend $\$\frac{1}{d+1}$, and they give $\$\frac{1}{d}$ back to you. After a long time, how much money have you gained by this arrangement?

(b) Evaluate $\sum_{d=1}^{\infty} \left(\frac{1}{d} - \frac{1}{(d+1)} \right)$.

(c) Starting on day $d = 1$, every day your friend gives you $\$(d+1)$, and they take $\$(d+2)$ from you. After a long time, how much money have you gained by this arrangement?

(d) Evaluate $\sum_{d=1}^{\infty} ((d+1) - (d+2))$.

Q[13]: hint answer solution

Suppose $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, and $\sum_{n=1}^{\infty} c_n = C$.

Evaluate $\sum_{n=1}^{\infty} (a_n + b_n + c_{n+1})$.

Q[14]: hint answer solution

Suppose $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B \neq 0$, and $\sum_{n=1}^{\infty} c_n = C$.

True or false: $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n} + c_n \right) = \frac{A}{B} + C$.

▸▸ Stage 2

Q[15](∗): hint answer solution

To what value does the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$ converge?

Q[16](∗): hint answer solution

Evaluate $\sum_{k=7}^{\infty} \frac{1}{8^k}$

Q[17](∗): hint answer solution

Show that the series $\sum_{k=1}^{\infty} \left(\frac{6}{k^2} - \frac{6}{(k+1)^2} \right)$ converges and find its limit.

Q[18](∗): hint answer solution

Find the sum of the convergent series $\sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)$.

Q[19](*): hint answer solution

The n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is known to have the formula $s_n = \frac{1+3n}{5+4n}$.

(a) Find an expression for a_n, valid for $n \geqslant 2$.

(b) Show that the series $\sum_{n=1}^{\infty} a_n$ converges and find its value.

Q[20](*): hint answer solution

Find the sum of the series $\sum_{n=2}^{\infty} \frac{3 \cdot 4^{n+1}}{8 \cdot 5^n}$. Simplify your answer completely.

Q[21](*): hint answer solution

Relate the number $0.2\bar{3} = 0.233333\ldots$ to the sum of a geometric series, and use that to represent it as a rational number (a fraction or combination of fractions, with no decimals).

Q[22](*): hint answer solution

Express $2.656565\ldots$ as a rational number, i.e. in the form p/q where p and q are integers.

Q[23](*): hint answer solution

Express the decimal $0.\overline{321} = 0.321321321\ldots$ as a fraction.

Q[24](*): hint answer solution

Find the *value* of the convergent series

$$\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n}+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

Simplify your answer completely.

Q[25](*): hint answer solution

Evaluate

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n+\left(-\frac{2}{5}\right)^{n-1}\right]$$

Q[26](*): hint answer solution

Find the sum of the series $\sum_{n=0}^{\infty} \frac{1+3^{n+1}}{4^n}$.

Q[27]: hint answer solution

Evaluate $\sum_{n=5}^{\infty} \ln\left(\frac{n-3}{n}\right)$.

Q[28]: hint answer solution

Evaluate $\sum_{n=2}^{\infty} \left(\frac{2}{n} - \frac{1}{n+1} - \frac{1}{n-1}\right)$.

▸▸ Stage 3

Q[29]: hint answer solution

A random variable X has PMF

$$f(x) = \begin{cases} \frac{1}{2^x} & x = 1, 2, 3, 4, 5, \ldots \\ 0 & \text{else} \end{cases}$$

Find the CDF of X.

Q[30]: hint answer solution

Find the combined volume of an infinite collection of spheres, where for each whole number $n = 1, 2, 3, \ldots$ there is exactly one sphere of radius $\frac{1}{\pi^n}$.

Q[31]: hint answer solution

Evaluate $\sum_{n=3}^{\infty} \left(\frac{\sin^2 n}{2^n} + \frac{\cos^2(n+1)}{2^{n+1}}\right)$.

Q[32]: hint answer solution

Theorem 3.1.6 claims the following:

- $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, for all integers $n \geqslant 1$.
- $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$, for all integers $n \geqslant 1$.
- $\sum_{i=1}^{n} i^3 = \left[\frac{1}{2}n(n+1)\right]^2$, for all integers $n \geqslant 1$.

The first equation was proved in the text. We can use telescoping series (actually - telescoping partial sums) to prove[3] the last two.

(a) Show that $3i^2 + 3i = i(i+1)(i+2) - (i-1)i(i+1)$.

3 There are many different ways to prove these identities; credit to https://proofwiki.org/wiki/Sum_of_Sequence_of_Squares/Proof_by_Sum_of_Differences_of_Cubes and https://math.stackexchange.com/questions/2938660/prove-summation-k-1-to-n-k3-with-telescoping-rule for inspiring this problem by discussing proofs that use ideas from calculus.

(b) From (a), and using the identity $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, use the principal behind telescoping series to show $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

(c) Following the same proof technique as above, show that $\sum_{i=1}^{n} i^3 = \left[\frac{1}{2}n(n+1)\right]^2$ by first showing that

$$i^2(i+1)^2 - (i-1)^2 i^2 = 4i^3$$

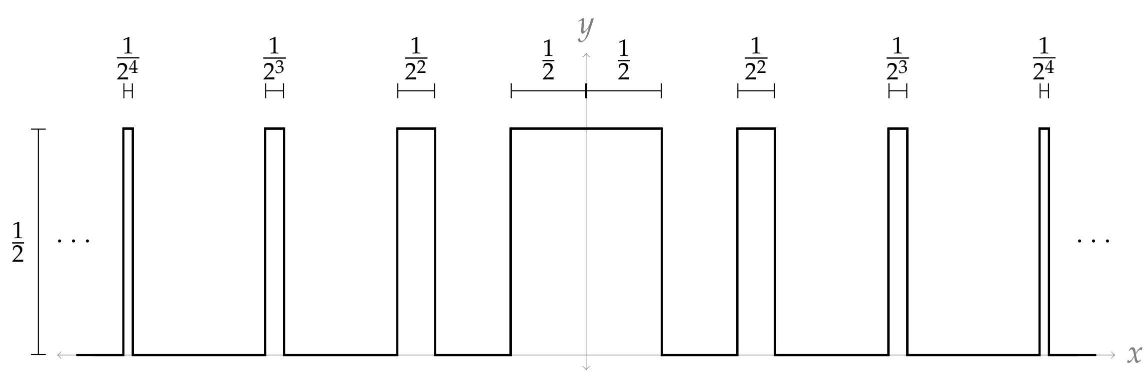
Q[33]: hint answer solution

Suppose a series $\sum_{n=1}^{\infty} a_n$ has sequence of partial sums $\{S_N\}$, and the series $\sum_{N=1}^{\infty} S_N$ has sequence of partial sums $\{\mathscr{S}_M\} = \left\{\sum_{N=1}^{M} S_N\right\}$.

If $\mathscr{S}_M = \dfrac{M+1}{M}$, what is a_n?

Q[34]: hint answer solution

The function $f(x)$ sketched below only takes on values of $f(x) = 0$ and $f(x) = \frac{1}{2}$. The pattern of "spikes" continues indefinitely. Determine whether $f(x)$ could be a PDF by deciding whether or not $\int_{-\infty}^{\infty} f(x) = 1$.



Q[35]: hint answer solution

Evaluate

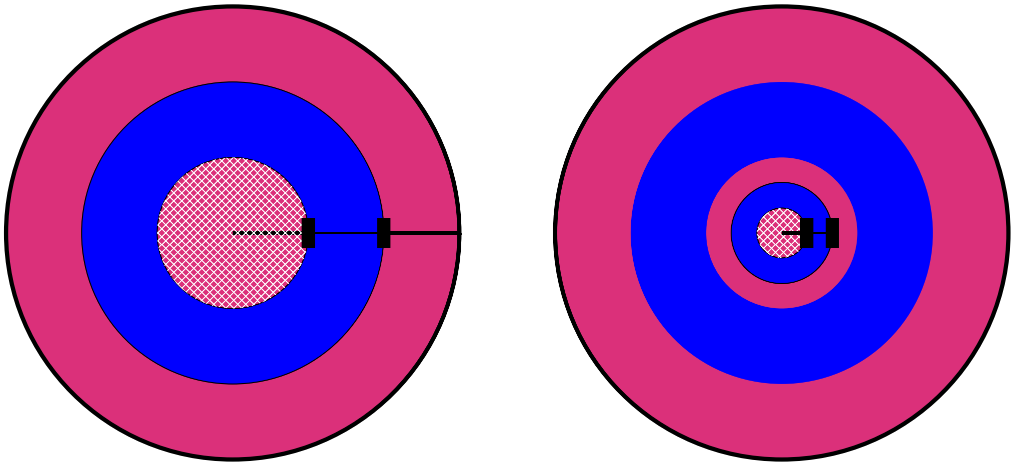
$$\sum_{n=1}^{1000} \frac{2}{n(n+1)(n+2)}$$

Q[36]: hint answer solution

Create a bullseye using the following method:

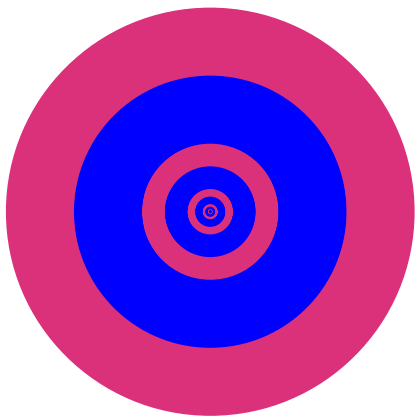
Starting with a red circle of area 1, divide the radius into thirds, creating two rings and a circle. Colour the middle ring blue.

Continue the pattern with the inside circle: divide its radius into thirds, and colour the middle ring blue.



Step 1 Step 2

Continue in this way indefinitely: dividing the radius of the innermost circle into thirds, creating two rings and another circle, and colouring the middle ring blue.



What is the area of the red portion?

5.3▲ Convergence Tests

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Select the series below that diverge by the divergence test.

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (B) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n+1}$ (C) $\displaystyle\sum_{n=1}^{\infty} \sin n$ (D) $\displaystyle\sum_{n=1}^{\infty} \sin(\pi n)$

Q[2]: hint answer solution

Select the series below whose terms satisfy the conditions to apply the integral test.

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (B) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n+1}$ (C) $\displaystyle\sum_{n=1}^{\infty} \sin n$ (D) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n + 1}{n^2}$

Q[3]: hint answer solution

Suppose there is some threshold after which a person is considered old, and before which they are young.

Let Olaf be an old person, and let Yuan be a young person.

(a) Suppose I am older than Olaf. Am I old?

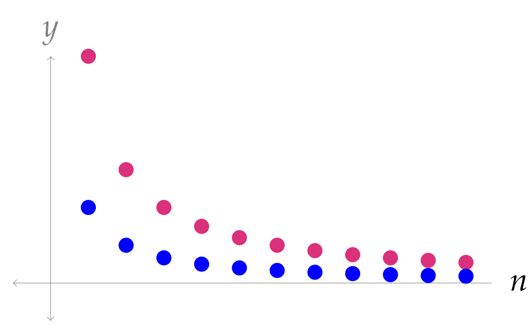
(b) Suppose I am younger than Olaf. Am I old?

(c) Suppose I am older than Yuan. Am I young?

(d) Suppose I am younger than Yuan. Am I young?

Q[4]: hint answer solution

Below are graphs of two sequences with positive terms. Assume the sequences continue as shown. Fill in the table with conclusions that can be made from the direct comparison test, if any.



	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$ ______	then $\sum b_n$ ______
and if $\{a_n\}$ is the blue series	then $\sum b_n$ ______	then $\sum b_n$ ______

Q[5]: hint answer solution

For each pair of series below, decide whether the second series is a valid comparison series to determine the convergence of the first series, using the *direct* comparison test and/or the *limit* comparison test.

(a) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n-1}$, compared to the divergent series $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2+1}$, compared to the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(c) $\displaystyle\sum_{n=5}^{\infty} \frac{n^3+5n+1}{n^6-2}$, compared to the convergent series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^3}$.

(d) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}}$, compared to the divergent series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt[4]{n}}$.

Q[6]: hint answer solution

Suppose a_n is a sequence with $\displaystyle\lim_{n\to\infty} a_n = \frac{1}{2}$. Does $\displaystyle\sum_{n=7}^{\infty} a_n$ converge or diverge, or is it not possible to determine this from the information given? Why?

Q[7]: hint answer solution

What flaw renders the following reasoning invalid?

Q: Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges or diverges.

A: First, we will evaluate $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n}$.

- Note $\dfrac{-1}{n} \leqslant \dfrac{\sin n}{n} \leqslant \dfrac{1}{n}$ for $n \geqslant 1$.
- Note also that $\displaystyle\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$.
- Therefore, by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = 0$ as well.

So, by the divergence test, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges.

Q[8]: hint answer solution

What flaw renders the following reasoning invalid?

Q: Determine whether $\displaystyle\sum_{n=1}^{\infty} (\sin(\pi n) + 2)$ converges or diverges.

A: We use the integral test. Let $f(x) = \sin(\pi x) + 2$. Note $f(x)$ is always positive, since $\sin(x) + 2 \geqslant -1 + 2 = 1$. Also, $f(x)$ is continuous.

$$\begin{aligned}
\int_1^\infty [\sin(\pi x) + 2]dx &= \lim_{b\to\infty} \int_1^b [\sin(\pi x) + 2]dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{\pi}\cos(\pi x) + 2x \Big|_1^b \right] \\
&= \lim_{b\to\infty} \left[-\frac{1}{\pi}\cos(\pi b) + 2b + \frac{1}{\pi}(-1) - 2 \right] \\
&= \infty
\end{aligned}$$

By the integral test, since the integral diverges, also $\sum_{n=1}^{\infty} (\sin(\pi n) + 2)$ diverges.

Q[9]: hint answer solution

What flaw renders the following reasoning invalid?

Q: Determine whether the series $\sum_{n=1}^{\infty} \frac{2^{n+1}n^2}{e^n + 2n}$ converges or diverges.

A: We want to compare this series to the series $\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n}$. Note both this series and the series in the question have positive terms.

First, we find that $\frac{2^{n+1}n^2}{e^n + 2n} > \frac{2^{n+1}}{e^n}$ when n is sufficiently large. The justification for this claim is as follows:

- We note that $e^n(n^2 - 1) > n^2 - 1 > 2n$ for n sufficiently large.
- Therefore, $e^n \cdot n^2 > e^n + 2n$
- Therefore, $2^{n+1} \cdot e^n \cdot n^2 > 2^{n+1}(e^n + 2n)$
- Since $e^n + 2n$ and e^n are both expressions that work out to be positive for the values of n under consideration, we can divide both sides of the inequality by these terms without having to flip the inequality. So, $\frac{2^{n+1}n^2}{e^n + 2n} > \frac{2^{n+1}}{e^n}$.

Now, we claim $\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n}$ converges.

Note $\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n} = 2\sum_{n=1}^{\infty} \frac{2^n}{e^n} = 2\sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n$. This is a geometric series with $r = \frac{2}{e}$.
Since $2/e < 1$, the series converges.

Now, by the Direct Comparison Test, we conclude that $\sum_{n=1}^{\infty} \frac{2^{n+1}n^2}{e^n + 2n}$ converges.

Q[10]: hint answer solution
Give an example of a convergent series for which the ratio test is inconclusive.

Q[11]: hint answer solution
Imagine you're taking an exam, and you momentarily forget exactly how the inequality in the ratio test works. You remember there's a ratio, but you don't remember which term goes on top; you remember there's something about the limit being greater than or less than one, but you don't remember which way implies convergence.

Explain why

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$$

or, equivalently,

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| < 1$$

should mean that the sum $\sum\limits_{n=1}^{\infty} a_n$ *diverges* (rather than converging).

Q[12]: hint answer solution
Give an example of a series $\sum\limits_{n=a}^{\infty} a_n$, with a function $f(x)$ such that $f(n) = a_n$ for all whole numbers n, such that:

- $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ diverges, while
- $\displaystyle\sum_{n=a}^{\infty} a_n$ converges.

Q[13](∗): hint answer solution
Suppose that you want to use the Limit Comparison Test on the series $\sum\limits_{n=0}^{\infty} a_n$ where $a_n = \dfrac{2^n + n}{3^n + 1}$. Write down a sequence $\{b_n\}$ such that $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}$ exists and is nonzero. (You don't have to carry out the Limit Comparison Test)

Q[14](∗): hint answer solution
Decide whether each of the following statements is true or false. If false, provide a counterexample. If true provide a brief justification.

(a) If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ converges.

(c) If $0 \leqslant a_n \leqslant b_n$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

▸▸ Stage 2

Q[15](*): hint answer solution

Does the series $\sum_{n=2}^{\infty} \frac{n^2}{3n^2+\sqrt{n}}$ converge?

Q[16](*): hint answer solution

Determine, with explanation, whether the series $\sum_{n=1}^{\infty} \frac{5^k}{4^k+3^k}$ converges or diverges.

Q[17](*): hint answer solution

Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}}$ is convergent or divergent. If it is convergent, find its value.

Q[18]: hint answer solution

Does the following series converge or diverge? $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\sqrt{k+1}}$

Q[19]: hint answer solution

Evaluate the following series, or show that it diverges: $\sum_{k=30}^{\infty} 3(1.001)^k$.

Q[20]: hint answer solution

Evaluate the following series, or show that it diverges: $\sum_{n=3}^{\infty} \left(\frac{-1}{5}\right)^n$.

Q[21]: hint answer solution

Does the following series converge or diverge? $\sum_{n=7}^{\infty} \sin(\pi n)$

Q[22]: hint answer solution

Does the following series converge or diverge? $\sum_{n=7}^{\infty} \cos(\pi n)$

Q[23]: hint answer solution

Does the following series converge or diverge? $\sum_{k=1}^{\infty} \frac{e^k}{k!}$.

Q[24]: hint answer solution

Evaluate the following series, or show that it diverges: $\sum_{k=0}^{\infty} \frac{2^k}{3^{k+2}}$.

Q[25]: hint answer solution

Does the following series converge or diverge? $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

Q[26]: hint answer solution

Does the following series converge or diverge? $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{2n^4+n}$.

Q[27](∗): hint answer solution

Show that the series $\displaystyle\sum_{n=3}^{\infty} \frac{5}{n(\ln n)^{3/2}}$ converges.

Q[28](∗): hint answer solution

Find the values of p for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges.

Q[29](∗): hint answer solution

Does $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converge or diverge?

Q[30](∗): hint answer solution

Use the comparison test (not the limit comparison test) to show whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{3n^2-7}}{n^3}$ converges or diverges.

Q[31](∗): hint answer solution

Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}}$ converges.

Q[32](∗): hint answer solution

Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 2^{n/3}}{(2n+7)^4}$ converge or diverge?

Q[33](∗): hint answer solution

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k^4-2k^3+2}{k^5+k^2+k}$$

converges or diverges.

Q[34](∗): hint answer solution

Determine whether each of the following series converge or diverge.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+n+1}{n^5-n}$

(b) $\displaystyle\sum_{m=1}^{\infty} \frac{3m+\sin\sqrt{m}}{m^2}$

Q[35]: hint answer solution

Evaluate the following series, or show that it diverges: $\displaystyle\sum_{n=5}^{\infty} \frac{1}{e^n}$.

Q[36](∗): hint answer solution

Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{6}{7^n}$ is convergent or divergent. If it is convergent, find its value.

Q[37](∗): hint answer solution

Determine, with explanation, whether each of the following series converge or diverge.

(a) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{2^{2n+1}}$

Q[38](∗): hint answer solution

Determine, with explanation, whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{\sqrt[3]{k}}{k^2-k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{10}10^k(k!)^2}{(2k)!}$.

(c) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)(\ln\ln k)}$.

Q[39](∗): hint answer solution

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-4}{2n^5-6n}$ is convergent or divergent.

▸▸ Stage 3

Q[40](∗): hint answer solution

Determine, with explanation, whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{9^n n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$

Q[41](∗): hint answer solution

(a) Prove that $\displaystyle\int_2^{\infty} \frac{x+\sin x}{1+x^2}\,dx$ diverges.

(b) Explain why you *cannot* conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges from part (a) and the Integral Test.

(c) Determine, with explanation, whether $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ converges or diverges.

Q[42](∗): hint answer solution

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges and find an interval of length 0.05 or less that contains its exact value.

Q[43](∗): hint answer solution

Suppose that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges and that $1 > a_n \geqslant 0$ for all n. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1-a_n}$ also converges.

Q[44](∗): hint answer solution

Suppose that the series $\sum_{n=0}^{\infty} (1-a_n)$ converges, where $a_n > 0$ for $n = 0, 1, 2, 3, \cdots$. Determine whether the series $\sum_{n=0}^{\infty} 2^n a_n$ converges or diverges.

Q[45](∗): hint answer solution

Assume that the series $\displaystyle\sum_{n=1}^{\infty} \frac{na_n - 2n + 1}{n+1}$ converges, where $a_n > 0$ for $n = 1, 2, \cdots$. Is the following series

$$-\ln a_1 + \sum_{n=1}^{\infty} \ln\left(\frac{a_n}{a_{n+1}}\right)$$

convergent? If your answer is NO, justify your answer. If your answer is YES, evaluate the sum of the series $-\ln a_1 + \sum_{n=1}^{\infty} \ln\left(\frac{a_n}{a_{n+1}}\right)$.

Q[46](*): hint answer solution

Prove that if $a_n \geqslant 0$ for all n and if the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} a_n^2$ also converges.

A number of phenomena roughly follow a distribution called Zipf's law. We discuss some of these in Questions 47 and 48.

Q[47]: hint answer solution

Suppose the frequency of word use in a language has the following pattern:

> The n-th most frequently used word accounts for $\dfrac{\alpha}{n}$ percent of the total words used.

So, in a text of 100 words, we expect the most frequently used word to appear α times, while the second-most-frequently used word should appear about $\frac{\alpha}{2}$ times, and so on.

If a text written in this language uses $20,000$ distinct words, then the most commonly used word accounts for roughly what percentage of total words used?

Q[48]: hint answer solution

Suppose the sizes of cities in a country adhere to the following pattern: if the largest city has population α, then the n-th largest city has population $\frac{\alpha}{n}$.

If the largest city in this country has 2 million people, what do you expect the population of the entire country is? Make your answer accurate to within 1 million people.

5.4▲ Absolute and Conditional Convergence

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1](*): hint answer solution

Decide whether the following statement is true or false. If false, provide a counterexample. If true provide a brief justification.

> If $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges.

Q[2]: hint answer solution

Describe the sequence $\sum_{n=1}^{\infty} a_n$ based on whether $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge or diverge, using vocabulary from this section where possible.

	$\sum a_n$ **converges**	$\sum a_n$ **diverges**
$\sum \lvert a_n \rvert$ **converges**		
$\sum \lvert a_n \rvert$ **diverges**		

▸▸ Stage 2

Q[3](*): hint answer solution

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{9n+5}$ is absolutely convergent, conditionally convergent, or divergent; justify your answer.

Q[4](*): hint answer solution

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{1+n}$ is absolutely convergent, conditionally convergent, or divergent.

Q[5](*): hint answer solution

The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1+4^n}{3+2^{2n}}$ either: converges absolutely; converges conditionally; diverges; or none of the above. Determine which is correct.

Q[6](*): hint answer solution

Does the series $\displaystyle\sum_{n=5}^{\infty} \frac{\sqrt{n}\cos n}{n^2-1}$ converge conditionally, converge absolutely, or diverge?

Q[7](*): hint answer solution

Determine (with justification!) whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-\sin n}{n^6+n^2}$ converges absolutely, converges but not absolutely, or diverges.

Q[8](*): hint answer solution

Determine (with justification!) whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(n^2+1)(n!)^2}$ converges absolutely, converges but not absolutely, or diverges.

Q[9](*): hint answer solution

Determine (with justification!) whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^{101}}$ converges absolutely, converges but not absolutely, or diverges.

Q[10]: hint answer solution

Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

Q[11]: hint answer solution

Show that the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sin n}{4}-\frac{1}{8}\right)^n$ converges.

Q[12]: hint answer solution

Show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n-\cos^2 n+\frac{1}{2}}{2^n}$ converges.

▸▸ Stage 3

Q[13](*): hint answer solution

Both parts of this question concern the series $S=\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}24n^2e^{-n^3}$.

(a) Show that the series S converges absolutely.

(b) Suppose that you approximate the series S by its fifth partial sum S_5. Give an upper bound for the error resulting from this approximation.

Q[14]: hint answer solution

You may assume without proof the following:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}=\cos(1)$$

Using this fact, approximate $\cos 1$ as a rational number, accurate to within $\frac{1}{1000}$.

Check your answer against a calculator's approximation of $\cos(1)$: what was your actual error?

Q[15]: hint answer solution

Let a_n be defined as

$$a_n=\begin{cases}-e^{n/2} & \text{if } n \text{ is prime}\\ n^2 & \text{if } n \text{ is \textbf{not} prime}\end{cases}$$

Show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{e^n}$ converges.

5.5▴ Power Series

Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Suppose $f(x) = \sum_{n=0}^{\infty} \left(\frac{3-x}{4}\right)^n$. What is $f(1)$?

Q[2]: hint answer solution

Suppose $f(x) = \sum_{n=1}^{\infty} \frac{(x-5)^n}{n!+2}$. Give a power series representation of $f'(x)$.

Q[3]: hint answer solution

Let $f(x) = \sum_{n=a}^{\infty} A_n(x-c)^n$ for some positive constants a and c, and some sequence of constants $\{A_n\}$. For which values of x does $f(x)$ definitely converge?

Q[4]: hint answer solution

Let $f(x)$ be a power series centred at $c = 5$. If $f(x)$ converges at $x = -1$, and diverges at $x = 11$, what is the radius of convergence of $f(x)$?

▸▸ Stage 2

Q[5](*): hint answer solution

(a) Find the radius of convergence of the series

$$\sum_{k=0}^{\infty} (-1)^k 2^{k+1} x^k$$

(b) You are given the formula for the sum of a geometric series, namely:

$$1 + r + r^2 + \cdots = \frac{1}{1-r}, \qquad |r| < 1$$

Use this fact to evaluate the series in part (a).

Q[6](*): hint answer solution

Find the radius of convergence for the power series $\sum_{k=0}^{\infty} \frac{x^k}{10^{k+1}(k+1)!}$

Q[7](*): hint answer solution

Find the radius of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$.

Q[8](*): hint answer solution

Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{\sqrt{n}}$, where x is a real number. Find the radius of convergence of this series.

Q[9](*): hint answer solution

Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{x+1}{3}\right)^n$$

Q[10](*): hint answer solution

Find the radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^{4/5}(5^n-4)}.$$

Q[11](*): hint answer solution

Find the largest possible open interval of values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$ converges.

Q[12](*): hint answer solution

Find the interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n}(x-1)^n$. Don't worry about convergence at the endpoints of the interval.

Q[13](*): hint answer solution

Find, with explanation, the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2^n(n+2)}$$

You don't need to include the endpoints of the interval of convergence.

Q[14](*): hint answer solution

Find the *interval of convergence* for the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2 (x-a)^{2n}$ where a is a constant. You don't need to worry about the endpoints of the interval.

Q[15](*): hint answer solution
Find the intervals of convergence of the following series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^k}{k 2 9^k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} a_k (x-1)^k$, where $a_k > 0$ for $k = 1, 2, \cdots$ and $\displaystyle\sum_{k=1}^{\infty} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right) = \frac{a_1}{a_2}$.

Don't worry about the endpoints of the intervals.

Q[16](*): hint answer solution

Find a power series representation for $\dfrac{x^3}{1-x}$.

Q[17]: hint answer solution

Suppose $f'(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n+2}$, and $\displaystyle\int_5^x f(t)\mathrm{d}t = 3x + \sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n(n+1)^2}$.
Give a power series representation of $f(x)$.

▸▸ Stage 3

Q[18](*): hint answer solution
Determine the values of x for which the series

$$\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \ln n}$$

converges absolutely, converges conditionally, or diverges.

Q[19](*): hint answer solution

(a) Show that $\displaystyle\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$ for $-1 < x < 1$.

(b) Express $\displaystyle\sum_{n=0}^{\infty} n^2 x^n$ as a ratio of polynomials. For which x does this series converge?
(Give the largest open interval.)

Q[20](*): hint answer solution
Suppose that you have a sequence $\{b_n\}$ such that the series $\sum_{n=0}^{\infty}(1-b_n)$ converges. Using the tests we've learned in class, prove that the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} b_n x^n$ is equal to 1.

Q[21](*): hint answer solution

Assume $\{a_n\}$ is a sequence such that na_n decreases to C as $n \to \infty$ for some real number $C > 0$

(a) Find the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$. Justify your answer carefully.

(b) Find the interval of convergence of the above power series, that is, find all x for which the power series in (a) converges. Justify your answer carefully, but don't bother with the endpoints of the interval of convergence.

Q[22]: hint answer solution

Let $f(x) = \sum_{n=0}^{\infty} A_n(x-c)^n$, for some constant c and a sequence of constants $\{A_n\}$. Further, let $f(x)$ have a positive radius of covergence.

If $A_1 = 0$, show that $y = f(x)$ has a critical point at $x = c$. What is the relationship between the behaviour of the graph at that point and the value of A_2?

Q[23]: hint answer solution

Evaluate $\sum_{n=3}^{\infty} \frac{n}{5^{n-1}}$.

5.6▲ Taylor Series

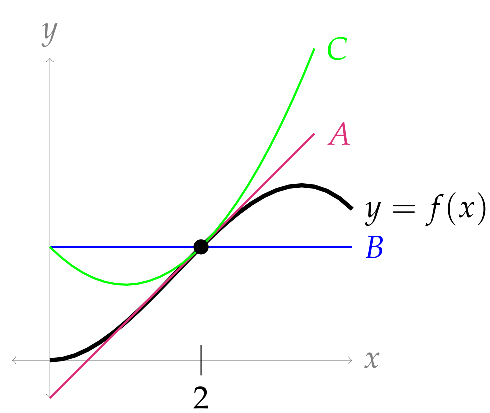
Exercises

Jump to HINTS, ANSWERS, SOLUTIONS or TABLE OF CONTENTS.

▸▸ Stage 1

Q[1]: hint answer solution

Below is a graph of $y = f(x)$, along with the constant approximation, linear approximation, and quadratic approximation centred at $a = 2$. Which is which?



Q[2]: hint answer solution

Suppose $T(x)$ is the Taylor series for $f(x) = \arctan^3(e^x + 7)$ centred at $a = 5$. What is $T(5)$?

Q[3]: hint answer solution

Below are a list of common functions, and their Taylor series representations. Match the function to the Taylor series.

function	series
A. $\dfrac{1}{1-x}$	I. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$
B. $\ln(1+x)$	II. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$
C. $\arctan x$	III. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$
D. e^x	IV. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$
E. $\sin x$	V. $\displaystyle\sum_{n=0}^{\infty}x^n$
F. $\cos x$	VI. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

Q[4]: hint answer solution

(a) Suppose $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{n^2}{(n!+1)}(x-3)^n$ for all real x. What is $f^{(20)}(3)$ (the twentieth derivative of $f(x)$ at $x = 3$)?

(b) Suppose $g(x) = \displaystyle\sum_{n=0}^{\infty}\frac{n^2}{(n!+1)}(x-3)^{2n}$ for all real x. What is $g^{(20)}(3)$?

(c) If $h(x) = \dfrac{\arctan(5x^2)}{x^4}$, what is $h^{(20)}(0)$? What is $h^{(22)}(0)$?

▸▸ Stage 2

In Questions 5 through 8, you will create Taylor series from scratch. In practice, it is often preferable to modify an existing series, rather than creating a new one, but you should understand both ways.

Q[5]: hint answer solution

Using the definition of a Taylor series, find the Taylor series for $f(x) = \ln(x)$ centred at $x = 1$.

Q[6]: hint answer solution

Find the Taylor series for $f(x) = \sin x$ centred at $a = \pi$.

Q[7]: hint answer solution

Using the definition of a Taylor series, find the Taylor series for $g(x) = \frac{1}{x}$ centred at $x = 10$. What is the interval of convergence of the resulting series?

Q[8]: hint answer solution

Using the definition of a Taylor series, find the Taylor series for $h(x) = e^{3x}$ centred at $x = a$, where a is some constant. What is the radius of convergence of the resulting series?

In Questions 9 through 16, practice creating new Taylor series by modifying known Taylor series, rather than creating your series from scratch.

Q[9](*): hint answer solution

Find the Maclaurin series for $f(x) = \frac{1}{2x-1}$.

Q[10](*): hint answer solution

Let $\sum_{n=0}^{\infty} b_n x^n$ be the Maclaurin series for $f(x) = \frac{3}{x+1} - \frac{1}{2x-1}$,

i.e. $\sum_{n=0}^{\infty} b_n x^n = \frac{3}{x+1} - \frac{1}{2x-1}$. Find b_n.

Q[11](*): hint answer solution

Find the coefficient c_5 of the fifth degree term in the Maclaurin series $\sum_{n=0}^{\infty} c_n x^n$ for e^{3x}.

Q[12](*): hint answer solution

Express the Taylor series of the function

$$f(x) = \ln(1 + 2x)$$

about $x = 0$ in summation notation.

Q[13](*): hint answer solution

The first two terms in the Maclaurin series for $x^2 \sin(x^3)$ are $ax^5 + bx^{11}$, where a and b are constants. Find the values of a and b.

Q[14](*): hint answer solution

Give the first two nonzero terms in the Maclaurin series for $\displaystyle\int \frac{e^{-x^2}-1}{x}\,dx$.

Q[15](*): hint answer solution

Find the Maclaurin series for $\displaystyle\int x^4 \arctan(2x)\,dx$.

Q[16](*): hint answer solution

Suppose that $\dfrac{df}{dx} = \dfrac{x}{1+3x^3}$ and $f(0) = 1$. Find the Maclaurin series for $f(x)$.

In past chapters, we were only able to exactly evaluate very specific types of series: geometric and telescoping. In Questions 17 through 25, we expand our range by relating given series to Taylor series.

Q[17](*): hint answer solution

The Maclaurin series for $\arctan x$ is given by

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

which has radius of convergence equal to 1. Use this fact to compute the exact value of the series below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

Q[18](*): hint answer solution

Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

Q[19](*): hint answer solution

Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{1}{e^k k!}$.

Q[20](*): hint answer solution

Evaluate the sum of the convergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\pi^k k!}$.

Q[21](*): hint answer solution

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}$.

Q[22](*): hint answer solution

Evaluate $\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n$.

Q[23]: hint answer solution

Evaluate $\sum_{n=1}^{\infty} \frac{2^n}{n}$, or show that it diverges.

Q[24]: hint answer solution

Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} \left(1 + 2^{2n+1}\right)$$

or show that it diverges.

Q[25](*): hint answer solution

(a) Show that the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ converges absolutely for all real numbers x.

(b) Evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n)!}$.

Q[26]: hint answer solution

Suppose you wanted to approximate the number e as a rational number using the Maclaurin expansion of e^x. How many terms would you need to add to get 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11}.)
You may assume without proof that $2 < e < 3$.

Q[27]: hint answer solution

Suppose you wanted to approximate the number $\ln(0.9)$ as a rational number using the Taylor expansion of $\ln(1-x)$. Which partial sum should you use to get 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11}.)

Q[28]: hint answer solution

Define the hyperbolic sine function as

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Suppose you wanted to approximate the number $\sinh(b)$ using the Maclaurin series of $\sinh x$, where b is some number in $(-2, 1)$. Which partial sum should you use to guarantee 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11}.)

You may assume without proof that $2 < e < 3$.

Q[29]: hint answer solution

Let $f(x)$ be a function with

$$f^{(n)}(x) = \frac{(n-1)!}{2} \left[(1-x)^{-n} + (-1)^{n-1}(1+x)^{-n}\right]$$

for all $n \geqslant 1$.

Give reasonable bounds (both upper and lower) on the error involved in approximating $f\left(-\frac{1}{3}\right)$ using the partial sum S_6 of the Taylor series for $f(x)$ centred at $a = \frac{1}{2}$.

Remark: One function with this quality is the inverse hyperbolic tangent function.[4]

▸▸ Stage 3

Q[30](*): hint answer solution

Use series to evaluate $\lim_{x\to 0} \dfrac{1-\cos x}{1+x-e^x}$.

Q[31](*): hint answer solution

Evaluate $\lim_{x\to 0} \dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$.

Q[32]: hint answer solution

Evaluate $\lim_{x\to 0} \left(1+x+x^2\right)^{2/x}$ using a Taylor series for the natural logarithm.

Q[33]: hint answer solution

Use series to evaluate

$$\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^x$$

Q[34]: hint answer solution

Evaluate the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{7^n}$ or show that it diverges.

Q[35]: hint answer solution

Write the series $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+1)(2n+2)}$ explicitly (without a series).

Q[36]: hint answer solution

(a) Find the Maclaurin series for $f(x) = (1-x)^{-1/2}$. What is its radius of convergence?

(b) Manipulate the series you just found to find the Maclaurin series for $g(x) = \arcsin x$. What is its radius of convergence?

Q[37](*): hint answer solution

Find the Taylor series for $f(x) = \ln(x)$ centred at $a = 2$. Find the interval of convergence for this series. (You may include endpoints for the interval of convergence if you wish.)

Q[38](*): hint answer solution

4 Of course it is! Actually, hyperbolic tangent is $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, and inverse hyperbolic tangent is its functional inverse.

Let $I(x) = \int_0^x \frac{1}{1+t^4}\,\mathrm{d}t$. Find the Maclaurin series for $I(x)$.

Q[39](*): hint answer solution

Let $I(x) = \int_0^x \frac{e^{-t}-1}{t}\,\mathrm{d}t$. Find the Maclaurin series for $I(x)$.

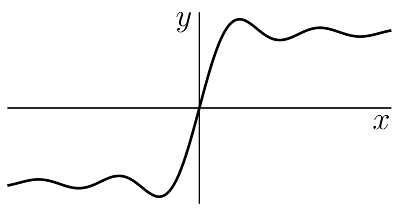
Q[40](*): hint answer solution

The function $\Sigma(x)$ is defined by $\Sigma(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$.

(a) Find the Maclaurin series for $\Sigma(x)$.

(b) It can be shown that $\Sigma(x)$ has an absolute maximum which occurs at its smallest positive critical point (see the graph of $\Sigma(x)$ below). Find this critical point.

(c) Use the previous information to find the maximum value of $\Sigma(x)$ to within ± 0.01.



Q[41](*): hint answer solution

Let $I(x) = \int_0^x \frac{\cos t - 1}{t^2}\,\mathrm{d}t$.

Find the Maclaurin series for $I(x)$.

Q[42](*): hint answer solution

Let $I(x) = \int_0^x \frac{\cos t + t\sin t - 1}{t^2}\,\mathrm{d}t$

Find the Maclaurin series for $I(x)$.

Q[43](*): hint answer solution

Define $f(x) = \int_0^x \frac{1-e^{-t}}{t}\,\mathrm{d}t$.

(a) Show that the Maclaurin series for $f(x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$.

(b) Use the ratio test to determine the values of x for which the Maclaurin series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$ converges.

Q[44](*): hint answer solution

Show that $\int_0^1 \frac{x^3}{e^x - 1}\,\mathrm{d}x \leqslant \frac{1}{3}$.

Q[45](*): hint answer solution

Let $\cosh(x) = \frac{e^x + e^{-x}}{2}$.

(a) Find the power series expansion of $\cosh(x)$ about $x_0 = 0$ and determine its interval of convergence.

(b) Show that $\frac{11}{3} \leqslant \cosh(2) \leqslant \frac{11}{3} + 0.1$.

(c) Show that $\cosh(t) \leqslant e^{\frac{1}{2}t^2}$ for all t.

Q[46]: hint answer solution

Consider the following function:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Sketch $y = f(x)$.

(b) Assume (without proof) that $f^{(n)}(0) = 0$ for all whole numbers n. Find the Maclaurin series for $f(x)$.

(c) Where does the Maclaurin series for $f(x)$ converge?

(d) For which values of x is $f(x)$ equal to its Maclaurin series?

Q[47]: hint answer solution

Suppose $f(x)$ is an odd function, and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Simplify $\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n}$.

Part II

HINTS TO QUESTIONS

Hints for Exercises 1.1. — Jump to TABLE OF CONTENTS.

H-1: The fill patterns are only included to distinguish different parts of the diagram.

H-2: Read the last page of Section 1.2 describing spheres in $\mathbb{R}^3$.

H-3: This is a review question to get you thinking about $\mathbb{R}^2$ in a way that will help you get used to $\mathbb{R}^3$.

H-4: Compare to Question 3. To visualize what's going on, it can help to consider what shapes you'd get if z were a constant.

If you're struggling to visualize $\mathbb{R}^3$, Appendix A.1 in the text shows you how to fold a model of its first octant.

H-5: From the text, the distance from the point (x, y, z) to the point (x', y', z') is

$$\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

H-6: From the text, the distance from the point (x, y, z) to the xy-plane is $|z|$.

H-7: From the text, the distance from the point (x, y, z) to the point (x', y', z') is

$$\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

100 metres is one-tenth of a kilometre.

H-8: From the text, the distance from the point (x, y, z) to the point (x', y', z') is

$$\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

Given the distance and the x and y coordinates, you can solve for the z coordinate.

H-9: At which part of the journey are you actually getting farther away from the wall?

H-10: The isobar is a curve of the form $x^2 - 2cx + y^2 = 1$, where c is a constant. These describe circles – figure out what their centres and radii are.

H-12: This centre must be equidistant from the three vertices.

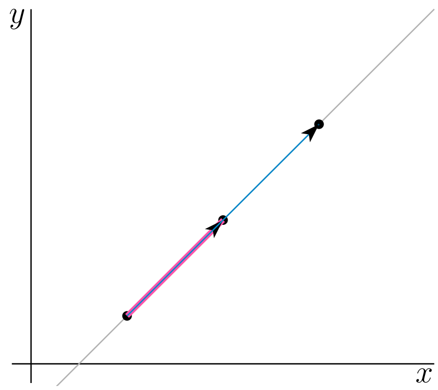
H-13: From the text, the distance from the point (x, y, z) to the point (x', y', z') is

$$\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

Also from the text, the distance from the point (x, y, z) to the xy-plane is $|z|$. Use a similar thought process to find the distance from a point (x, y, z) to the plane $z = -1$.

Hints for Exercises 1.2. — Jump to TABLE OF CONTENTS.

H-2: If three points are collinear, then the vector from the first point to the second point, and the vector from the first point to the third point must both be parallel to the line, and hence must be parallel to each other (i.e. must be multiples of each other).



H-3: Review Theorem 1.2.9 in the text.

H-5: the statement is false

H-6: You can think of the name of a vector as directions from its tail to its head. The vector $\langle 1, 2\rangle$ has a head that is one unit to the right, and two units up, from its tail.

H-7: Recall $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$

H-8: If a vector $\mathbf{a}$ has length ℓ, then $\frac{1}{\ell}\mathbf{a}$ is its unit vector.

H-9: Multiplying a vector by -1 gives a parallel vector of the same length.

H-10: $|\mathbf{a} - \mathbf{b}| \neq |\mathbf{a}| - |\mathbf{b}|$

H-11: The centre of the sphere is the midpoint of the diameter.

H-12: Parallel vectors are nonzero scalar multiples of one another. Perpendicular vectors have a dot product of 0.

H-13: Two vectors are perpendicular when their dot product is 0.

H-14: Remember $\hat{\imath} = \langle 1, 0\rangle$ and $\hat{\jmath} = \langle 0, 1\rangle$.

H-15: Set up a parallelogram so that one vertex is at the origin and the two sides touching that vertex are the vectors $\mathbf{a}$ and $\mathbf{b}$. Then, find an expression for the midpoint of each diagonal.

Hints for Exercises 1.3. — Jump to TABLE OF CONTENTS.

H-1: You are looking for a vector that is perpendicular to $z = 0$ and hence is parallel to $\hat{\mathbf{k}}$.

H-2: See Example 1.3.2 in the text.

H-3: Guess.

H-4: See Question 3 — or just have a guess!

H-5: Three points don't always determine a plane — why?

H-6: All vectors normal to a given plane are parallel to one another.

H-7: Write the equation of the plane.

H-8: Use the two points to find a vector in the plane.

H-9: Your intersections will all be lines.

H-10: Planes are perpendicular if their normal vectors are orthogonal; they are parallel (or identical) if their normal vectors are parallel; they are identical if their equations are equivalent.

H-11: Your answer will be a plane.

H-12: Review Example 1.3.6 in the text; be extra careful with (c).

H-13: Compare to Question 11. You can describe your plane by giving a point and a vector, without explicitly writing an equation of the form "$ax + by + ca = d$."

Hints for Exercises 1.4. — Jump to TABLE OF CONTENTS.

H-1: Once you pick the number for the range, you're basically done....

H-2: This is a review of high-school material, since we have functions of only one variable. We want you to think about it to get in the right mindset.

H-3: If you set $x = y = 1$, is there a solution to the equation?

H-4: To find the range, consider all points in the domain with $x = 0$.

H-5: For the range, consider $h(x, 0)$.

H-6: The domain of the function $\arcsin(x)$ is $[-1, 1]$, and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

H-7: One way of thinking of $xy > 0$ is that x and y must have the same sign (and both be nonzero).

H-8: y doesn't impact the final value of $f(x, y)$, so think of this as a problem from last semester. What are the maximum and minimum values of the function $f(x) = \frac{x^2}{x^2+1}$? Can you sketch its graph?

H-9: Consider the functions $f_1(x) = \frac{x}{x^2+1}$ and $f_2(y) = \sin y$ separately.

H-10: Do you see any *signs* that might point you in the right direction?

H-11: The domain will look like a ring

H-12: First work with the function

$$h(t) = 72t^2 - t^4$$

Then, think about the implications of $t = x^2 - y$.

Hints for Exercises 1.5. — Jump to TABLE OF CONTENTS.

H-1: Consider the traces. That is, if you set one variable equal to a constant, what will the resulting cross-sections look like?

H-2: Draw in the plane $z = C$ for several values of C.

H-3: Remember when you set $f(x, y)$ equal to a constant, the result is a curve with only x's and y's.

H-4: The circle centred at $(0, a)$ with radius r has equation

$$x^2 + (y - a)^2 = r^2$$

Rearranged, this is

$$x^2 + y^2 - (2a)y = r^2 = a^2$$

Use this to describe the level curves of the function given.

H-5: If z is constant, then the entire expression $-z^2 + 2z$ is one big constant.

H-6: For each fixed z, $4x^2 + y^2 = 1 + z^2$ is an ellipse. So the surface consists of a stack of ellipses one on top of the other. The

H-7: Start by determining what convenient traces look like. For (a), the level curves are less instructive at first than are the traces found by setting y equal to a constant.

H-9: To solve (say) $\sin(x + y) = 0$, you get lots of solutions: $x + y = 0$, $x + y = \pi$, $x + y = 2\pi$, etc.

H-10: Since the level curves are circles centred at the origin (in the xy-plane), the equation will have the form $x^2 + y^2 = g(z)$, where $g(z)$ is a function depending only on z.

Hints for Exercises 2.1. — Jump to TABLE OF CONTENTS.

H-1: What happens if you move "backwards," in the negative y direction?

H-2: Use the definition of the derivative:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \approx \frac{f(x + 0.1, y) - f(x, y)}{0.1}$$

H-4: Just evaluate $x\frac{\partial z}{\partial x}(x, y) + y\frac{\partial z}{\partial y}(x, y)$.

H-5: This is an implicit differentiation question. Implicit differentiation, as you'll recall from first-semester calculus, is more-or-less just an application of the chain rule.

H-6: Differentiate implicitly.

H-9: Just evaluate $y\frac{\partial z}{\partial x}(x, y)$ and $x\frac{\partial z}{\partial y}(x, y)$.

H-11: You can find an equation for the surface, or just look at the diagram.

H-12: For (a) and (b), remember $\frac{\partial f}{\partial x}(x, y) = \lim\limits_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$ and $\frac{\partial f}{\partial y}(x, y) = \lim\limits_{h \to 0} \frac{f(x,y+h)-f(x,y)}{h}$. For (c), you're finding the derivative of a function of one variable, say $g(t)$, where

$$g(t) = f(t,t) = \begin{cases} \frac{t^2 t}{t^2+t^2} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Hints for Exercises 2.2. — Jump to TABLE OF CONTENTS.

H-1: Repeatedly use (Clairaut's) Theorem 2.2.5 in the CLP-3 text.

H-2: If $f(x,y)$ obeying the specified conditions exists, then it is necessary that $f_{xy}(x,y) = f_{yx}(x,y)$.

H-3: Save yourself time by using Theorem 2.2.5.

H-4: Remember there are *four* second partial derivatives: f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

H-5: (a) This higher order partial derivative can be evaluated extremely efficiently by carefully choosing the order of evaluation of the derivatives.

(b) This higher order partial derivative can be evaluated extremely efficiently by carefully choosing a *different order of evaluation* of the derivatives for each of the three terms.

(c) Set $g(x) = f(x,0,0)$. Then $f_{xx}(1,0,0) = g''(1)$.

H-7: Check whether the above utility functions satisfy all the properties mentioned in the question for x and y (in place of t). If any of properties is not satisfied then you do not need to check the rest.

H-8: A similar method as Question 3 in Section 2.1, but iterated.

Hints for Exercises 2.3. — Jump to TABLE OF CONTENTS.

H-2: Write down the equations of specified level curves.

H-3: Remember $a^2 < 1$ means $|a| < 1$, i.e. $-1 < a < 1$.

H-4: Use the Second Derivative Test

H-5: Use the Second Derivative Test

H-6: Use the Second Derivative Test

H-7: Use the second derivative test

H-8: Use the Second Derivative Test

H-9: When you're looking for critical points, remember you need *both* $f_x = 0$ and $f_y = 0$. So if it's hard to solve (say) $f_x = 0$, then first solve $f_y = 0$; then you can narrow your search of $f_x = 0$.

H-14: "Explain your reasoning" is test-speak for "show your work."

H-17: Check Example 2.3.12 in the text.

Hints for Exercises 2.4. — Jump to TABLE OF CONTENTS.

H-1: What is an endpoint of a circle?

H-2: Interpret the height $\sqrt{x^2+y^2}$ geometrically.

H-3: Check the boundary of the square as well as critical points inside the square.

H-5: There are five places to check: the interior and four boundaries.

H-7: Since the region is a triangle, your boundary will have three separate parts to check.

H-9: There are two boundary lines. You'll want to find their intersections.

H-10: Plugging in the boundaries should be quite easy if you choose your variables wisely

H-11: When you see "classify critical points," think "second derivative test."

H-12: Suppose that the bends are made a distance x from the ends of the fence and that the bends are through an angle θ. Draw a sketch of the enclosure and figure out its area, as a function of x and θ.

H-13: Suppose that the box has side lengths x, y and z.

H-15: If (x, y, z) is on the plane, then you know $z = 5 - 2x - y$. So, you can write x^2y^2z as a function of only x and y by eliminating z.

H-17: The answer will be piecewise, depending on what exactly a is.

H-18: Instead of maximizing the total profit function, maximize the profit functions of each type of paper.

H-19: Profit is (revenue) minus (costs). If Ayan and Pipe work separately, then each seller only sees the cost and revenue from the lemonade that they themselves sold.

To find how much each seller will sell when they are working separately, find out which values of q_A and q_P end up with both individual profit functions being maximized.

To find out how much they'll sell when they're working together, use your assumption from part (c) to make the solving smoother.

Hints for Exercises 2.5. — Jump to TABLE OF CONTENTS.

H-1: Interpret $f(x, y)$ as a distance squared, and sketch $xy = 1$ in the xy-plane. You might also want to review section 2.5.1 in the text.

H-2: The easiest way out is to find a function $z = k(x)$ with local but not absolute extrema, then affix that to the plane $y = 0$.

H-3: Find all solutions to

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ x^2 y &= 1 \end{aligned} \tag{E3}$$

H-4: This is a straightforward application of the method of Lagrange multipliers, Theorem 2.5.2 in the text.

H-5: This is a straightforward application of the method of Lagrange multipliers, Theorem 2.5.2 in the text.

H-6: When you set your two equations for λ equal to one another, you should get something that you can easily plug into the constraint function.

H-7: We want to minimize $\sqrt{x^2+y^2}$; it's easier to minimize $f(x,y) = x^2+y^2$. The minima will occur at the same point (x,y).

Note the system has no maximum, since we can keep travelling along the parabola to end up arbitrarily far from the origin.

H-8: To find extrema over a region, we check critical points *and* the boundary.

H-9: You can check your answer from (a) by using a method *other* than Lagrange multipliers.

H-10: Since $x \geqslant 0$ and $y \geqslant 0$, our constraint function has endpoints $(x,y) = (0,400)$ and $(x,y) = (25,0)$. Absolute extrema will occur at these endpoints or at points that solve the system of Lagrange equations.

H-11: The constraint tells you $a + 2b = 1$. So, your variables are a and b.

H-12: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point $(1,2)$ if and only if $\frac{1}{a^2} + \frac{4}{b^2} = 1$.

H-13: You may choose your coordinate system so the cylinder is oriented vertically along the z-axis. Then you can write the volume of the cylinder as a function of two variables.

H-14: The volume is your constraint function.

Hints for Exercises 2.6. — Jump to TABLE OF CONTENTS.

H-1: See Definitions 2.6.3 and 2.6.10, and Example 2.6.12, in the text.

H-2: See Definition 2.6.7 in the text.

H-3: See Definition 2.6.5 in the text.

H-4: You want to optimize utility, so that's your objective function. Your constraint is your budget. Lagrange multipliers are helpful but not strictly necessary.

H-5: You want to optimize utility, so that's your objective function. Your constraint is your budget. Lagrange multipliers are helpful but not strictly necessary.

H-6: u is the objective function

H-8: To determine if the combo is worth Coral's money, check utility levels of each option.

H-10: Use the Marshallian demand method explained in the Section 2.6 of the text.

H-11: You're finding Marshallian demand

H-12:

(a) No need for optimization here. Look at the utility function and think about any necessary domain constraints.

Hints for Exercises 3.1. — Jump to TABLE OF CONTENTS.

H-1: Draw a rectangle that *encompasses* the entire shaded area, and one that is *encompassed by* the shaded area. The shaded area is no more than the area of the bigger rectangle, and no less than the area of the smaller rectangle.

H-2: We can improve on the method of Question 1 by using three rectangles that together encompass the shaded region, and three rectangles that together are encompassed by the shaded region.

H-3: Four rectangles suffice.

H-4: Try drawing a picture.

H-5: Try an oscillating function.

H-6: The ordering of the parts is intentional: each sum can be written by changing some small part of the sum before it.

H-7: If we raise -1 to an even power, we get $+1$, and if we raise it to an odd power, we get -1.

H-8: Sometimes a little anti-simplification can make the pattern more clear.

(a) Re-write as $\frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + \frac{9}{243}$.

(b) Compare to the sum in the hint for (a).

(c) Re-write as $1 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + \frac{4}{1} + \frac{5}{10} + \frac{6}{100} + \frac{7}{1000}$.

H-9: (a), (b) These are geometric sums.
(c) You can write this as three separate sums.
(d) You can write this as two separate sums. Remember that e is a constant. Don't be thrown off by the index being n instead of i.

H-10:

(a) Write out the terms of the two sums.

(b) A change of index is an easier option than expanding the cubic.

(c) Which terms cancel?

(d) Remember $2n+1$ is odd for every integer n. The index starts at $n=2$, not $n=1$.

H-11: Since the sum adds four pieces, there will be four rectangles. However, one might be extremely small.

H-12: Write out the general formula for the left Riemann sum from Definition 3.1.10 in the text and choose a, b and n to make it match the given sum.

H-13: Since the sum runs from 1 to 3, there are three intervals. Suppose $2 = \Delta x = \frac{b-a}{n}$. You may assume the sum given is a right Riemann sum (as opposed to left or midpoint).

H-14: Let $\Delta x = \dfrac{\pi}{20}$. Then what is $b-a$?

H-15: Notice that the index starts at $k=0$, instead of $k=1$. Write out the given sum explicitly without using summation notation, and sketch where the rectangles would fall on a graph of $y=f(x)$. Then try to identify $b-a$, and n, followed by "right", "left", or "midpoint", and finally a.

H-16: The area is a triangle.

H-17: There is one triangle of positive area, and one of negative area.

H-18: Review Definition 3.1.10 in the text.

H-20: You'll want the limit as n goes to infinity of a sum with n terms. If you're having a hard time coming up with the sum in terms of n, try writing a sum with a finite number of terms of your choosing. Then, think about how that sum would change if it had n terms.

H-21: The main step is to express the given sum as the right Riemann sum,

$$\sum_{i=1}^{n} f(a+i\Delta x)\Delta x.$$

Don't be afraid to guess Δx and $f(x)$ (review Definition 3.1.10 in the text). Then write out explicitly $\sum\limits_{i=1}^{n} f(a+i\Delta x)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

H-22: The main step is to express the given sum as the right Riemann sum $\sum\limits_{k=1}^{n} f(a+k\Delta x)\Delta x$. Don't be afraid to guess Δx and $f(x)$ (review Definition 3.1.10 in the text). Then write out explicitly $\sum\limits_{k=1}^{n} f(a+k\Delta x)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

H-23: The main step is to express the given sum in the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$. Don't be afraid to guess Δx, x_i^* (for either a left or a right or a midpoint sum — review Definition 3.1.10 in the text) and $f(x)$. Then write out explicitly $\sum_{i=1}^{n} f(x_i^*)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

H-24: The main step is to express the given sum in the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$. Don't be afraid to guess Δx, x_i^* (probably, based on the symbol R_n, assuming we have a right Riemann sum — review Definition 3.1.10 in the text) and $f(x)$. Then write out explicitly $\sum_{i=1}^{n} f(x_i^*)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

H-25: Try several different choices of Δx and x_i^*.

H-26: Let $x = r^3$, and re–write the sum in terms of x.

H-27: Note the sum does not start at $r^0 = 1$.

H-28: Draw a picture. See Example 3.1.15 in the text.

H-29: Draw a picture. Remember $|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$.

H-30: Draw a picture: the area we want is a trapezoid. If you don't remember a formula for the area of a trapezoid, think of it as the difference of two triangles.

H-31: You can draw a very similar picture to Question 30, but remember the areas are negative.

H-32: If $y = \sqrt{16 - x^2}$, then y is nonnegative, and $y^2 + x^2 = 16$.

H-33: Sketch the graph of $f(x)$.

H-34: At which time in the interval, for example, $0 \leqslant t \leqslant 0.5$, is the car moving the fastest?

H-35: What are the possible speeds the car could have reached at time $t = 0.25$?

H-36: You need to know the speed of the plane at the midpoints of your intervals, so (for example) noon to 1pm *is not* one of your intervals.

H-37: Sure looks like a Riemann sum.

H-38: For part (b): don't panic! Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given identity. The third step is to evaluate the limit $n \to \infty$.

H-39: The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formulas. The third step is to evaluate the limit as $n \to \infty$.

H-40: The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formulas. The third step is to evaluate the limit $n \to \infty$.

H-41: You've probably seen this hint before. It is worth repeating. Don't panic! Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formula. The third step is to evaluate the limit $n \to \infty$.

H-42: Using the definition of a right Riemann sum, we can come up with an expression for $f(-5 + 10i)$. In order to find $f(x)$, set $x = -5 + 10i$.

H-43: Recall that for a positive constant a, $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \log a$, where $\log a$ is the natural logarithm (base e) of a.

H-44: Part (a) follows the same pattern as Question 43–there's just a little more algebra involved, since our lower limit of integration is not 0.

H-45: Your area can be divided into a section of a circle and a triangle. Then you can use geometry to find the area of each piece.

H-46:

(a) The difference between the upper and lower bounds is the area that is outside of the smaller rectangles but inside the larger rectangles. Drawing both sets of rectangles on one picture might make things clearer. Look for an easy way to compute the area you want.

(b) Use your answer from Part (a). Your answer will depend on f, a, and b.

H-47: Since $f(x)$ is linear, there exist real numbers m and c such that $f(x) = mx + c$. It's a little easier to first look at a single triangle from each sum, rather than the sums in their entirety.

H-48: Let $f(n)$ be the number of stitches in the nth round. Find a formula for $f(n)$. Use sigma notation to figure out how many stitches have been made after N rounds.

Hints for Exercises 3.2. — Jump to TABLE OF CONTENTS.

H-1:

(a) What is the length of this figure?

(b) Think about cutting the area into two pieces vertically.

(c) Think about cutting the area into two pieces another way.

H-2: Use the identity $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.

H-4: Note that the limits of the integral given are in the opposite order from what we might expect: the smaller number is the top limit of integration.

Recall $\Delta x = \frac{b-a}{n}$.

H-5: Remember that a definite integral is the *signed* area

H-6: Split the "target integral" up into pieces that can be evaluated using the given integrals.

H-7: Split the "target integral" up into pieces that can be evaluated using the given integrals.

H-8: Split the "target integral" up into pieces that can be evaluated using the given integrals.

H-9: For part (a), use the symmetry of the integrand. For part (b), the area $\int_0^1 \sqrt{1-x^2}\,dx$ is easy to find–how is this useful to you?

H-10: The evaluation of this integral was also the subject of Question 10 in Section 1.1. This time try using the method of Example 3.2.6 in the text.

H-11: If $x^2 \leqslant x$, then $e^{x^2} \leqslant e^x$.

H-12: Use symmetry.

H-13: Check Theorem 3.2.11 in the text.

H-14: Split the integral into a sum of two integrals. Interpret each geometrically.

H-15: For example: if Student 1 finds that a value is less than 3, and Student 2 finds that the same value is less than 4, which is more useful? With the information from Student 1, we no longer need the information from Student 2. So, Student 1 gave a more useful bound.

This is a situation where some thoughtfulness up front can save a lot of time.

H-16: Hmmmm. Looks like a complicated integral. It's probably a trick question. Check for symmetries.

H-17: Check for symmetries again.

H-18: What does the integrand look like to the left and right of $x = 3$?

H-19: In part (b), you'll have to factor a constant out through a square root. Remember the upper half of a circle looks like $\sqrt{r^2 - x^2}$.

H-20: For two functions $f(x)$ and $g(x)$, define $h(x) = f(x) \cdot g(x)$. If $h(-x) = h(x)$, then the product is even; if $h(-x) = -h(x)$, then the product is odd.

The table will *not* be the same as if we were multiplying even and odd *numbers*.

H-21: Note $f(0) = f(-0)$.

H-22: If $f(x)$ is even and odd, then $f(x) = -f(x)$ for every x.

H-23: Think about mirroring a function across an axis. What does this do to the slope?

Hints for Exercises 3.3. — Jump to TABLE OF CONTENTS.

H-2: First find the general antiderivative by guessing and checking.

H-3: Be careful. Two of these make no sense at all.

H-4: Check by differentiating.

H-5: Check by differentiating.

H-6: Use the Fundamental Theorem of Calculus Part 1.

H-7: Use the Fundamental Theorem of Calculus, Part 1.

H-8: You already know that $F(x)$ is an antiderivative of $f(x)$.

H-9: (a) Recall $\frac{d}{dx}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$.
(b) All antiderivatives of $\sqrt{1-x^2}$ differ from one another by a constant. You already know one antiderivative.

H-10: In order to apply the Fundamental Theorem of Calculus Part 2, the integrand must be continuous over the interval of integration.

H-11: Use the definition of $F(x)$ as an area.

H-12: $F(x)$ represents net *signed* area.

H-13: Note $G(x) = -F(x)$, when $F(x)$ is defined as in Question 12.

H-14: Using the definition of the derivative, $F'(x) = \lim_{h \to 0} \dfrac{F(x+h) - F(x)}{h}$.

The area of a trapezoid with base b and heights h_1 and h_2 is $\frac{1}{2}b(h_1 + h_2)$.

H-15: There is only one!

H-16: If $\frac{d}{dx}\{F(x)\} = f(x)$, that tells us $\int f(x)\, dx = F(x) + C$.

H-17: When you're differentiating, you can leave the e^x factored out.

H-18: After differentiation, you can simplify pretty far. Keep at it!

H-19: This derivative also simplifies considerably. You might need to add fractions by finding a common denominator.

H-20: Guess a function whose derivative is the integrand, then use the Fundamental Theorem of Calculus Part 2.

H-21: Split the given integral up into two integrals.

H-22: The integrand is similar to $\dfrac{1}{1+x^2}$, so something with arctangent seems in order.

H-23: The integrand is similar to $\dfrac{1}{\sqrt{1-x^2}}$, so factoring out $\sqrt{2}$ from the denominator will make it look like some flavour of arcsine.

H-24: We know how to antidifferentiate $\sec^2 x$, and there is an identity linking $\sec^2 x$ with $\tan^2 x$.

H-25: Recall $2 \sin x \cos x = \sin(2x)$.

H-26: $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$

H-28: There is a good way to test where a function is increasing, decreasing, or constant, that also has something to do with topic of this section.

H-29: See Example 3.3.5 in the text.

H-30: See Example 3.3.5 in the text.

H-31: See Example 3.3.5 in the text.

H-32: See Example 3.3.5 in the text.

H-33: See Example 3.3.6 in the text.

H-34: Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides.

H-35: What is the title of this section?

H-36: See Example 3.3.6 in the text.

H-37: See Example 3.3.6 in the text.

H-38: See Example 3.3.6 in the text.

H-39: See Example 3.3.6 in the text.

H-40: Split up the domain of integration.

H-41: It is possible to guess an antiderivative for $f'(x)f''(x)$ that is expressed in terms of $f'(x)$.

H-42: When does the car stop? What is the relation between velocity and distance travelled?

H-43: See Example 3.3.5 in the text. For the absolute maximum part of the question, study the sign of $f'(x)$.

H-44: See Example 3.3.5 in the text. For the "minimum value" part of the question, study the sign of $f'(x)$.

H-45: See Example 3.3.5 in the text. For the "maximum" part of the question, study the sign of $F'(x)$.

H-46: Review the definition of the definite integral and in particular Definitions 3.1.8 and 3.1.10 in the text.

H-47: Review the definition of the definite integral and in particular Definitions 3.1.8 and 3.1.10 in the text.

H-48: Carefully check the Fundamental Theorem of Calculus: as written, it only applies directly to $F(x)$ when $x \geqslant 0$.

Is $F(x)$ even or odd?

H-49: In general, the equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is $y = f(a) + f'(a)\,(x - a)$.

H-50: Recall $\tan^2 x + 1 = \sec^2 x$.

H-51: Since the integration is with respect to t, the x^3 term can be moved outside the integral.

H-52: Remember that antiderivatives may have a constant term.

H-53: It's actually quite hard to find q_e and p_e explicitly, so leave them as they are. The objective of this exercise is for you to practice integration. Furthermore, you may want to make use of the rectangle formed by p_e and q_e to find the areas required.

H-54: If you find B first, then you don't need an integral to find A – you can just subtract.

H-55: Using the Fundamental Theorem of Calculus,

$$\int MC\mathrm{d}q = TC + C$$

for some constant C. To find C, remember $TC(0) = FC$.

H-56: By the Fundamental Theorem of Calculus,

$$\int \mathrm{MR}\,\mathrm{d}q = \mathrm{TR} + C$$

To find the constant of integration C, you must make sure that at $q = 0$ we have $TR = 0$.

Hints for Exercises 3.4. — Jump to TABLE OF CONTENTS.

H-1: One is true, the other false.

H-2: You can check whether the final answer is correct by differentiating.

H-3: Check the limits.

H-4: Check every step. Do they all make sense?

H-6: What is $\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\}$?

H-7: What is the derivative of the argument of the cosine?

H-8: What is the title of the current section?

H-9: What is the derivative of $x^3 + 1$?

H-10: What is the derivative of $\ln x$?

H-11: What is the derivative of $1 + \sin x$?

H-12: $\cos x$ is the derivative of what?

H-13: What is the derivative of the exponent?

H-14: What is the derivative of the argument of the square root?

H-15: What is $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{\ln x}\right\}$?

H-16: There is a short, slightly sneaky method — guess an antiderivative — and a really short, still-more-sneaky method.

H-17: Review the definition of the definite integral and in particular Definitions 3.1.8 and 3.1.10 in the text.

H-18: If $w = u^2 + 1$, then $u^2 = w - 1$.

H-19: Using a trigonometric identity, this is similar (though not identical) to $\int \tan\theta \cdot \sec^2\theta \, \mathrm{d}\theta$.

H-20: If you multiply the top and the bottom by e^x, what does this look like the antiderivative of?

H-21: You know methods other than substitution to evaluate definite integrals.

H-22: $\tan x = \dfrac{\sin x}{\cos x}$

H-23: Review the definition of the definite integral and in particular Definitions 3.1.8 and 3.1.10 in the text.

H-24: Review the definition of the definite integral and in particular Definitions 3.1.8 and 3.1.10 in the text.

H-25: Find the right Riemann sum for both definite integrals.

H-26:

(a) Remember to integrate in terms of q. Furthermore, do not forget to replace the constant c with the value provided for FC.

(b) Remember that to find the constant c after integrating, you must make sure that at $q = 0 \to TR = 0$.

(c) Profit equals revenue minus costs.

(d) Check the domain of the function.

Hints for Exercises 3.5. — Jump to TABLE OF CONTENTS.

H-1: Read back over Sections 3.4 and 3.5 of the text. When these methods are introduced, they are justified using the corresponding differentiation rules.

H-2: Remember our rule: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$. So, we take u and use it to make $\mathrm{d}u$, and we take $\mathrm{d}v$ and use it to make v.

H-3: According to the quotient rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Antidifferentiate both sides of the equation, then solve for the expression in the question.

H-4: Remember all the antiderivatives differ only by a constant, so you can write them all as $v(x) + C$ for some C.

H-5: What integral do you have to evaluate, after you plug in your choices to the integration by parts formula?

H-6: You'll probably want to use integration by parts. (It's the title of the section, after all). You'll break the integrand into two parts, integrate one, and differentiate the other. Would you rather integrate $\ln x$, or differentiate it?

H-7: This problem is similar to Question 6.

H-8: Example 3.5.5 in the text shows you how to find the antiderivative. Then the Fundamental Theorem of Calculus Part 2 gives you the definite integral.

H-9: Compare to Question 8. Try to do this one all the way through without peeking at another solution!

H-10: If at first you don't succeed, try using integration by parts a few times in a row. Eventually, one part will go away.

H-11: Similarly to Question 10, look for a way to use integration by parts a few times to simplify the integrand until it is antidifferentiatable.

H-12: Use integration by parts twice to get an integral with only a trigonometric function in it.

H-13: If you let $u = \ln t$ in the integration by parts, then $\mathrm{d}u$ works quite nicely with the rest of the integrand.

H-14: Those square roots are a little disconcerting– get rid of them with a substitution.

H-15: This can be solved using the same ideas as Example 3.5.8 in the text.

H-16: Not every integral should be evaluated using integration by parts.

H-17: You know, or can easily look up, the derivative of arccosine. You can use a similar trick as the book did when antidifferentiating other inverse trigonometric functions in Example 3.5.9 of the text.

H-18: After integrating by parts, do some algebraic manipulation to the integral until it's clear how to evaluate it.

H-19: After integration by parts, use a substitution.

H-20: This example is similar to Example 3.5.10 in the text. The functions $e^{x/2}$ and $\cos(2x)$ both do not substantially alter when we differentiate or antidifferentiate them. If we use integration by parts twice, we'll end up with an expression that includes our original integral. Then we can just solve for the original integral in the equation, without actually integrating.

H-21: This looks a bit like a substitution problem, because we have an "inside function."

It might help to review Example 3.5.11 in the text.

H-22: Start by simplifying.

H-23: $\sin(2x) = 2 \sin x \cos x$

H-24: What is the derivative of xe^{-x}?

H-25: You'll want to do an integration by parts for (a)–check the end result to get a guess as to what your parts should be. A trig identity and some amount of algebraic manipulation will be necessary to get the final form.

H-26: See Examples 3.5.9 in the text for refreshers on integrating arctangent.

Remember $\tan^2 x + 1 = \sec^2 x$, and $\sec^2 x$ is easy to integrate.

H-27: Your integral can be broken into two integrals, which yield to two different integration methods.

H-28: Think, first, about how to get rid of the square root in the argument of f'', and, second, how to convert f'' into f'. Note that you are told that $f'(2) = 4$ and $f(0) = 1$, $f(2) = 3$.

H-29: Interpret the limit as a right Riemann sum.

H-30: Start by finding the value of p_e. Proceed to find the area C, using the square created by p_e and q_e.

H-31: Using the Fundamental Theorem of Calculus,

$$TC = \int \mathrm{MC}\mathrm{d}q + C$$

for some constant C.

Remember to integrate in terms of q. You might want to use substitution followed by integration by parts. Be careful with your constants: you need $\mathrm{TC}(0) = $FC.

The average cost per unit of producing 10 units is one-tenth the total cost of producing 10 units.

Hints for Exercises 3.6. — Jump to TABLE OF CONTENTS.

H-1: Go ahead and try it!

H-2: Use the substitution $u = \sec x$.

H-3: Divide both sides of the second identity by $\cos^2 x$.

H-4: See Example 3.6.6 in the text. Note that the power of cosine is odd, and the power of sine is even (it's zero).

H-5: See Example 3.6.7 in the text. All you need is a helpful trig identity.

H-6: The power of cosine is odd, so we can reserve one cosine for $\mathrm{d}u$, and turn the rest into sines using the identity $\sin^2 x + \cos^2 x = 1$.

H-7: Since the power of sine is odd (and positive), we can reserve one sine for $\mathrm{d}u$, and turn the rest into cosines using the identity $\sin^2 + \cos^2 x = 1$.

H-8: When we have even powers of sine and cosine both, we use the identities in the last two lines of Equation 3.6.3 in the text.

H-9: Since the power of sine is odd, you can use the substitution $u = \cos x$.

H-10: Which substitution will work better: $u = \sin x$, or $u = \cos x$?

H-11: Try a substitution.

H-12: For practice, try doing this in two ways, with different substitutions.

H-13: A substitution will work. See Example 3.6.14 in the text for a template for integrands with even powers of secant.

H-14: Try the substitution $u = \sec x$.

H-15: Compare to Question 14.

H-16: What is the derivative of tangent?

H-17: Don't be scared off by the non-integer power of secant. You can still use the strategies in the notes for an odd power of tangent.

H-18: Since there are no secants in the problem, it's difficult to use the substitution $u = \sec x$ that we've enjoyed in the past. Example 3.6.12 in the text provides a template for antidifferentiating an odd power of tangent.

H-19: Integrating even powers of tangent is surprisingly different from integrating odd powers of tangent. You'll want to use the identity $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$ on (parhaps only a part of) the resulting integral. Example 3.6.16 in the text show you how this can be accomplished.

H-20: Since there is an even power of secant in the integrand, we can use the substitution $u = \tan x$.

H-21: How have we handled integration in the past that involved an odd power of tangent?

H-22: Remember e is some constant. What are our strategies when the power of secant is even and positive? We've seen one such substitution in Example 3.6.15 of the text.

H-23: See Example 3.6.16 in the text for a strategy for integrating powers of tangent.

H-24: Write $\tan x = \dfrac{\sin x}{\cos x}$.

H-25: $\dfrac{1}{\cos\theta} = \sec\theta$

H-26: $\cot x = \dfrac{\cos x}{\sin x}$

H-27: Try substituting.

H-28: To deal with the "inside function," start with a substitution.

H-29: Try an integration by parts.

Hints for Exercises 3.7. — Jump to TABLE OF CONTENTS.

H-1: The beginning of this section has a template for choosing a substitution. Your goal is to use a trig identity to turn the argument of the square root into a perfect square, so you can cancel $\sqrt{(\text{something})^2} = |\text{something}|$.

H-2: You want to do the same thing you did in Question 1, but you'll have to complete the square first.

H-3: Since θ is acute, you can draw it as an angle of a right triangle. The given information will let you label two sides of the triangle, and the Pythagorean Theorem will lead you to the third.

H-4: You can draw a right triangle with angle θ, and use the given information to label two of the sides. The Pythagorean Theorem gives you the third side.

H-5: As in Question 1, choose an appropriate substitution. Your answer should be in terms of your original variable, x, which can be achieved using the methods of Question 3.

H-6: As in Question 1, choose an appropriate substitution. Your answer will be a number, so as long as you change your limits of integration when you substitute, you don't need to bother changing the antiderivative back into the original variable x. However, you might want to use the techniques of Question 4 to simplify your final answer.

H-7: Question 1 guides the way to finding the appropriate substitution. Since the integral is definite, your final answer will be a number. Your limits of integration should be common reference angles.

H-8: Question 1 guides the way to finding the appropriate substitution. Since you have in indefinite integral, make sure to get your answer back in terms of the original variable, x. Question 3 gives a reliable method for this.

H-9: A trig substitution is not the easiest path.

H-10: To antidifferentiate, change your trig functions into sines and cosines.

H-11: The integrand should simplify quite far after your substitution.

H-12: In part (a) you are asked to integrate an even power of $\cos x$. For part (b) you can use a trigonometric substitution to reduce the integral of part (b) almost to the integral of part (a).

H-13: What is the symmetry of the integrand?

H-14: See Example 3.7.3 in the text.

H-15: To integrate an even power of tangent, use the identity $\tan^2 x = \sec^2 x - 1$.

H-16: A trig substitution is not the easiest path.

H-17: Complete the square. Your final answer will have an inverse trig function in it.

H-18: To antidifferentiate even powers of cosine, use the formula $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$. Then, remember $\sin(2\theta) = 2\sin\theta\cos\theta$.

H-19: After substituting, use the identity $\tan^2 x = \sec^2 x - 1$ more than once. Remember $\int \sec x \mathrm{d}x = \ln\left|\sec x + \tan x\right| + C$.

H-20: There's no square root, but we can still make use of the substitution $x = \tan\theta$.

H-21: You'll probably want to use the identity $\tan^2\theta + 1 = \sec^2\theta$ more than once.

H-22: Complete the square — refer to Question 2 if you want a refresher. The constants aren't pretty, but don't let them scare you.

H-23: After substituting, use the identity $\sec^2 u = \tan^2 u + 1$. It might help to break the integral into a few pieces.

H-24: Make use of symmetry, and integrate with respect to y (rather than x). The limits of integration should be reference angles.

H-25: Use the symmetry of the function to re-write your integrals without an absolute value.

H-26: Think of e^x as $\left(e^{x/2}\right)^2$, and use a trig substitution. Then, use the identity $\sec^2\theta = \tan^2\theta + 1$.

H-27:

(a) Use logarithm rules to simplify first.

(b) Think about domains.

(c) What went wrong in part (b)? At what point in the work was that problem introduced?
There is a subtle but important point mentioned in the introductory text to Section 3.7 of the text that may help you make sense of things.

H-28: Consider the ranges of the inverse trigonometric functions. For (c), also consider the domain of $\sqrt{x^2 - a^2}$.

Hints for Exercises 3.8. — Jump to TABLE OF CONTENTS.

H-1: If a quadratic function has root a, then $(x - a)$ can be factored out of it.

H-2: Review Equation 3.8.7 of the text. Be careful to *fully* factor the denominator.

H-3: Review Example 3.8.1 in the text. Is the "Algebraic Method" or the "Sneaky Method" going to be easier?

H-4: For each part, use long division as in Example 3.8.3 of the text.

H-5: (a) Look for a pattern you can exploit to factor out a linear term.
(b) If you set $y = x^2$, this is quadratic. Remember $(x^2 - a) = (x + \sqrt{a})(x - \sqrt{a})$ as long as a is positive.

H-6: Look for integer roots, then use long division.

H-7: Why do we do partial fraction decomposition at all?

H-8: What is the title of this section?

H-9: Fill in the blank: the integrand is a ____ function.

H-10: The integrand is yet another ____ function.

H-11: Since the degree of the numerator is the same as the degree of the denominator, we can't do our partial fraction decomposition before we simplify the integrand.

H-12: In the partial fraction decomposition, several constants turn out to be 0.

H-13: The denominators for your expansion will be $3x-5$, $(3x-5)^2$, and $(3x-5)^3$.

H-14: The decomposition will have the form $\frac{A}{x-4}+\frac{B}{2x-1}+\frac{C}{(2x-1)^2}$.

H-15: The denominator has two repeated linear factors

H-16: The coefficients are all integers. If you're getting fractions, check your partial fractions.

H-17: Factor $(2x-1)$ out of the denominator to get started. You don't need long division for this step.

H-18: Try renaming x^2 to y and comparing the result with Question 8.

H-19: $\csc x = \dfrac{1}{\sin x} = \dfrac{\sin x}{\sin^2 x}$

H-20: Use the partial fraction decomposition from Queston 19 to save yourself some time.

H-21: $\cos^2\theta = 1-\sin^2\theta$

H-22: If you're having a hard time making the substitution, multiply the numerator and the denominator by e^t.

H-23: Try the substitution $u=\sqrt{1+e^x}$. You'll need to do long division before you can use partial fraction decomposition.

H-24: You'll need to use two regions, because the curves cross.

H-25: For (b), use the Fundamental Theorem of Calculus Part 1.

H-26: Don't forget the $\mathrm{d}x$ when you substitute.

H-27: After your substitution, factor the denominator using the difference of two squares.

H-28: Use the substitution $u^n = x$, where n is a power that will result in a rational integrand.

H-29: The equation $u=\sqrt{x-5}$ leads us to $x=5+u^2$, and so $\mathrm{d}x = 2u\mathrm{d}u$.

After the substitution, you'll need to simplify the integrand before you can use the method of partial fraction decomposition.

Hints for Exercises 3.9. — Jump to TABLE OF CONTENTS.

H-1: The absolute error is the difference of the two values; the relative error is the absolute error divided by the exact value; the percent error is one hundred times the relative error.

H-2: You should have four rectangles in one drawing, and four trapezoids in another.

H-3: Sketch the second derivative–it's quadratic.

H-4: You don't have to find the actual, exact maximum the second derivative achieves–you only have to give a reasonable "ceiling" that it never breaks through.

H-5: To compute the upper bound on the error, find an upper bound on the fourth derivative of cosine, then use Theorem 3.9.12 in the text.

To find the actual error, you need to find the actual value of A.

H-6: Find a function with $f''(x) = 3$ for all x in $[0, 1]$.

H-7: You're allowed to use common sense for this one.

H-8: For part (b), consider Question 7.

H-9: Draw a sketch.

H-10: The error bound for the approximation is given in Theorem 3.9.12 in the text. You want this bound to be zero.

H-11: Follow the formulas in Equations 3.9.2, 3.9.6, and 3.9.9 in the text.

H-12: See Section 3.9.1 in the text. You should be able to simplify your answer to an exact value (in terms of π).

H-13: For the trapezoid rule, see Section 3.9.2 in the text.

Note the dimensions given for the cross sections are diameters, not radii.

Approximate the volume of the solid by slicing it horizontally into disks that look somewhat like cylinders.

H-14: See Section 3.9.3 in the text, and compare to Question 13. Note the table gives diameters, not radii.

H-15: See §3.9.3 in the text for Simpson's rule. To set up the volume integral, see Question 14.

Note that the table gives the circumference, not radius, of the tree at a given height.

H-18: The main step is to find an appropriate value of M. It is *not necessary* to find the smallest possible M.

H-19: The main step is to find M. This question is unusual in that its wording requires you to find the smallest possible allowed M.

H-20: The main steps in part (b) are to find the smallest possible values of M and L.

H-21: As usual, the biggest part of this problem is finding L. Don't be thrown off by the error bound being given slightly differently from Theorem 3.9.12 in the text: these expressions are equivalent, since $\Delta x = \frac{b-a}{n}$.

H-22: The function $e^{-2x} = \dfrac{1}{e^{2x}}$ is positive and decreasing, so its maximum occurs when x is as small as possible.

H-23:

H-24: The "best ... approximations that you can" means using the maximum number of intervals, given the information available.

The final sentence in part (b) is just a re-statement of the error bounds we're familiar with from Theorem 3.9.12 in the text. The information $\left|s^{(k)}(x)\right| \leqslant \dfrac{k}{1000}$ gives you values of M and L when you set $k = 2$ and $k = 4$, respectively.

H-25: Set the error bound to be less than 0.001, then solve for n.

H-26: See Section 3.9.3 in the text for Simpson's rule.

Since the cross-sections of the pool are semi-circular disks, a section that is d metres across will have area $\frac{1}{2}\pi\left(\frac{d}{2}\right)^2$ square feet. Based on the drawing, you may assume the very ends of the pool have distance 0 feet across.

H-27: See Example 3.9.14 in the text.

Don't get caught up in the interpretation of the integral. It's nice to see how integrals can be used, but for this problem, you're still just approximating the integral given, and bounding the error.

When you find the second derivative to bound your error, pay attention to the difference between the integrand and $g(r)$.

H-28: See Example 3.9.15 in the text. You'll want to use a calculator for the approximation in (a), and for finding the appropriate number of intervals in (b). Remember that Simpson's rule requires an even number of intervals.

H-29: See Example 3.9.15 in the text.

Rather than calculating the fourth derivative of the integrand, use the graph to find the largest absolute value it attains over our interval.

H-30: See Example 3.9.14 in the text.

You'll have to differentiate $f(x)$. To that end, you may also want to review the fundamental theorem of calculus and, in particular, Example 3.3.5 in the text.

You don't have to find the best possible value for M. A reasonable upper bound on $|f''(x)|$ will do.

To have five decimal places of accuracy, your error must be less than 0.000005. This ensures that, if you round your approximation to five decimal places, they will all be correct.

H-31: To find the maximum value of $|f''(x)|$, check its critical points and endpoints.

H-32: In using Simpson's rule to approximate $\displaystyle\int_1^x \frac{1}{t}\,\mathrm{d}t$ with n intervals, $a = 1$, $b = x$, and $\Delta x = \dfrac{x-1}{n}$.

H-33:

- $\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(2) - \frac{\pi}{4}$, so $\arctan(2) = \frac{\pi}{4} + \int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x$
- If an approximation A of the integral $\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x$ has error at most ε, then $A - \varepsilon \leqslant \int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x \leqslant A + \varepsilon$.
- Looking at our target interval will tell you how small ε needs to be, which in turn will tell you how many intervals you need to use.
- You can show, by considering the numerator and denominator separately, that $|f^{(4)}(x)| \leqslant 30.75$ for every x in $[1, 2]$.
- If you use Simpson's rule to approximate $\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x$, you won't need very many intervals to get the requisite accuracy.

Hints for Exercises 3.10. — Jump to TABLE OF CONTENTS.

H-1: There are two kinds of impropreity in an integral: an infinite discontinuity in the integrand, and an infinite limit of integration.

H-2: The integrand is continuous for all x.

H-3: What matters is which function is bigger for large values of x, not near the origin.

H-4: Read both the question and Theorem 3.10.17 in the text *very* carefully.

H-5: (a) What if $h(x)$ is negative? What if it's not?
(b) What if $h(x)$ is very close to $f(x)$ or $g(x)$, rather than right in the middle?
(c) Note $|h(x)| \leqslant 2f(x)$.

H-6: First: is the integrand unbounded, and if so, where?

Second: when evaluating integrals, always check to see if you can use a simple substitution before trying a complicated procedure like partial fractions.

H-7: Is the integrand bounded?

H-8: See Example 3.10.21 in the text. Rather than antidifferentiating, you can find a nice comparison.

H-9: Which of the two terms in the denominator is more important when $x \approx 0$? Which one is more important when x is very large?

H-10: Remember to break the integral into two pieces.

H-11: Remember to break the integral into two pieces.

H-12: The easiest test in this case is limiting comparison, Theorem 3.10.22 in the text.

H-13: Not all discontinuities cause an integral to be improper–only infinite discontinuities.

H-14: Which of the two terms in the denominator is more important when x is very large?

H-15: Which of the two terms in the denominator is more important when $x \approx 0$? Which one is more important when x is very large?

H-16: Review Example 3.10.8 in the text. Remember the antiderivative of $\frac{1}{x}$ looks very different from the antiderivative of other powers of x.

H-17: Compare to Example 3.10.14 in the text. You can antidifferentiate with a u-substitution.

H-18: Break up the integral. The absolute values give you a nice even function, so you can replace $|x - a|$ with $x - a$ if you're careful about the limits of integration.

H-19: Use integration by parts twice to find the antiderivative of $e^{-x} \sin x$, as in Example 3.5.10 of the text. Be careful with your signs — it's easy to make a mistake with all those negatives.

If you're having a hard time taking the limit at the end, review the Squeeze Theorem from last semester.

H-20: What is the limit of the integrand when $x \to 0$?

H-21: The only "source of impropriety" is the infinite domain of integration. Don't be afraid to be a little creative to make a comparison work.

H-22: There are two things that contribute to your error: using t as the upper bound instead of infinity, and using n intervals for the approximation.

First, find a t so that the error introduced by approximating $\int_0^\infty \frac{e^{-x}}{1+x}\,\mathrm{d}x$ by $\int_0^t \frac{e^{-x}}{1+x}\,\mathrm{d}x$ is at most $\frac{1}{2}10^{-4}$. Then, find your n.

H-23: Look for a place to use Theorem 3.10.20 of the text.

Examples 3.2.9 and 3.2.10 in the text have nice results about the area under an even/odd curve.

H-24: x should be a real number

Hints for Exercises 3.11. — Jump to TABLE OF CONTENTS.

H-1: Each option in each column should be used exactly once.

H-2: The integrand is the product of sines and cosines. See how this was handled with a substitution in Section 3.6.1 of the text.

After your substitution, you should have a polynomial expression in u–but it might take some simplification to get it into a form you can easily integrate.

H-3: We notice that the integrand has a quadratic polynomial under the square root. If that polynomial were a perfect square, we could get rid of the square root: try a trig substitution, as in Section 3.7 of the text.

The identity $\sin(2\theta) = 2\sin\theta\cos\theta$ might come in handy.

H-4: Notice the integral is improper. When you compute the limit, l'Hôpital's rule might help.

If you're struggling to think of how to antidifferentiate, try writing $\frac{x-1}{e^x} = (x-1)e^{-x}$.

H-5: Which method usually works for rational functions (the quotient of two polynomials)?

H-6: It would be nice to replace logarithm with its derivative, $\frac{1}{x}$.

H-7: The integrand is a rational function, so it is possible to use partial fractions. But there is a much easier way!

H-8: You should prepare your own personal internal list of integration techniques ordered from easiest to hardest. You should have associated to each technique your own personal list of signals that you use to decide when the technique is likely to be useful.

H-9: Despite both containing a trig function, the two integrals are easiest to evaluate using different methods.

H-10: For the integral of secant, see See Section A.8 or Example 3.8.4 in the text.

In (c), notice the denominator is not yet entirely factored.

H-11: Part (a) can be done by inspection – use a little highschool geometry! Part (b) is reminiscent of the antiderivative of logarithm–how did we find that one out? Part (c) is an improper integral.

H-12: Use the substitution $u = \sin\theta$.

H-13: For (c), try a little algebra to split the integral into pieces that are easy to antidifferentiate.

H-14: If you're stumped, review Sections 3.6, 3.7, and 3.8 in the text.

H-15: For part (a), see Example 3.5.11 in the text. For part (d), see Example 3.8.3 in the text.

H-16: For part (b), first complete the square in the denominator. You can save some work by first comparing the derivative of the denominator with the numerator. For part (d) use a simple substitution.

H-17: For part (b), complete the square in the denominator. You can save some work by first comparing the derivative of the denominator with the numerator.

H-18: For part (a), the numerator is the derivative of a function that appears in the denominator.

H-19: The integral is improper.

H-20: For part (a), can you convert this into a partial fractions integral? For part (b), start by completing the square inside the square root.

H-21: For part (b), the numerator is the derivative of a function that is embedded in the denominator.

H-22: Try a substitution.

H-23: Note the quadratic function under the square root: you can solve this with trigonometric substitution, as in Section 3.7 of the text.

H-24: Try a u-substitution, as in Section 3.6.2 of the text

H-25: What's the usual trick for evaluating a rational function (quotient of polynomials)?

H-26: If the denominator were x^2+1, the antiderivative would be arctangent.

H-27: Simplify first.

H-28: $x^3+1=(x+1)(x^2-x+1)$

H-29: You have the product of two quite dissimilar functions in the integrand–try integration by parts.

H-30: Use the identity $\cos(2x)=2\cos^2 x-1$.

H-31: Using logarithm rules can make the integrand simpler.

H-32: What is the derivative of the function in the denominator? How could that be useful to you?

H-33: For part (a), the substitution $u=\ln x$ gives an integral that you have seen before.

H-34: For part (a), split the integral in two. One part may be evaluated by interpreting it geometrically, without doing any integration at all. For part (c), multiply both the numerator and denominator by e^x and then make a substitution.

H-35: Let $u=\sqrt{1-x}$.

H-36: Use the substitution $u=e^x$.

H-37: Use integration by parts. If you choose your parts well, the resulting integration will be very simple.

H-38: $\frac{\sin x}{\cos^2 x}=\tan x\sec x$

H-39: The cases $n=-1$ and $n=-2$ are different from all other values of n.

H-40: $x^4+1=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$

Hints for Exercises 3.12. — Jump to TABLE OF CONTENTS.

H-1: You don't need to solve the differential equation from scratch, only verify whether the given function $y=f(x)$ makes it true. Find $\frac{dy}{dx}$ and plug it into the differential equation.

H-2: For (d), note the equation given is quadratic in the variable $\frac{dy}{dx}$.

H-3: The step $\int g(y)\,dy = \int f(x)\,dx$ shows up whether we're using our mnemonic or not.

H-4: Note $\frac{d}{dx}\{f(x)\} = \frac{d}{dx}\{f(x)+C\}$. Plug in $y = f(x) + C$ to the equation $\frac{dy}{dx} = xy$ to see whether it makes the equation is true.

H-5: If a function is differentiable at a point, it is also continuous at that point.

H-6: Let $Q(t)$ be the quantity of morphine in a patient's bloodstream at time t, where t is measured in minutes.

Using the definition of a derivative,

$$\frac{dQ}{dt} = \lim_{h\to 0} \frac{Q(t+h) - Q(t)}{h} \approx \frac{Q(t+1) - Q(t)}{1}$$

So, $\frac{dQ}{dt}$ is roughly the change in the amount of morphine in one minute, from t to $t+1$.

H-7: If $p(t)$ is the proportion of the new form, then $1 - p(t)$ is the proportion of the old form.

When we say two quantities are proportional, we mean that one is a constant multiple of the other.

H-8: The red marks show the slope $y(x)$ would have at a point *if* it crosses that point. So, pick a value of $y(0)$; based on the red marks, you can see how fast $y(x)$ is increasing or decreasing at that point, which leads you roughly to a value of $y(1)$; again, the red marks tell you how fast $y(x)$ is increasing or decreasing, which leads you to a value of $y(2)$, etc (unless you're already off the graph).

H-9: Use Theorem 3.12.10 in the text.

H-10: Use Theorem 3.12.10 in the text.

H-11: Use Theorem 3.12.10 in the text.

H-12: Use Definition 3.12.11 in the text.

H-13: Use Definition 3.12.11 in the text.

H-14: Start by multiplying both sides of the equation by e^y and dx, pretending that $\frac{dy}{dx}$ is a fraction, according to our mnemonic.

H-15: You need to solve for your function $y(x)$ explicitly. Be careful with absolute values: if $|y| = F$, then $y = F$ or $y = -F$. However, $y = \pm F$ *is not a function*. You have to choose one: $y = F$ or $y = -F$.

H-16: If your answer doesn't quite look like the answer given, try manipulating it with logarithm rules: $\ln a + \ln b = \ln(ab)$, and $a \ln b = \ln(b^a)$.

H-17: Simplify the equation.

H-18: Be careful with the arbitrary constant.

H-19: Start by cross-multiplying.

H-20: Be careful about signs. If $y^2 = F$, then possibly $y = \sqrt{F}$, and possibly $y = -\sqrt{F}$. However, $y = \pm\sqrt{F}$ is not a function.

H-21: Be careful about signs.

H-22: Be careful about signs. If $\ln|y| = F$, then $|y| = e^F$. Since you should give your answer as an explicit function $y(x)$, you need to decide whether $y = e^F$ or $y = -e^F$.

H-23: Move the y from the left hand side to the right hand side, then use partial fractions to integrate.

Be careful about the signs. Remember that we need $y = -1$ when $x = 1$. This suggests how to deal with absolute values.

H-24: The unknown function $f(x)$ satisfies an equation that involves the derivative of f.

H-25: Try guessing the partial fractions expansion of $\dfrac{1}{x(x+1)}$.

Since $x = 1$ is in the domain and $x = 0$ is not, you may assume $x > 0$ for all x in the domain.

H-26: $\dfrac{\mathrm{d}}{\mathrm{d}x}\{\sec x\} = \sec x \tan x$

H-27: The general solution to the differential equation will contain the constant k and one other constant. They are determined by the data given in the question.

H-28:

- When you're solving the differential equation, you should have an integral that you can massage to look something like arctangent.
- What is the velocity of the object at its highest point?
- Your final answer will depend on the (unspecified) constants v_0, m, g and k.

H-29: The general solution to the differential equation will contain the constant k and one other constant. They are determined by the data given in the question.

H-30: The method of partial fractions will help you integrate.

To solve $\frac{x-a}{x-b} = Y$ for x, move the terms containing x out of the denominator, then gather them on one side of the equals sign and factor out the x.

$$\begin{aligned}
\frac{x-a}{x-b} &= Y \\
x - a &= Y(x-b) = Yx - Yb \\
x - Yx &= a - Yb \\
x(1-Y) &= a - Yb \\
x &= \frac{a - Yb}{1 - Y}
\end{aligned}$$

To find the limit, you can avoid l'Hôpital's rule using some clever algebra–but you can also just use l'Hôpital's rule.

H-31: Be careful about signs.

Part (a) has some algebraic similarities to Question 30.

H-32: The general solution to the differential equation will contain a constant of proportionality and one other constant. They are determined by the data given in the question.

H-34: You do not need to know anything about investing or continuous compounding to do this problem. You are given the differential equation explicitly. The whole first sentence is just window dressing.

H-35: Differentiate the given integral equation. Plugging in $x = 0$ gives you $y(0)$.

H-36: The fundamental theorem of calculus will be useful in part (b).

H-37: For (a), think of a very simple function.

The equation in the question statement is equivalent to the equation

$$\frac{1}{\sqrt{x-a}} \int_a^x f(t)\,\mathrm{d}(t) = \sqrt{\int_a^x f^2(t)\,\mathrm{d}t}$$

which is, in some cases, easier to use.

For (d), you'll want to let $Y(x) = \int_a^x f(t)\,\mathrm{d}t$, and use the quadratic equation.

H-38: Start by antidifferentiating both sides of the equation with respect to x.

H-39:

(a) Find when the derivatives are zero.

(b) Start with the inequality $\frac{\mathrm{d}W}{\mathrm{d}t} = rW + a > 0$, thinking about how it changed with $r > 0$ vs. $r < 0$.

(c) With the given restrictions, you can re-write the first differential equation in terms of W only.

H-40: This question reviews material from Section 3.9.

(a) Note $\Delta x = 1$ and $x_0 = 0$. That gives us the very nice relationship $x_i = i$.

(b) $f''(x)$ is a positive, increasing function, so its max on the interval $[a, b]$ will occur when $x = b$.

(c) Subtract the two expressions

(d) The sum from (a) will be in the interval from (approx $-$ error bound) to (approx $+$ error bound). Then use this to bound the entire expression.

(e) To find the max error, remember that the actual value is anywhere in the interval you just found – take the worst-case scenario that gives the biggest error, and that's your bound.

H-41: Follow Example 3.12.18 from the textbook, replacing numbers with parameters P_0, r, and N.

Hints for Exercises 4.1. — Jump to TABLE OF CONTENTS.

H-1: Notation 4.1.4 in the text

H-2: see Definition 4.1.12

H-3: see Definition 4.1.16

H-4: Consider two disjoint outcomes.

H-5: Compare to Question 4

H-6: Out of the six values in the sample space, how many are we concerned with?

H-7: How many numbers in the sample space correspond to the desired outcome?

Hints for Exercises 4.2. — Jump to TABLE OF CONTENTS.

H-1: Look in the text if you've forgotten

H-2: Look in the text if you've forgotten

H-3: Values not in the sample space are not included in the table.

H-4: Check Theorem 4.2.3

H-5: find the value with the highest probability

H-6: Remember $f(x) = Pr(X = x)$; so, for instance, $f(2) = Pr(X = 2) = \frac{1}{4}$.

Hints for Exercises 4.3. — Jump to TABLE OF CONTENTS.

H-1: $Pr(T \leqslant x) = 1 - Pr(T > x)$.

H-2: Remember "uniformly distributed" is the continuous analogue of "equally likely."

H-3: Dots should be most densely packed around $x = 2.5$, and sparse near $x = 0$ and $x = 5$.

H-4: See Definition 4.3.6

H-5: review Corollary 4.3.10

H-6: Corollary 4.3.10,

H-7: This system is a simpler version of the random variable of Example 4.3.4 in the text.

H-8: This might be easier to sketch first – then use the sketch to find the function.

H-9: Compare to Question 8.

H-10: Properties of a CDF are given in Corollary 4.3.10 in the text. Multiple values of B are possible.

H-11: Properties of a CDF are given in Corollary 4.3.10 in the text. Note

$$\lim_{x\to\infty}\frac{Bx}{x+1}=B \quad \text{and} \quad \lim_{x\to-\infty}\frac{Dx}{1-x}=-D$$

H-12: Corollary 4.3.3 is helpful. For example: $F(8)-F(6)=Pr(6<W\leqslant 8)$

H-13: See Question 12.

Hints for Exercises 4.4. — Jump to TABLE OF CONTENTS.

H-1: First sketch the probability density function (PDF). Where the probability density function (PDF) is higher, the dots are densest – remember, it's the probability *density* function.

H-2: See Corollaries 4.4.8 and 4.3.10

H-3: Use Question 2.

H-4: If the cumulative distribution function (CDF) $F(x)$ is continuous, then the random variable is continuous, and the probability density function (PDF) is $f(x)=F'(x)$.

H-5: In each case, you're plotting $F(x)=\int_{-\infty}^{x} f(t)\mathrm{d}t$, where $f(t)$ is the function shown in the problem. In each case shown, since $f(t)$ is zero when $t<0$, so $F(x)$ is an antiderivative of $f(x)$ with $F(0)=0$.

H-6: Try the substitution $u=10x$

H-7: Use a sketch of $f(x)$ to find the area under the curve. Remember that on all non-specified intervals, $f(x)=0$ or $f(x)$ DNE.

H-8: Because the two events are disjoint,
$Pr\Big(0<M<1 \text{ OR } 9<M<10\Big)=Pr\Big(0<M<1\Big)+Pr\Big(9<M<10\Big)$

H-9: The cumulative distribution function (CDF) will have the form

$$F(x)=\begin{cases} F_1(x) & x\leqslant 0\\ F_2(x) & 0<x<10\\ F_3(x) & x\geqslant 10\end{cases}$$

H-10: The cumulative distribution function (CDF) will have the form

$$F(x)=\begin{cases} F_1(x) & x\leqslant 0\\ F_2(x) & 0<x\leqslant 3\\ F_3(x) & 3<x\leqslant 4\\ F_4(x) & 4\leqslant x<6\\ F_5(x) & 6\leqslant x<7\\ F_6(x) & 7\leqslant x<8\\ F_7(x) & 8\leqslant x\end{cases}$$

It can be helpful to sketch $y = f(t)$ when you're finding $F(x)$ for various values of x.

H-11: X is continuous if $F(X)$ is continuous (Definition 4.3.6 in the text).

H-12: X is continuous if $F(X)$ is continuous (Definition 4.3.6 in the text).

H-13: Properties of a PDF are given in Corollary 4.4.8 in the text.

H-14: Properties of a PDF are given in Corollary 4.4.8 in the text.

H-15: Very similar to Question 10.

H-16: Remember we define $|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$. So, you'll be considering the intervals $[-1, 0]$ and $(0, 1]$ separately.

H-17: For part c, treat the largest and smallest values in the sample space separately.

H-18: In the text, Simpson's Rule is Equation 3.9.9. It says

$$\int_a^b f(x)\,\mathrm{d}x \approx \Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots$$
$$\cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \Big] \tfrac{\Delta x}{3}$$

where $\Delta x = \frac{b-a}{2}$.

Hints for Exercises 4.5. — Jump to TABLE OF CONTENTS.

H-1: Imagine X results from a coin flip, where $X = 0$ when the flip is tails and $X = 1$ when the flip is heads. What is $\mathbb{E}(X)$? Is this a likely outcome?

H-2: Let X be the random variable resulting in a dice roll. If the dice is fair, what is $\mathbb{E}(X)$?

H-3: If you ran X a lot of times and took the average, what would that average be?

H-4: Can you make $\mathbb{E}(Z) = 1$ if $\mathcal{S} = \{-1, 1\}$?

H-5: Definition 4.5.1: Given a discrete random variable X, the **expected value of** X, denoted $\mathbb{E}(X)$, is given by

$$\sum x \cdot Pr(X = x)$$

where the sum is taken over every possible value of X.

H-6: Compare to Question 5.

H-7: Let X be the amount of money you win after one round, where $X = -1$ if you loose (because your total money went down by one dollar).

Once you have the long-term average value of X, you can guess how much money you'll win after N plays, where N is some large number.

H-8: Let X be the amount of money won after one game, where X is negative if you lose.

Once you have the long-term average value of X, you can guess how much money you'll win after N plays, where N is some large number.

H-9: $f(x)$ is increasing, so use both Theorems 4.5.7 and 4.5.8.

H-10: Make use of symmetry to avoid a long computation.

H-11: You'll compute two integrals separately.

H-12: Use a substitution.

H-13: Create the probability density function (PDF) of Y.

H-14: See Example 3.10.10 in the text. Corollary 4.4.8 has properties of probability density functions (PDFs).

H-15: Use integration by parts.

H-16: Use integration by parts to evaluate $\int \ln x \mathrm{d}x$.

H-17: Use long division to get the integrand into the correct form for partial fractions.

H-18: Start with $\tan^2 x = \sec^2 x - 1$; then integration by parts.

H-19: Take advantage of symmetry, or use the substitution rule. Remember that b is negative.

H-20: First, find the expected amount that Sonic will pay to Fred.

H-21: Employment insurance benefits don't count towards "salary" in this scenario.

H-22: To get you started on (a), the expected return from the x dollars invested in Asset A is $1.2x$ dollars.

For (b), first consider the worst case scenario. If Asset B has a bad-return year, how much money does Riley have to have in safe Asset A to make sure her total return is at least \$350? Then, think about which asset leads to a higher expected return for each dollar invested.

Hints for Exercises 4.6. — Jump to TABLE OF CONTENTS.

H-1: "Weight giving relative likelihood" isn't necessarily a probability, but "how likely" is a probability.

H-2: Let $\mathbb{S} = \{-100, 100\}$

H-3: $f(x) = \frac{1}{b-a}$, $a \leqslant x \leqslant b$

H-4: $Pr(X = s) = 1$

H-5: If $f(x)$ has even symmetry, then $xf(x)$ has odd symmetry.

H-6: To find the cumulative distribution function (CDF), remember $F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$

H-7: See Question 5 for the expectation;

$$f(x) = \begin{cases} 1+x & -1 \leqslant x \leqslant 0 \\ 1-x & 0 \leqslant x \leqslant 1 \end{cases}$$

H-8: The probability density function (PDF) of X is $f(x) = F'(x)$.

H-9: start with $\mathbb{E}(T) = \sum_{x=1}^{3} x \cdot Pr(T = x)$

H-10: just like Question 9

H-11: U is discrete. Look to the locations of the jump discontinuities to find the sample space; the height of the jumps gives you the probability mass function (PMF).

H-12: $\sin^3 x = (1 - \cos^2 x) \sin x$

Hints for Exercises 5.1. — Jump to TABLE OF CONTENTS.

H-1: Not every limit exists.

H-2: 100 isn't all that big when you're contemplating infinity. (Neither is any other number.)

H-3: $\lim_{n\to\infty} a_{2n+5} = \lim_{n\to\infty} a_n$

H-4: The sequence might be defined by different functions when n is large than when n is small.

H-5: Recall $(-1)^n$ is positive when n is even, and negative when n is odd.

H-6: Modify your answer from Question 5, but make the terms approach zero.

H-7: $(-n)^{-n} = \dfrac{(-1)^n}{n^n}$

H-8: What might cause your answers in (a) and (b) to differ? Carefully read Theorem 5.1.6 in the text about convergent functions and their corresponding sequences.

H-9: You can use the fact that π is somewhat close to $\dfrac{22}{7}$, or you can use trial and error.

H-10: For part (d), you're still writing out only five terms: d_0, d_1, d_2, d_3, and d_4.

H-11: Recall $\tan 0 = \tan \pi = 0$

H-12: Writing out the first few terms and looking for a pattern is the usual way to start these.

For the last part, you're given that the formula will look like $d_n = pn + qn + r$. You can set up a system of equations with three equations and three unknowns.

H-13: These types of questions are a little dodgy, because different answers are always possible. Just go for the simplest relationship you can find.

You can assume the indices start at 0.

H-14: You can compare the leading terms, or factor a high power of n from the numerator and denominator.

H-15: This isn't a rational expression, but you can treat it in a similar way. Recall $e < 3$.

H-16: The techniques of evaluating limits of rational sequences are again useful here.

H-17: Use the Squeeze Theorem.

H-18: $\dfrac{1}{n} \leqslant n^{\sin n} \leqslant n$

H-19: $e^{-1/n} = \dfrac{1}{e^{1/n}}$; what happens to $\dfrac{1}{n}$ as n grows?

H-20: Use the Squeeze Theorem.

H-21: L'Hôpital's rule might help you decide what happens if you are unsure.

H-22: Simplify a_k.

H-23: What happens to $\dfrac{1}{n}$ as n gets very big?

H-24: $\cos 0 = 1$

H-25: The first three terms of each follow a simple pattern.

H-26: $5x - x^2$ is negative for values of x in $(-\infty, 0) \cup (5, \infty)$.

H-27: This is trickier than it looks. Write $\dfrac{1}{n} = x$ and look at the limit as $x \to 0$.

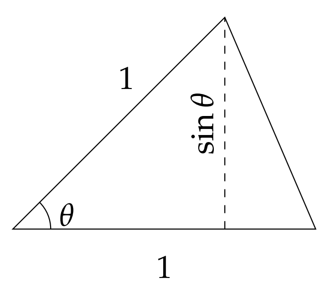
H-28: Multiply and divide by the conjugate.

H-29: Compared to Question 28, there's an easier path.

H-30: Consider $f'(x)$, when $f(x) = x^{100}$.

H-31: Look to Question 30 for inspiration.

H-32: The area of an isosceles triangle with two sides of length 1, meeting at an angle θ, is $\frac{1}{2}\sin\theta$.



H-33: If you lose one-third of your readers, you're left with two-thirds remaining.

H-34: Every term of A_n is the same, and $g(x)$ is a constant function.

H-35: You'll need to use a logarithm before you can apply l'Hôpital's rule.

H-36: (a) Write out the first few terms of the sequence.
(c) Consider how $a_{n+1} - L$ relates to $a_n - L$. What should happen to these numbers if a_n converges to L?

H-37: Your answer from (b) will help you a lot with the subsequent parts.

H-38: When simplifying, factor out P from all terms.

Hints for Exercises 5.1.1. — Jump to TABLE OF CONTENTS.

H-1: Remember an interval is the ratio between two frequencies. It's the ratios that need to stay the same.

H-2: The octave in question goes from 100 Hz to 200 Hz.

H-3: The common ratio between consecutive notes is $2^{1/12}$

H-4: How will the common ratio between consecutive notes be differen from Question 3?

H-5: Keep the ratios between notes the same to find out what notes should appear in the song.

H-6: The interval from a_0 to a_n is $2^{n/12}$.

H-7: What *whole numbers* n and m have $\frac{m}{n} = \frac{11}{10}$?

H-8: What's an octave?

H-9: Take care that the part of the string we're measuring is the part to the right.

Hints for Exercises 5.2. — Jump to TABLE OF CONTENTS.

H-1: S_N is the sum of the terms corresponding to $n = 1$ through $n = N$.

H-2: Note C_k is the cumulative number of cookies.

H-3: How is (a) related to Question 2?

H-4: You'll have to calculate a_1 separately from the other terms.

H-5: When does adding a number decrease the total sum?

H-6: For (b), imagine cutting up the triangle into its black and white parts, then sharing it equally among a certain number of friends. What is the easiest number of friends to share with, making sure each has the same area in their pile?

H-7: Compare to Question 6.

H-8: Iteratively divide a shape into thirds.

H-9: Theorem 5.2.5 in the text tells us $\displaystyle\sum_{n=0}^{N} ar^n = a\frac{1-r^{N+1}}{1-r}$, for $r \neq 1$.

H-10: Note C_k is the cumulative number of cookies.

H-11: To adjust the starting index, either factor out the first term in the series, or subtract two series. For the subtraction option, consider Question 10.

H-12: Express your gains in (a) and (c) as series.

H-13: To find the difference between $\displaystyle\sum_{n=1}^{\infty} c_n$ and $\displaystyle\sum_{n=1}^{\infty} c_{n+1}$, try writing out the first few terms.

H-14: You might want to first consider a simpler true or false: $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{b_n} \stackrel{?}{=} \frac{A}{B}$.

H-15: What kind of a series is this?

H-16: This is a special kind of series, that you should recognize.

H-17: When you see $\displaystyle\sum_k \Big(\cdots k \cdots \; - \; \cdots k+1 \cdots \Big)$, you should think "telescoping series."

H-18: When you see $\displaystyle\sum_n \Big(\cdots n \cdots \; - \; \cdots n+1 \cdots \Big)$, you should immediately think "telescoping series". But be careful not to jump to conclusions — evaluate the n^{th} partial sum explicitly.

H-19: Review Definition 5.2.3 in the text.

H-20: This is a special case of a general series whose sum we know.

H-21: Review Example 5.2.7 in the text. To write the number as a geometric series, the first few terms might not fit the pattern of the rest of the terms.

H-22: Start by writing it as a geometric series.

H-23: Review Example 5.2.7 in the text. Since the pattern repeats every three decimals, your common ratio r will be $\dfrac{1}{10^3}$.

H-24: Split the series into two parts.

H-25: Split the series into two parts.

H-26: Split the series into two parts.

H-27: Use logarithm rules to turn this into a more obvious telescoping series.

H-28: This is a telescoping series.

H-29: See Definitions 4.2.1 and 4.3.1.

Helpful notation: rounding x down to the nearest integer is $\lfloor x \rfloor$, and rounding x up to the nearest integer is $\lceil x \rceil$.

H-30: The volume of a sphere of radius r is $\dfrac{4}{3}\pi r^3$.

H-31: Use the properties of a telescoping series to simplify the terms.
Recall $\sin^2\theta + \cos^2\theta = 1$.

H-32:

(a) is straight algebra

(b) On the one hand, $\sum(3i^2+3i) = 3\sum i^2 + 3\sum i$. On the other hand, $\sum(3i^2+3i) = \sum\big(i(i+1)(i+2) - (i-1)i(i+1)\big)$. Use telescoping sums to simplify $\sum\big(i(i+1)(i+2) - (i-1)i(i+1)\big)$, then solve for $\sum i^2$.

(c) $\sum_{i=1}^{n} i^3 = \frac{1}{4}\sum_{i=1}^{n}\Big(i^2(i+1)^2 - (i-1)^2 i^2\Big)$, and this is a telescoping series.

H-33: Review Question 3 for using the sequence of partial sums.

H-34: Write the area under the curve as a geometric series. Be careful about the starting index, and make use of symmetry.

H-35: Start with partial fractions.

H-36: What is the ratio of areas between the outermost (red) ring and the next (blue) ring?

Hints for Exercises 5.3. — Jump to TABLE OF CONTENTS.

H-1: That is, which series have terms whose limit is not zero?

H-2: That is, if $f(x)$ is a function with $f(n) = a_n$ for all whole numbers n, is $f(x)$ nonnegative and decreasing?

H-3: This isn't a trick. It's meant to give you intuition to the direct comparison test.

H-4: The comparison test is Theorem 5.3.8 in the text. However, rather than trying to memorize which way the inequalities go in all cases, you can use the same reasoning as Question 3.

H-5: Think about Question 4 to remind yourself which way the inequalities have to go for direct comparison.

Note that all the comparison series have positive terms, so we don't need to worry about that part of the limit comparison test.

H-6: The divergence test is Theorem 5.3.1 in the text.

H-7: The limit is calculated correctly.

H-8: It is true that $f(x)$ is positive. What else has to be true of $f(x)$ for the integral test to apply?

H-9: Refer to Question 4.

H-10: For the ratio test to be inconclusive, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$ should be 1 or nonexistent.

H-11: By the divergence test, for a series $\sum a_n$ to *converge*, we need $\lim\limits_{n\to\infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller.

H-12: If $f(x)$ is positive and decreasing, then the integral test tells you that the integral and the series either both increase or both decrease. So, in order to find an example with the properties required in the question, you need $f(x)$ to *not* be both positive and decreasing.

H-13: Review Theorem 5.3.11 and Example 5.3.12 in the text.

H-14: Don't jump to conclusions about properties of the a_n's.

H-15: Always try the divergence test first (in your head).

H-16: Which test should you always try first (in your head)?

H-17: Review the integral test, which is Theorem 5.3.5 in the text.

H-18: A comparison might be helpful–try some algebraic manipulation to find a likely series to compare it to.

H-19: This is a geometric series.

H-20: Notice that the series is geometric, but it doesn't start at $n = 0$.

H-21: Note n only takes integer values: what's $\sin(\pi n)$ when n is an integer?

H-22: Note n only takes integer values: what's $\cos(\pi n)$ when n is an integer?

H-23: What's the test that you should always think of when you see a factorial?

H-24: This is a geometric series, but you'll need to do a little algebra to figure out r.

H-25: Which test fits most often with factorials?

H-26: Try finding a nice comparison.

H-27: With the substitution $u = \ln x$, the function $\dfrac{1}{x(\ln x)^{3/2}}$ is easily integrable.

H-28: Combine the integral test with the results about p-series, Example 5.3.6 in the text.

H-29: Try the substitution $u = \sqrt{x}$.

H-30: Review Example 5.3.9 in the text for developing intuition about comparisons, and Example 5.3.10 for an example where finding an appropriate comparison series calls for some creativity.

H-31: What does the summand look like when k is very large?

H-32: What does the summand look like when n is very large?

H-33: What is the behaviour for large k?

H-34: When m is large, $3m + \sin\sqrt{m} \approx 3m$.

H-35: This is a geometric series, but it doesn't start at $n = 0$.

H-36: The series is geometric.

H-37: The first series can be written as $\sum_{n=1}^{\infty} \frac{1}{2n-1}$.

H-39: What does the summand look like when n is very large?

H-40: $\lim_{n\to\infty} (1+1/n)^n = e$

H-41: For part (a), see Example 3.10.23 in the text.

For part (b), review Theorem 5.3.5 in the text.

For part (c), see Example 5.3.12 in the text.

H-42: The truncation error arising from the approximation $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \approx \sum_{n=1}^{N} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ is precisely $E_N = \sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$. You'll want to find a bound on this sum using the integral test.

A key observation is that, since $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$ is decreasing, we can show that

$$\frac{e^{-\sqrt{n}}}{\sqrt{n}} \leqslant \int_{n-1}^{n} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, \mathrm{d}x$$

for every $n \geqslant 1$.

H-43: What does the fact that the series $\sum_{n=0}^{\infty} a_n$ converges guarantee about the behavior of a_n for large n?

H-44: What does the fact that the series $\sum_{n=0}^{\infty} (1-a_n)$ converges guarantee about the behavior of a_n for large n?

H-45: What does the fact that the series $\sum_{n=1}^{\infty} \frac{na_n - 2n + 1}{n+1}$ converges guarantee about the behavior of a_n for large n?

H-46: What does the fact that the series $\sum_{n=1}^{\infty} a_n$ converges guarantee about the behavior of a_n for large n? When is $x^2 \leqslant x$?

H-47: If we add together the frequencies of all the words, they should amount to 100%. We can approximate this sum using ideas from Example 5.3.4 in the text.

H-48: No city has fewer than one person, so we are approximating a finite sum–not an infinite series. To get greater accuracy, use exact values for the first several terms in the sum, and use an integral to approximate the rest.

Hints for Exercises 5.4. — Jump to TABLE OF CONTENTS.

H-1: What is conditional convergence?

H-2: If $\sum |a_n|$ converges, then $\sum a_n$ is guaranteed to converge as well. (That's Theorem 5.4.2 in the text.) So, one of the blank spaces describes an impossible sequence.

H-4: Be careful about the signs.

H-5: Does the alternating series test really apply?

H-6: What does the summand look like when n is very large?

H-7: What does the summand look like when n is very large?

H-8: This is a trick question. Be sure to verify *all* of the hypotheses of any convergence test you apply.

H-9: Try the substitution $u = \ln x$.

H-10: Show that it converges absolutely.

H-11: Use a similar method to Queston 10.

H-12: Show it converges absolutely using a direct comparison test.

H-13: For part (a), replace n by x in the absolute value of the summand. Can you integrate the resulting function?

H-14: You don't need to add up very many terms for this level of accuracy.

H-15: Use the direct comparison test to show that the series converges absolutely.

Hints for Exercises 5.5. — Jump to TABLE OF CONTENTS.

H-1: $f(1)$ is the sum of a geometric series.

H-2: Calculate $\frac{d}{dx}\left\{\frac{(x-5)^n}{n!+2}\right\}$ when n is a constant.

H-3: There is only one.

H-4: Use Theorem 5.5.9 in the text.

H-5: Review the discussion immediately following Definition 5.5.1 in the text.

H-6: Review the discussion immediately following Definition 5.5.1 in the text.

H-7: Review the discussion immediately following Definition 5.5.1 in the text.

H-8: See Example **??** in the text.

H-9: See Example **??** in the text.

H-11: If a power series is centred at a and has radius of convergence R, then the largest possible open interval of values for which it converges is $(a-R, a+R)$.

H-15: Start part (b) by computing the partial sums of $\displaystyle\sum_{k=1}^{\infty}\left(\frac{a_k}{a_{k+1}}-\frac{a_{k+1}}{a_{k+2}}\right)$

H-16: You should know a power series representation for $\dfrac{1}{1-x}$. Use it.

H-17: You can safely ignore one of the given equations, but not the other.

H-18: $n \geqslant \ln n$ for all $n \geqslant 1$.

H-19: You know the geometric series expansion of $\frac{1}{1-x}$. What (calculus) operation(s) can you apply to that geometric series to convert it into the given series?

H-20: First show that the fact that the series $\sum_{n=0}^{\infty}(1-b_n)$ converges guarantees that $\lim_{n\to\infty} b_n = 1$.

H-21: What does a_n look like for large n?

H-22: Use the second derivative test.

H-23: What function has $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$ as its power series representation?

Hints for Exercises 5.6. — Jump to TABLE OF CONTENTS.

H-1: Which of the functions are constant, linear, and quadratic?

H-2: You don't have to actually calculate the entire series $T(x)$ to answer the question.

H-3: If you don't have these memorized, it's good to be able to derive them. For instance, $\ln(1+x)$ is the antiderivative of $\dfrac{1}{1+x}$, whose Taylor series can be found by modifying the geometric series $\sum x^n$.

H-4: See Example 5.6.12 in the text.

H-5: The series will bear some resemblance to the Maclaurin series for $\ln(1+x)$.

H-6: The terms $f^{(n)}(\pi)$ are going to be similar to the terms $f^{(n)}(0)$ that we used in the Maclaurin series for sine.

H-7: The Taylor series will look similar to a geometric series.

H-8: Your answer will depend on a.

H-9: You should know the Maclaurin series for $\dfrac{1}{1-x}$. Use it.

H-10: You should know the Maclaurin series for $\dfrac{1}{1-x}$. Use it.

H-11: You should know the Maclaurin series for e^x. Use it.

H-12: Review Example 5.5.19 in the text.

H-13: You should know the Maclaurin series for $\sin x$. Use it.

H-14: You should know the Maclaurin series for e^x. Use it.

H-15: You should know the Maclaurin series for $\arctan(x)$. Use it.

H-16: You should know the Maclaurin series for $\dfrac{1}{1-x}$. Use it.

H-17: Set $(-1)^n \dfrac{x^{2n+1}}{2n+1} = C\dfrac{(-1)^n}{(2n+1)3^n}$, for some constant C. What are x and C?

H-18: There is an important Taylor series, one of the series in Theorem 5.6.5 of the text, that looks a lot like the given series.

H-19: There is an important Taylor series, one of the series in Theorem 5.6.5 of the text, that looks a lot like the given series.

H-20: There is an important Taylor series, one of the series in Theorem 5.6.5 of the text, that looks a lot like the given series. Be careful about the limits of summation.

H-21: There is an important Taylor series, one of the series in Theorem 5.6.5 of the text, that looks a lot like the given series.

H-22: Split the series into a sum of two series. There is an important Taylor series, one of the series in Theorem 5.6.5 of the text, that looks a lot like each of the two series.

H-23: Try the ratio test.

H-24: Write it as the sum of two Taylor series.

H-25: Can you think of a way to eliminate the odd terms from $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$?

H-26: Use Theorem 5.6.1-b in the text to bound the error in a partial-sum approximation.

H-27: Use Theorem 5.6.1-b in the text to bound the error in a partial-sum approximation.

H-28: Use Theorem 5.6.1-b in the text to bound the error in a partial-sum approximation. This theorem requires you to consider values of c between x and $x = 0$; since x could be anything from -2 to 1, you should think about values of c between -2 and 1.

H-29: Use Theorem 5.6.1-b in the text to bound the error in a partial-sum approximation.

To bound the derivative over the appropriate range, remember how to find absolute extrema.

H-30: See Example 5.6.15 in the text

H-31: See Example 5.6.15 in the text

H-32: Set $f(x) = \left(1 + x + x^2\right)^{2/x}$, and find $\lim\limits_{x\to 0} \ln\left(f(x)\right)$.

H-33: Use the substitution $y = \dfrac{1}{x}$, and compare to Question 32.

H-34: Start by differentiating $\sum\limits_{n=0}^{\infty} x^n$.

H-35: The series bears a resemblance to the Taylor series for arctangent.

H-36: For simplification purposes, note $(1)(3)(5)(7)\cdots(2n-1) = \dfrac{(2n)!}{2^n\, n!}$.

H-37: You know the Maclaurin series for $\ln(1+y)$. Use it! You also know its interval of convergence, endpoints and all.

Remember that you are asked for a series expansion in powers of $x-2$. So you want y to be some constant times $x-2$.

H-38: See Example 5.5.20 in the text.

H-39: See Example 5.6.10 in the text.

H-40: See Example 5.6.10 in the text. For part (b), review the Fundamental Theorem of Calculus in §3.3 of the text. For part (c), review §A.12.1 in the text.

H-41: See Example 5.6.10 in the text.

H-42: See Example 5.6.10 in the text.

H-44: Use the Maclaurin series for e^x and the fact that $1+x \leqslant e^x$ for all $x \geqslant 0$.

H-45: For part (c), compare two power series term-by-term.

H-46: Remember e^x is never negative for any real number x.

H-47: Since $f(x)$ is odd, $f(-x) = -f(x)$ for all x in its domain. Consider the even-indexed terms and odd-indexed terms of the Taylor series.

Part III

Answers to Questions

Answers to Exercises 1.1 — Jump to TABLE OF CONTENTS

A-1:

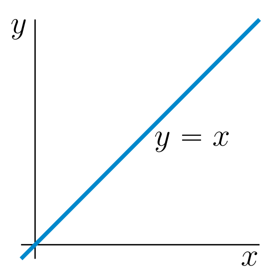
The xz plane is filled with vertical lines; the yz plane is crosshatched; and the xy plane is solid.

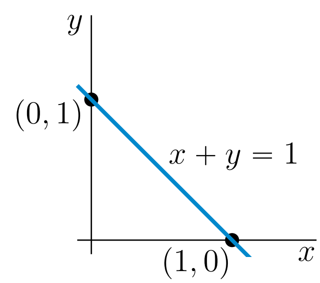
The left bottom triangle vertex is $(1,0,0)$; the right bottom triangle vertex is $(0,1,0)$; the top triangle vertex is $(0,0,1)$.

A-2: (a) The sphere of radius 3 centered on $(1,-2,0)$.

(b) The interior of the sphere of radius 3 centered on $(1,-2,0)$.

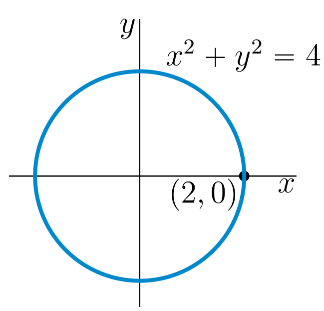
A-3: (a) $x = y$ is the straight line through the origin that makes an angle 45° with the x– and y–axes. It is sketched in the figure on the left below.

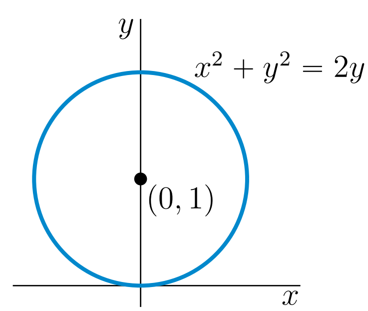




(b) $x + y = 1$ is the straight line through the points $(1,0)$ and $(0,1)$. It is sketched in the figure on the right above.

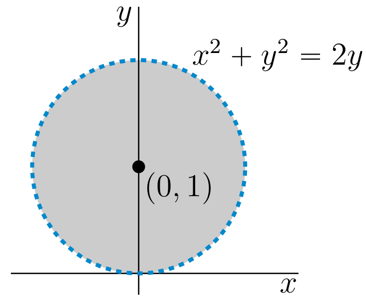
(c) $x^2 + y^2 = 4$ is the circle with centre $(0,0)$ and radius 2. It is sketched in the figure on the left below.



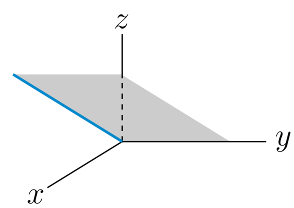


(d) $x^2 + y^2 = 2y$ is the circle with centre $(0,1)$ and radius 1. It is sketched in the figure on the right above.

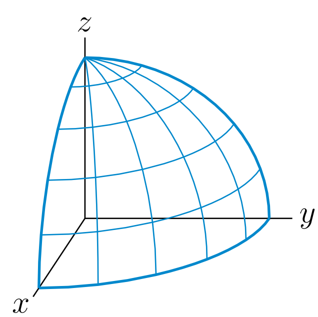
(e) $x^2 + y^2 < 2y$ is the set of points that are strictly inside the circle with centre $(0,1)$ and radius 1. It is the shaded region (not including the dashed circle) in the sketch below.



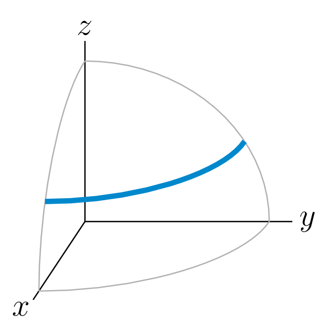
A-4: (a) The set $z = x$ is the plane which contains the y–axis and which makes an angle 45° with the xy–plane. Here is a sketch of the part of the plane that is in the first octant.



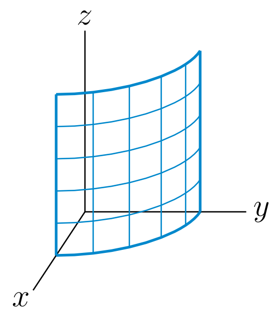
(b) $x^2 + y^2 + z^2 = 4$ is the sphere with centre $(0, 0, 0)$ and radius 2. Here is a sketch of the part of the sphere that is in the first octant.



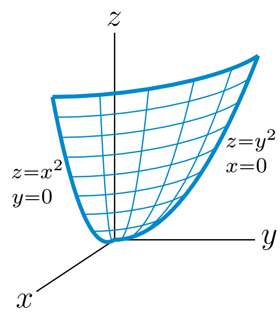
(c) $x^2 + y^2 + z^2 = 4$, $z = 1$ is the circle in the plane $z = 1$ that has centre $(0, 0, 1)$ and radius $\sqrt{3}$. The part of the circle in the first octant is the heavy quarter circle in the sketch



(d) $x^2 + y^2 = 4$ is the cylinder of radius 2 centered on the z–axis. Here is a sketch of the part of the cylinder that is in the first octant.



(e) $z = x^2 + y^2$ is a paraboloid consisting of a vertical stack of horizontal circles. The intersection of the surface with the yz–plane is the parabola $z = y^2$. Here is a sketch of the part of the paraboloid that is in the first octant.



A-5: $\sqrt{67}$

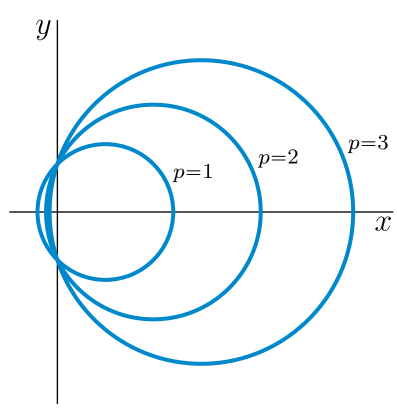
A-6: 9

A-7: $\sqrt{5.01}$ km

A-8: 1 km

A-9: 2 km

A-10:



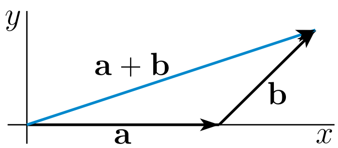
A-11: The sphere has radius 3 and is centered on $(1, 2, -1)$.

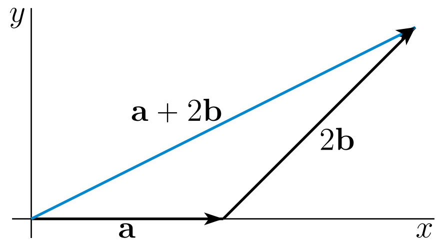
A-12: The circumscribing circle has centre $(\bar{x}, \bar{y})$ and radius r with $\bar{x} = \frac{a}{2}$, $\bar{y} = \frac{b^2+c^2-ab}{2c}$ and $r = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b^2+c^2-ab}{2c}\right)^2}$.

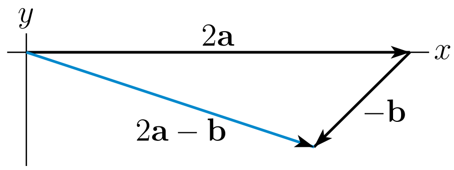
A-13: $x^2 + y^2 = 4z$ The surface is a paraboloid consisting of a stack of horizontal circles, starting with a point at the origin and with radius increasing vertically. The circle in the plane $z = z_0$ has radius $2\sqrt{z_0}$.

Answers to Exercises 1.2 — Jump to TABLE OF CONTENTS

A-1: $\mathbf{a} + \mathbf{b} = \langle 3, 1\rangle$, $\mathbf{a} + 2\mathbf{b} = \langle 4, 2\rangle$, $2\mathbf{a} - \mathbf{b} = \langle 3, -1\rangle$







A-2: (a) not collinear (b) collinear

A-3: (a) perpendicular (b) perpendicular (c) not perpendicular

A-4: Yes.

A-5: This statement is false. One counterexample is $\mathbf{a} = \langle 1, 0, 0\rangle$, $\mathbf{b} = \langle 0, 1, 0\rangle$, $\mathbf{c} = \langle 0, 0, 1\rangle$. Then $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 0$, but $\mathbf{b} \neq \mathbf{c}$. There are *many* other counterexamples.

A-6:

(a) $\langle -1, 1\rangle$

(b) $\langle -3, -3, -1\rangle$.

(c) $\langle -2, 2\rangle$.

(d) $\langle -2, 1\rangle$

A-7:

(a) $\sqrt{5}$

(b) $\sqrt{5}$

(c) $2\sqrt{5}$

A-8:

(a) $\langle \frac{3}{5}, \frac{4}{5}\rangle$

(b) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$

(c) $\langle 0, 1, 0\rangle$

(d) $\left\langle \frac{7}{\sqrt{162}}, \frac{7}{\sqrt{162}}, \frac{8}{\sqrt{162}} \right\rangle$

(e) $\hat{\boldsymbol{\imath}}$

A-9: $\pm\left\langle \frac{7047}{\sqrt{130}}, 0, \frac{5481}{\sqrt{130}} \right\rangle$

A-10: $\mathbf{a} + 3\mathbf{b} = \langle 10, 14, 16 \rangle, \quad |\mathbf{a} - \mathbf{b}| = \sqrt{24}$

A-11: $(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$

A-12:

(a) 4

(b) 0, perpendicular

(c) 4, parallel

(d) 2

(e) 0, perpendicular

A-13: (a) -1 (b) 0, 4 (c) -2, -3

A-14: (a) -5 (b) 0.8

A-15: See the solution.

A-16: All 6 edges have length $\sqrt{2}s$.

A-17: (a) $AA'B'B$ is a parallelogram, but not a rectangle.
$AA'C'C$ is a rectangle.
$BB'C'C$ is a parallelogram, but not a rectangle.

(b) $\sqrt{17}$

Answers to Exercises 1.3 — Jump to TABLE OF CONTENTS

A-1: Any vector of the form $c\,\hat{\mathbf{k}}$ with $c \neq 0$ and $c \neq 1$ works. Three possible choices are $-\hat{\mathbf{k}}$, $2\,\hat{\mathbf{k}}$, $7.12345\,\hat{\mathbf{k}}$.

A-2:

(a) $x + 4y + z = 0$

(b) $7x + 8y + 9z = 48$

(c) $7x + 8y + 9z = 75$

A-3: $x + y + z = 1$

A-4: $x + y + z = 2$

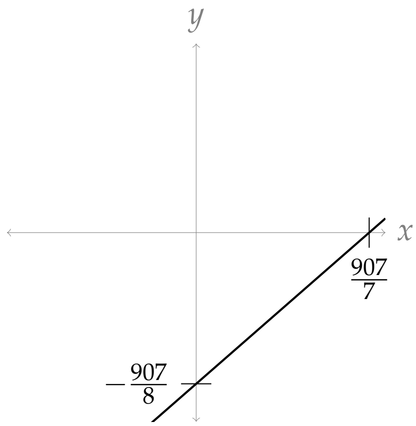
A-5: All three points $(1,2,3)$, $(2,3,4)$ and $(3,4,5)$ are on the line $\mathbf{x}(t) = (1,2,3) + t(1,1,1)$. There are many planes through that line.

A-6: $a = 8$, $b = 14$

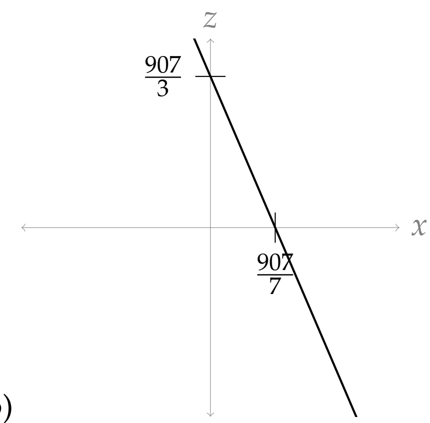
A-7: yes

A-8: $a = \dfrac{31}{111}$

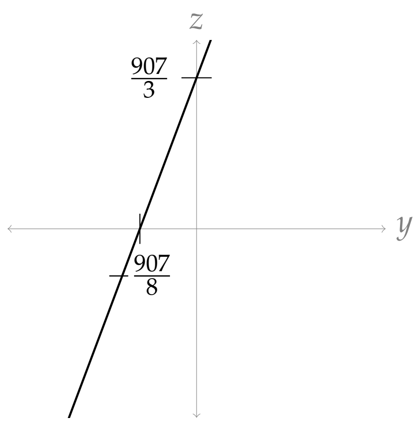
A-9: (a)



(b)



(c)



A-10: P and Q are identical; P and Q are perpendicular to R; S is neither parallel, nor perpendicular, nor identical to any of the other planes.

A-11: It is the plane $x + z = 8$

A-12: (a) $9x - y - z = 8$ (b) $14x - 7y - 8z = 52$

(c) For any real number a, the plane with equation

$$ax + (3 - 4a)\, y + (3a - 3)\, z = 3$$

contains the three given points

A-13: It is the plane $2(\mathbf{b} - \mathbf{a}) \cdot \mathbf{x} = |\mathbf{b}|^2 - |\mathbf{a}|^2$, which is the plane through $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ with normal vector $\mathbf{b} - \mathbf{a}$.

A-14: (a) $3x + 2y + z = 8$ (b) $\big(3\,,\, -1\,,\, 1\big)$

A-15: $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 3$

Answers to Exercises 1.4 — Jump to TABLE OF CONTENTS

A-1: Any constant function, for example $f(x, y) = 0$.

A-2:

(a) $[-10, 10]$

(b) $[0, 1]$

(c) $[-1, 1]$

(d) $[0, 10]$

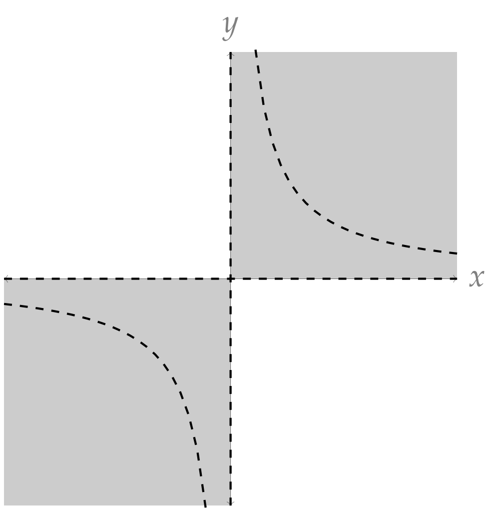
A-3: yes

A-4: Domain: all of $\mathbb{R}^2$. Range: $[0, \infty)$

A-5: Domain: all of $\mathbb{R}^2$. Range: $[0, \infty)$.

A-6: Domain: interior of the unit circle. Range: $[0, \pi/2]$.

A-7: Domain: all points (x, y) such that x and y have the same sign; x and y are nonzero; and $y \neq \frac{1}{x}$.



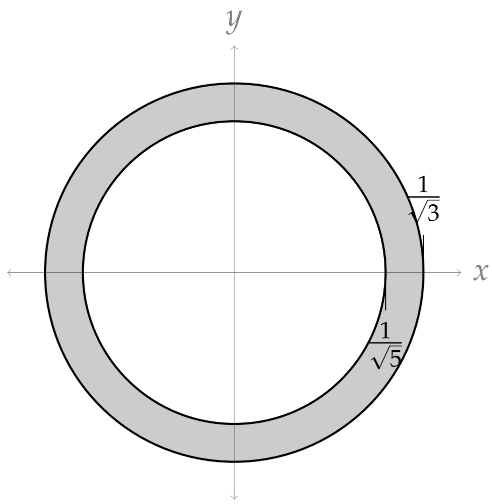
Range: $(-\infty, 0) \cup (0, \infty)$.

A-8: Domain: all of $\mathbb{R}^2$. Range: $[0, 1)$.

A-9: Domain: all of $\mathbb{R}^2$. Range: $\left[-\frac{3}{2}, \frac{3}{2}\right]$.

A-10: For example: domain should be all (a, p) where $a \geqslant 0$ and $p > 0$; range should be $[0, \infty)$.

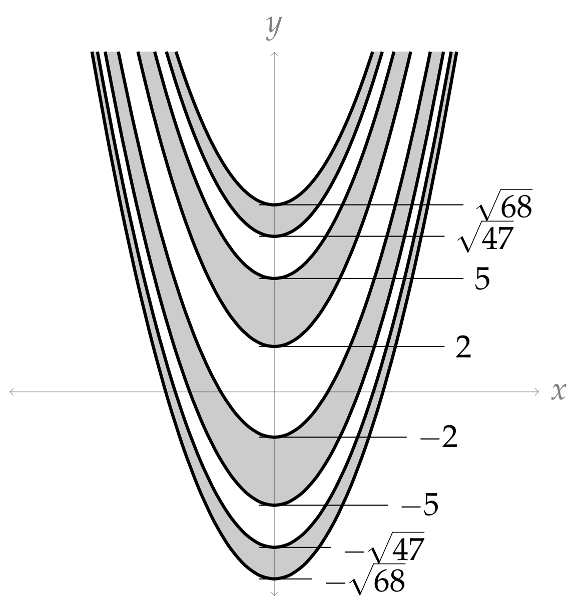
A-11: $\frac{1}{5} \leqslant x^2 + y^2 \leqslant \frac{1}{3}$: that is, the points (x, y) that are inside or on the circle centred at the origin with radius $\frac{1}{\sqrt{3}}$, but not inside the circle centred at the origin with radius $\frac{1}{\sqrt{5}}$.



A-12:

The point (x, y) must be in one of the following regions:

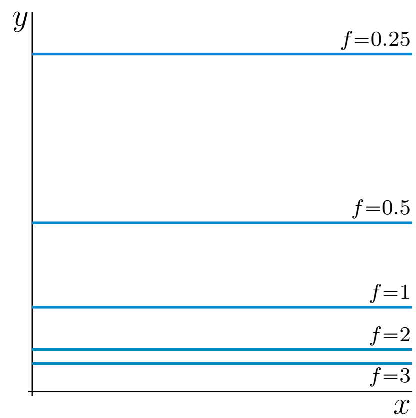
- $x^2 - \sqrt{68} \leqslant y \leqslant x^2 - \sqrt{47}$
- $x^2 - 5 \leqslant y \leqslant x^2 - 2$
- $x^2 + 2 \leqslant y \leqslant x^2 + 5$
- $x^2 + \sqrt{47} \leqslant y \leqslant x^2 + \sqrt{68}$



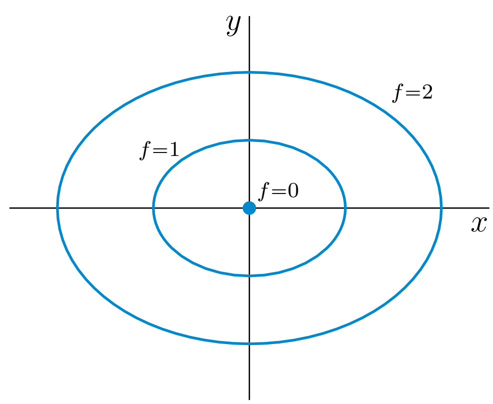
Answers to Exercises 1.5 — Jump to TABLE OF CONTENTS

A-1: (a) $\leftrightarrow$ (B) (b) $\leftrightarrow$ (A) (c) $\leftrightarrow$ (C)

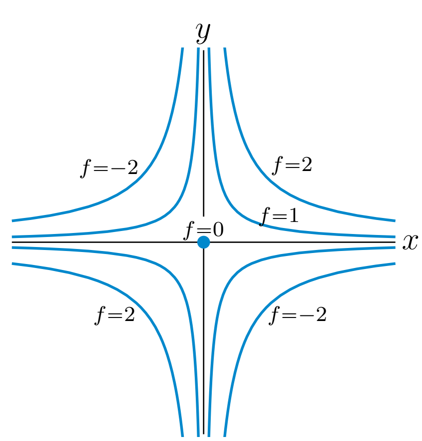
A-2:



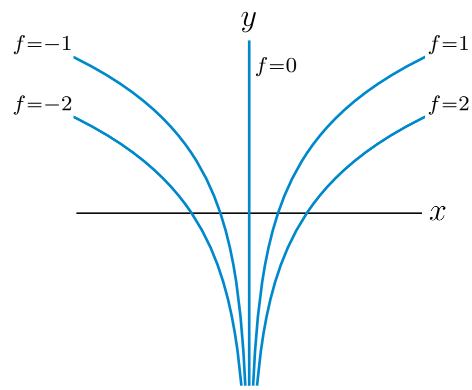
A-3: (a)



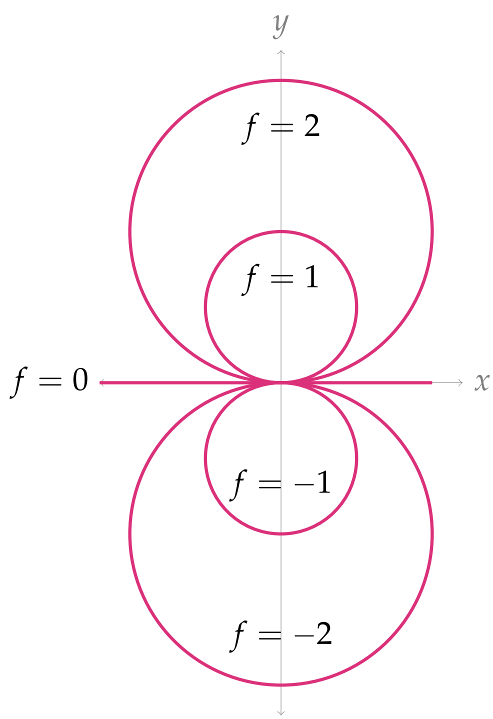
(b)



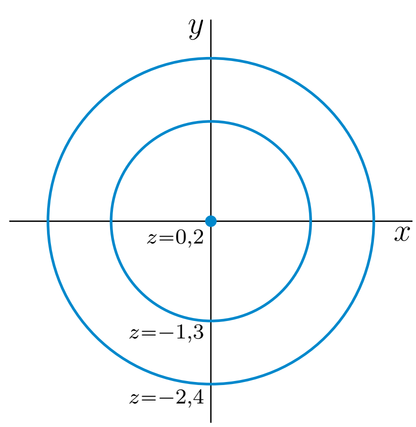
(c)



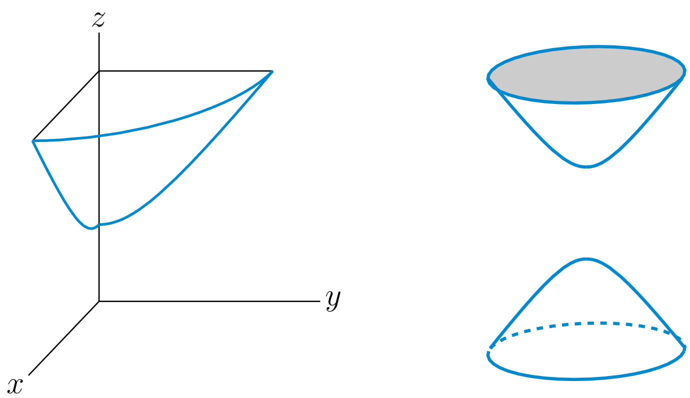
A-4:



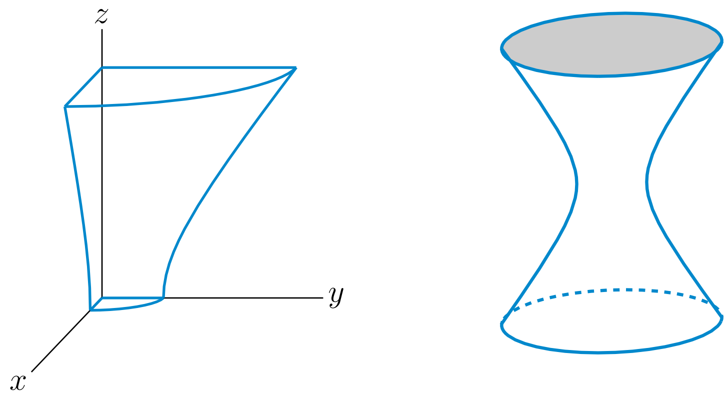
A-5: (a)



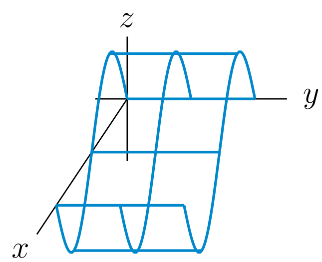
(b)



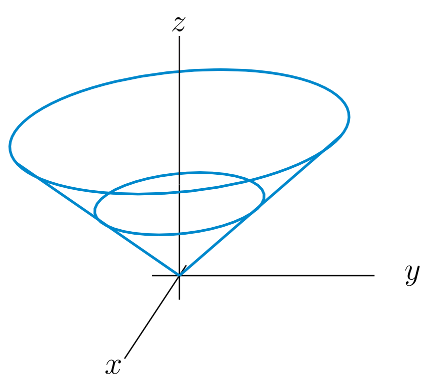
A-6:



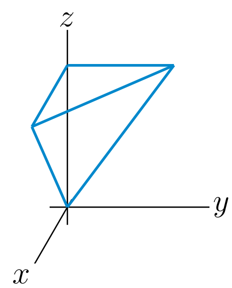
A-7: (a)

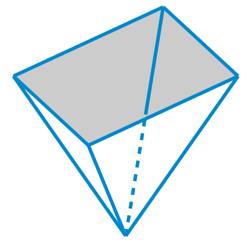


(b)

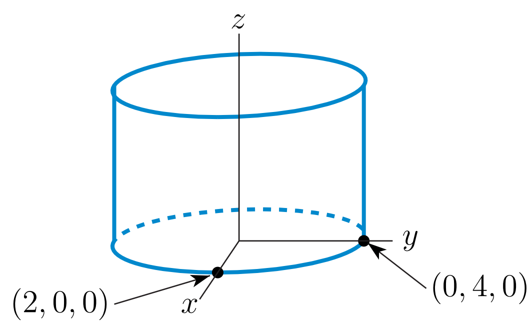


(c)

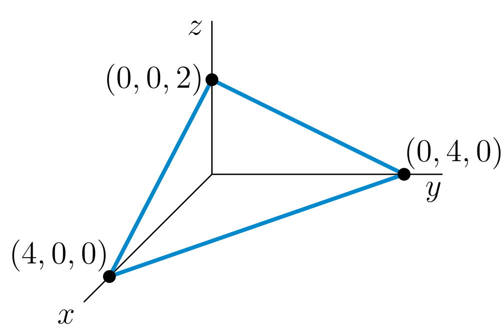




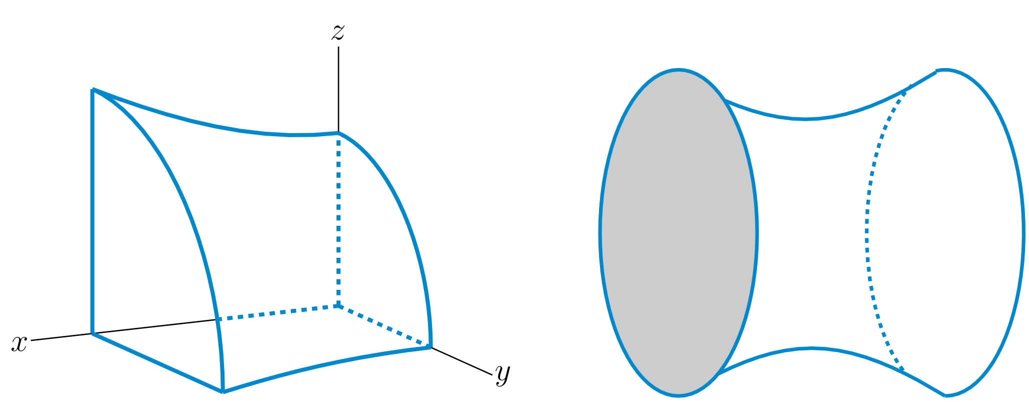
A-8: (a) This is an elliptic cylinder parallel to the z-axis. Here is a sketch of the part of the surface above the xy–plane.



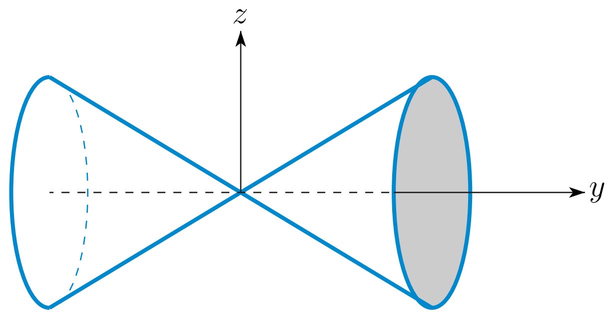
(b) This is a plane through $(4, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 2)$. Here is a sketch of the part of the plane in the first octant.



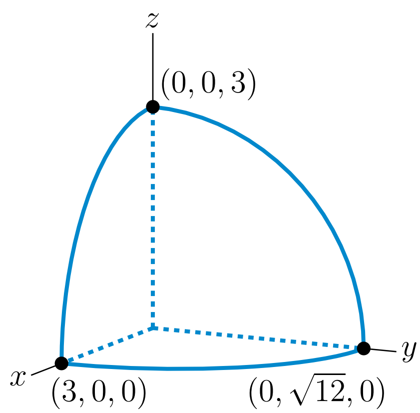
(c) This is a hyperboloid of one sheet with axis the x-axis.

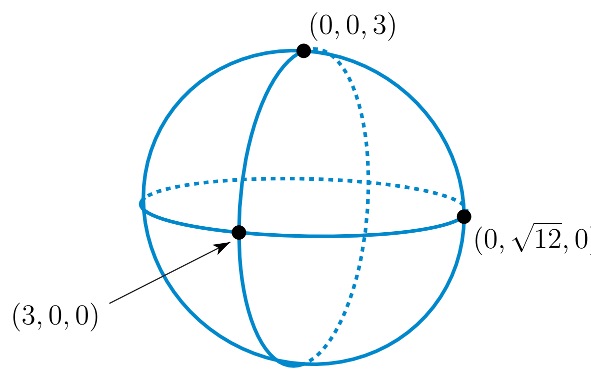


(d) This is a circular cone centred on the y-axis.

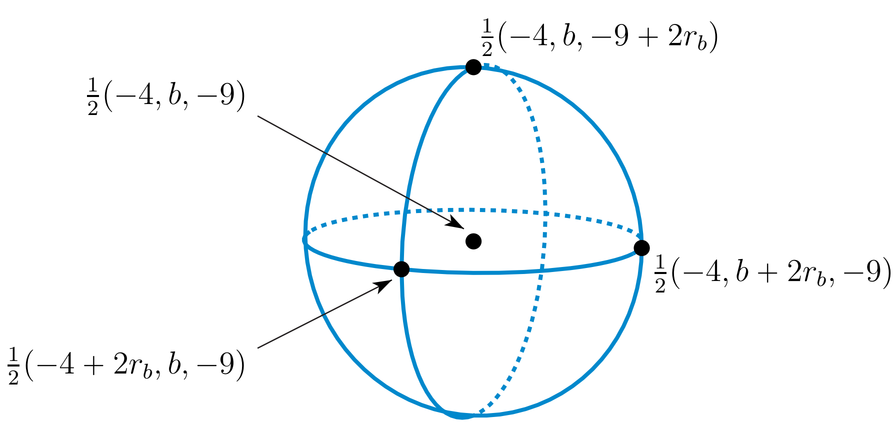


(e) This is an ellipsoid centered on the origin with semiaxes 3, $\sqrt{12} = 2\sqrt{3}$ and 3 along the x, y and z-axes, respectively.

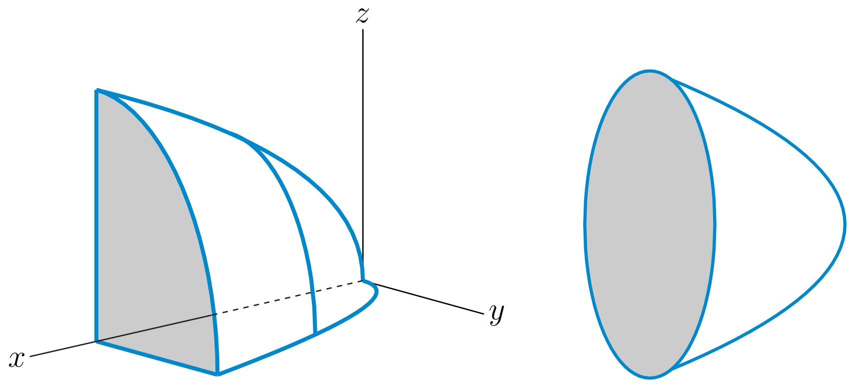




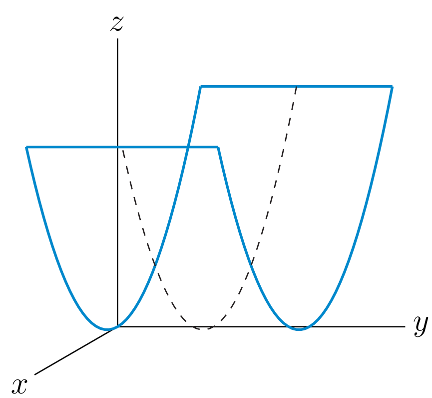
(f) This is a sphere of radius $r_b = \frac{1}{2}\sqrt{b^2 + 4b + 97}$ centered on $\frac{1}{2}(-4, b, -9)$.



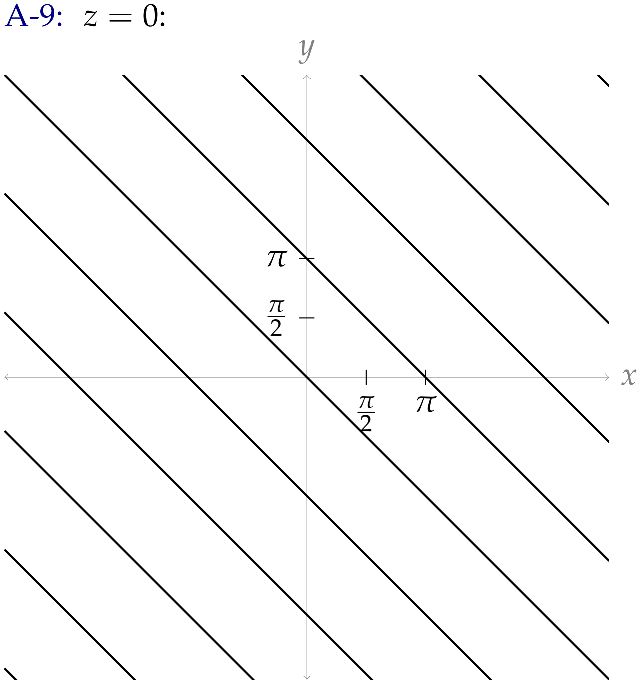
(g) This is an elliptic paraboloid with axis the x-axis.



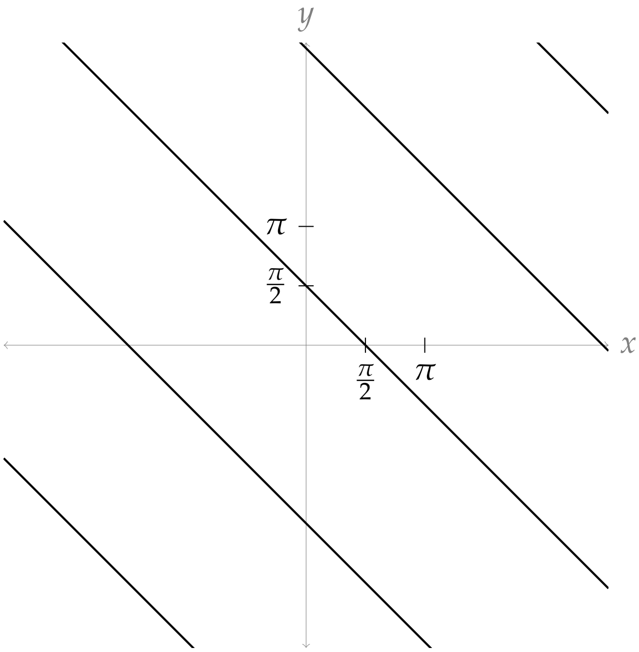
(h) This is an upward openning parabolic cylinder.



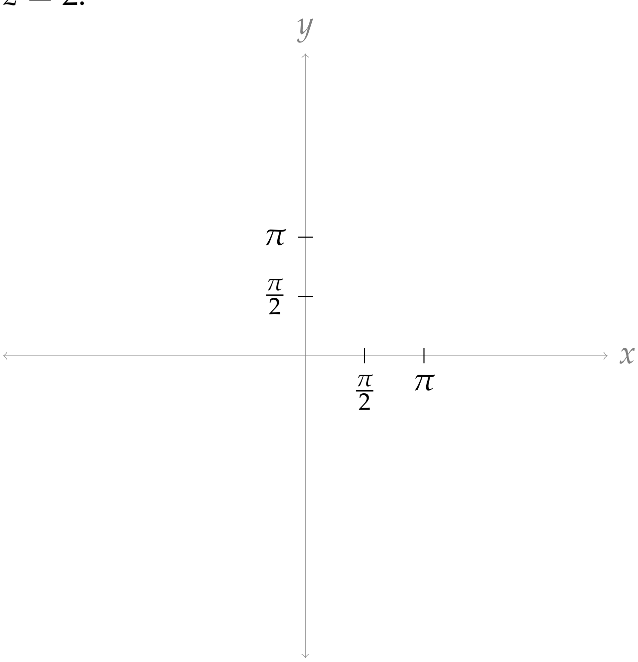
A-9: $z = 0$:



$z = 1$:



$z = 2$:



A-10: $x^2 + y^2 = \left(\frac{|z|}{3} + 1\right)^2$

Answers to Exercises 2.1 — Jump to TABLE OF CONTENTS

A-1: No: you can go higher by moving in the negative y direction.

A-2:

(a) $f_y(1.5, 2.4) \approx -2$

(b) $f_x(1.7, 1.7) \approx 11$

(c) $f_y(1.7, 1.7) \approx -3$

(d) $f_x(1.1, 2) \approx 9$

A-3: (a)

$$\begin{aligned} f_x(x,y,z) &= 3x^2y^4z^5 & \qquad f_x(0,-1,-1) &= 0 \\ f_y(x,y,z) &= 4x^3y^3z^5 & f_y(0,-1,-1) &= 0 \\ f_z(x,y,z) &= 5x^3y^4z^4 & f_z(0,-1,-1) &= 0 \end{aligned}$$

(b)

$$\begin{aligned} w_x(x,y,z) &= \frac{yze^{xyz}}{1+e^{xyz}} & \qquad w_x(2,0,-1) &= 0 \\ w_y(x,y,z) &= \frac{xze^{xyz}}{1+e^{xyz}} & w_y(2,0,-1) &= -1 \\ w_z(x,y,z) &= \frac{xye^{xyz}}{1+e^{xyz}} & w_z(2,0,-1) &= 0 \end{aligned}$$

(c)

$$\begin{aligned} f_x(x,y) &= -\frac{x}{(x^2+y^2)^{3/2}} & \qquad f_x(-3,4) &= \frac{3}{125} \\ f_y(x,y) &= -\frac{y}{(x^2+y^2)^{3/2}} & f_y(-3,4) &= -\frac{4}{125} \end{aligned}$$

A-4: By the quotient rule

$$\begin{aligned} \frac{\partial z}{\partial x}(x,y) &= \frac{(1)(x-y)-(x+y)(1)}{(x-y)^2} &= \frac{-2y}{(x-y)^2} \\ \frac{\partial z}{\partial y}(x,y) &= \frac{(1)(x-y)-(x+y)(-1)}{(x-y)^2} &= \frac{2x}{(x-y)^2} \end{aligned}$$

Hence

$$x\frac{\partial z}{\partial x}(x,y) + y\frac{\partial z}{\partial y}(x,y) = \frac{-2xy+2yx}{(x-y)^2} = 0$$

A-5: (a) $\frac{\partial z}{\partial x} = \frac{z(1-x)}{x(yz-1)}$, $\quad \frac{\partial z}{\partial y} = \frac{z(1+y-yz)}{y(yz-1)}$

(b) $\frac{\partial z}{\partial x}(-1,-2) = \frac{1}{2}$, $\quad \frac{\partial z}{\partial y}(-1,-2) = 0$.

A-6: $\frac{\partial U}{\partial T}(1,2,4) = -\frac{2\ln(2)}{1+2\ln(2)} \qquad \frac{\partial T}{\partial V}(1,2,4) = 1 - \frac{1}{4\ln(2)}$

A-7: 24

A-8: $f_x(0,0) = 1, \qquad f_y(0,0) = 2$

A-9: Yes.

A-10: (a) $\frac{\partial f}{\partial x}(0,0) = 1$, $\frac{\partial f}{\partial y}(0,0) = 4 \qquad$ (b) Nope.

A-11: 1 resp. 0

A-12: (a) 0 (b) 0 (c) $\frac{1}{2}$

Answers to Exercises 2.2 — Jump to TABLE OF CONTENTS

A-1: See the solution.

A-2: No such $f(x, y)$ exists.

A-3: (a) $f_{xx}(x, y) = 2y^3$ $\quad f_{yxy}(x, y) = f_{xyy}(x, y) = 12xy$

(b) $f_{xx}(x, y) = y^4 e^{xy^2}$ $\quad f_{xy}(x, y) = \big(2y + 2xy^3\big)e^{xy^2}$ $\quad f_{xxy}(x, y) = \big(4y^3 + 2xy^5\big)e^{xy^2}$
$f_{xyy}(x, y) = \big(2 + 10xy^2 + 4x^2y^4\big)e^{xy^2}$

(c) $\dfrac{\partial^3 f}{\partial u\, \partial v\, \partial w}(u, v, w) = -\dfrac{36}{(u + 2v + 3w)^4}$ $\qquad \dfrac{\partial^3 f}{\partial u\, \partial v\, \partial w}(3, 2, 1) = -0.0036 = -\dfrac{9}{2500}$

A-4: $f_{xx} = \frac{5y^2}{(x^2+5y^2)^{3/2}}$ $\quad f_{xy} = f_{yx} = -\frac{5xy}{(x^2+5y^2)^{3/2}}$ $\quad f_{yy} = \frac{5x^2}{(x^2+5y^2)^{3/2}}$

A-5: (a) $f_{xyz}(x, y, z) = 0$ (b) $f_{xyz}(x, y, z) = 0$ (c) $f_{xx}(1, 0, 0) = 0$

A-6: See the solution.

A-7: Only $u(x, y) = x^{0.5}y^{0.5}$ satisfies all of the properties described in the question.

A-8:

$f_{xy}(1.8, 2.0) \approx 0$

Answers to Exercises 2.3 — Jump to TABLE OF CONTENTS

A-1: (a) (i) T, U

(a) (ii) U

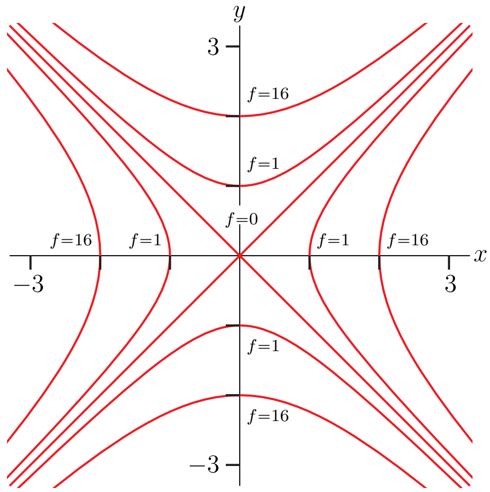
(a) (iii) S

(b) (i) $F_x(1, 2) > 0$

(b) (ii) F does *not* have a critical point at $(2, 2)$.

(b) (iii) $F_{xy}(1, 2) < 0$

A-2: (a)



(b) $(0,0)$ is a local (and also absolute) minimum.

(c) No. See the solutions.

A-3: $|c| > 2$

A-4:

critical point	type
$(0,0)$	saddle point
$(-\frac{2}{3}, \frac{2}{3})$	local max

A-5:

critical point	type
$(0,3)$	saddle point
$(0,-3)$	saddle point
$(-2,1)$	local max
$(2,-1)$	local min

A-6:

critical point	type
$(0,0)$	local min
$(\sqrt{2},-1)$	saddle point
$(-\sqrt{2},-1)$	saddle point

A-7:

critical point	type
$(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	local min
$-(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	saddle point

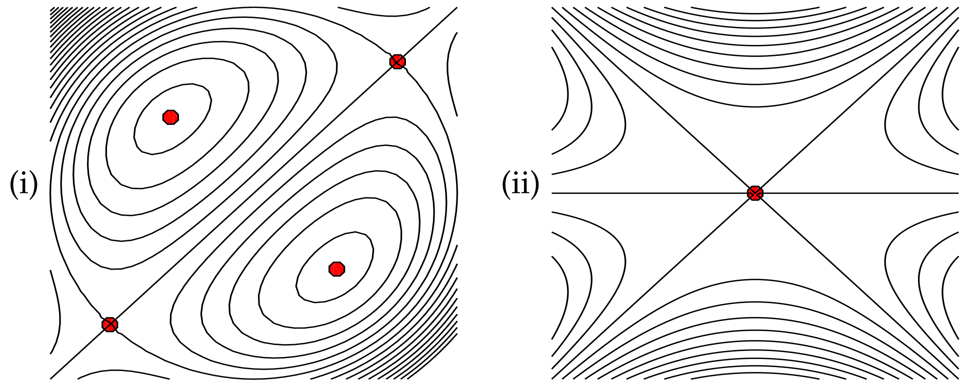
A-8:

critical point	type
$(0,0)$	local max
$(2,0)$	saddle point

A-9: (a)

critical point	type
$(\frac{3}{2}, -\frac{1}{4})$	local min
$(-1, 1)$	saddle point

(b)



A-10:

critical point	type
$(\frac{3}{2}, -\frac{1}{4})$	local min
$(-1, 1)$	saddle point

A-11: (0,0) is a local max

(0,2) is a local min

(1,1) and (-1,1) are saddle points

A-12: $(0,0)$ is a saddle point and $\pm(1,1)$ are local mins

A-13: $(0,0)$ is a saddle point and $\pm(1,1)$ are local mins

A-14: $(0,\pm 1)$ are saddle points, $(\frac{1}{\sqrt{3}}, 0)$ is a local min and $(-\frac{1}{\sqrt{3}}, 0)$ is a local max

A-15: $(-1, \pm\sqrt{3})$ and $(2,0)$ are saddle points and $(0,0)$ is a local max.

A-16: *Case* $k < \frac{1}{2}$*:*

critical point	type
$(0,0)$	local max
$(0,2)$	saddle point

Case $k = \frac{1}{2}$*:*

critical point	type
$(0,0)$	local max
$(0,2)$	unknown

Case $k > \frac{1}{2}$*:*

critical point	type
$(0,0)$	local max
$(0,2)$	local min
$\left(\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$	saddle point
$\left(-\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$	saddle point

A-17: $m = \frac{nS_{xy}-S_xS_y}{nS_{x^2}-S_x^2}$ and $b = \frac{S_yS_{x^2}-S_xS_{xy}}{nS_{x^2}-S_x^2}$ where $S_y = \sum\limits_{i=1}^{n} y_i$, $S_{x^2} = \sum\limits_{i=1}^{n} x_i^2$ and $S_{xy} = \sum\limits_{i=1}^{n} x_iy_i$.

Answers to Exercises 2.4 — Jump to TABLE OF CONTENTS

A-1: false

A-2: The minimum height is zero at $(0,0,0)$. The derivatives z_x and z_y do not exist there. The maximum height is $\sqrt{2}$ at $(\pm 1, \pm 1, \sqrt{2})$. There z_x and z_y exist but are not zero — those points would not be the highest points if it were not for the restriction $|x|, |y| \leqslant 1$.

A-3: $\min = 0$ $\qquad$ $\max = \frac{2}{3\sqrt{3}} \approx 0.385$

A-4: (a)

critical point	type
$\left(0, \frac{2}{\sqrt{3}}\right)$	local max
$\left(0, -\frac{2}{\sqrt{3}}\right)$	local min
$(2,0)$	saddle point
$(-2,0)$	saddle point

(b) The maximum and minimum values of $h(x,y)$ in $x^2+y^2 \leqslant 1$ are 3 (at $(0,1)$) and -3 (at $(0,-1)$), respectively.

A-5: The minimum is -2 and the maximum is 6.

A-6: $6-2\sqrt{5}$

A-7: (a) (0,0) and (3,0) and (0,3) are saddle points
(1,1) is a local min

(b) The minimum is -1 at $(1,1)$ and the maximum is 80 at $(4,4)$.

A-8: (a) $(1,1)$ is a saddle point and $(2,4)$ is a local min

(b) The min and max are $\frac{19}{27}$ and 5, respectively.

A-9: (a) $(0,0)$, $(6,0)$, $(0,3)$ are saddle points and $(2,1)$ is a local min

(b) The maximum value is 0 and the minimum value is $4(4\sqrt{2}-6) \approx -1.37$.

A-10: The coldest temperture is -0.391 and the coldest point is $(0,2)$.

A-11: (a) $(0,-5)$ is a saddle point

(b) The smallest value of g is 0 at $(0,0)$ and the largest value is 21 at $(\pm 2\sqrt{3},-1)$.

A-12: $\frac{2500}{\sqrt{3}}$

A-13: The box has dimensions $(2V)^{1/3}\times(2V)^{1/3}\times 2^{-2/3}V^{1/3}$.

A-14: (a) The maximum and minimum values of $T(x,y)$ in $x^2+y^2\leqslant 4$ are 20 (at $(0,0)$) and 4 (at $(\pm 2,0)$), respectively.

(b) $(0,2)$

A-15: The minimum value is 0 on

$$\{\ (x,y,z)\ \big|\ x\geqslant 0,\ y\geqslant 0,\ z\geqslant 0,\ 2x+y+z=5,\ \text{at least one of } x,y,z \text{ zero}\ \}$$

The maximum value is 4 at $(1,2,1)$.

A-16: (a) $x=1, y=\frac{1}{2}, f\big(1,\frac{1}{2}\big)=6$ (b) local minimum

(c) As x or y tends to infinity (with the other at least zero), $2x+4y$ tends to $+\infty$. As (x,y) tends to any point on the first quadrant part of the x- and y–axes, $\frac{1}{xy}$ tends to $+\infty$. Hence as x or y tends to the boundary of the first quadrant (counting infinity as part of the boundary), $f(x,y)$ tends to $+\infty$. As a result $\big(1,\frac{1}{2}\big)$ is a global (and not just local) minimum for f in the first quadrant. Hence $f(x,y)\geqslant f\big(1,\frac{1}{2}\big)=6$ for all $x,y>0$.

A-17: If $a<\frac{1}{2}$, then the closest point is the origin. If $a\geqslant\frac{1}{2}$, then the closest points are the level curve where $z=a-\frac{1}{2}$.

A-18:

(a) The total profit is given by

$$\Pi(x,y)=(15x^{0.8}-x)+(80y^{0.6}-3y)$$

(b) The optimal production: $x=248,832$ leading to 51840 reams of A4 and $y=1,024$ leading to 640 reams of A3

(c) In this case, the optimal production is still 640 reams of A3

A-19:

(a) $\Pi_A(q_A)=-2q_A^2+120q_A-2q_Aq_P$; maximum profit when $q_A=30-\frac{1}{2}q_P$

(b) $\Pi_P(q_P)=-2q_P^2+120q_P-2q_Pq_A$; maximum profit when $q_P=30-\frac{1}{2}q_A$

(c) Their businesses are identical, so we predict they will sell the same amounts of lemonade.

(d) If Ayan and Pipe sell 20 pitchers they will maximize their respective profit functions.

(e) They would each make 800 dollars in profit.

(f) Their optimal joint profit will be $1,800$ dollars. But, they need to share this profit among the two of them. So if they collaborate, they will each earn 900 dollars. This is more than their individual optimal profit in the scenario where they are competing found in part (e) (we found this to be \$800). So it is better for them to collaborate!

(g) Collaborating sellers lead to higher prices and fewer goods, so it's better for consumers with the sellers compete

Answers to Exercises 2.5 — Jump to TABLE OF CONTENTS

A-1: (a) f does not have a maximum. It does have a minimum.

(b) The minima are at $\pm(1,1)$, where f takes the value 2.

A-2: One possible answer: $g(x,y) = y$, $f(x,y) = x^3 - x$.

A-3: The minimum value is $2^{\frac{1}{3}} + 2^{-\frac{2}{3}} = \frac{3}{2}\sqrt[3]{2} = \frac{3}{\sqrt[3]{4}}$ at $\left(\pm 2^{\frac{1}{6}}, 2^{-\frac{1}{3}}\right)$.

A-4: The maximum and minimum values of f are $\frac{1}{2\sqrt{2}}$ and $-\frac{1}{2\sqrt{2}}$, respectively.

A-5: min$= 1$, max$= \sqrt{2}$.

A-6: absolute min $\frac{13-8\sqrt{2}}{3}$, absolute max $\frac{5}{3}$

A-7: $(\pm 1, 1/2)$

A-8: Largest $\frac{\sqrt{5}}{10-2\sqrt{5}}$, smallest $\frac{-\sqrt{5}}{10-2\sqrt{5}}$

A-9: (a) (i)

$$\begin{aligned} 2x\,e^y &= \lambda(2x) \\ e^y\left(x^2 + y^2 + 2y\right) &= \lambda(2y) \\ x^2 + y^2 &= 100 \end{aligned}$$

(a) (ii) The warmest point is $(0,10)$ and the coolest point is $(0,-10)$.

(b) (i)

$$\begin{aligned} 2x\,e^y &= 0 \\ e^y\left(x^2 + y^2 + 2y\right) &= 0 \end{aligned}$$

(b) (ii) $(0,0)$ and $(0,-2)$

(c) $(0,0)$

A-10: Min 0; max $75 \cdot 2^{10/3}$

A-11: 4

A-12: $a = b = \sqrt{5}$

A-13: radius $= \sqrt{\frac{2}{3}}$ and height $= \frac{2}{\sqrt{3}}$.

A-14: $3 \times 6 \times 4$

A-15: See the solution.

Answers to Exercises 2.6 — Jump to TABLE OF CONTENTS

A-1: A: Marshallian; B: Hicksian

A-2: $$\frac{\partial x^m(p_x, p_y, I)}{\partial p_x} = \begin{cases} \frac{-I}{2(p_x - p_y)^2} & \text{if } p_x \geqslant 2p_y \\ \frac{-I}{p_x^2} & \text{if } p_x < p_y \end{cases}$$

$$\frac{\partial x^m(p_x, p_y, I)}{\partial p_y} = \begin{cases} \frac{I}{2(p_x - p_y)^2} & \text{if } p_x \geqslant 2p_y \\ 0 & \text{if } p_x < p_y \end{cases}$$

A-3: inferior

A-4:

(a) $c = \dfrac{5}{3}$.

(b) $f = \dfrac{25}{9}$.

(c) She will spend \$ 8.33 on food and \$ 1.67 on coffee.

A-5: Four shares of Inter de Milan and thirty two shares of La Spezia.

$$m = 4, \qquad s = 34$$

A-6: Laura's optimal consumption is $\frac{60}{7}$ units of cheese and $\frac{40}{7}$ units of strawberries.

A-7: Alessio will buy 15 packages for Keitu and 2 packages for Nefret.

A-8:

(a) 320 gm of popcorn and 280 ml of soda.

(b) (i) No, it would cost 22.1 dollars.

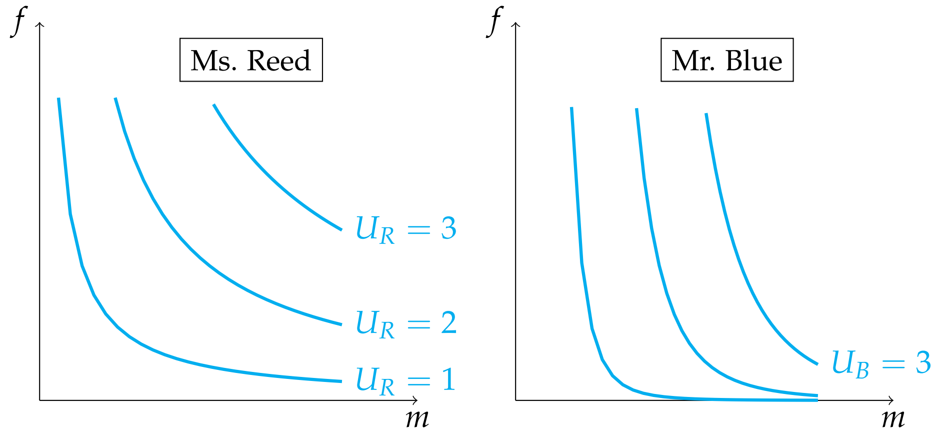
(ii) Yes! Her utility would be around 5.097 with the combo, and approximately 4.666 without it.

A-9:

(a) The budget constraint is given by

$$30m + 30f = I$$

(b) Mr. Reed prefers male officers to female officers while Ms. Reed does not prefer one to the other.



(c) We denote m and f that maximize U_B by m_B and f_B (to avoid confusion):

$$f_B = \frac{I}{300} \qquad m_B = \frac{9I}{300}$$

and m and f that maximize U_R by m_R and f_R:

$$f_R = \frac{I}{60} \qquad m_R = \frac{I}{60}$$

Mr. Blue will hire a higher proportion of male police officers than Ms. Reed as $m_R < m_B$ for any value of I.

(d) We distinguish the new points (m, f) that maximize U_B and U_R by (m_B, m_B) and (m_R, f_R), respectively.

$$f_B^* = \frac{I}{300} \qquad f_R^* = \frac{I}{60}$$
$$m_B^* = \frac{9I}{350} \qquad m_R^* = \frac{I}{70}$$

Because it is cheaper to hire female officers, both hire a higher proportion of female officers. However, Mr. Blue has such a strong bias that he still hires more male officers than female officers.

A-10:

(a) $c^*(p_c, p_f, I) = \dfrac{Ip_f - 25p_c^2}{p_f p_c}$.

(b) $f^*(p_c, p_f, I) = \left(\dfrac{10p_c}{2p_f}\right)^2$.

A-11: the optimal consumption is buying $\frac{5I}{7p_k}$ expansion packs for Keitu, and $\frac{2I}{7p_n}$ for Nefret.

A-12:

(a) $1 \leqslant c \leqslant 49$ and $k \geqslant 2$

(b) $k^*(p_k, p_c, I) = \dfrac{50p_c - I}{p_k} - 2$ and $c^*(p_k, p_c, I) = \dfrac{6p_k}{p_c} + \dfrac{2I}{p_c} - 50$.

(c) kraft dinner is an inferior good while chicken is a normal good.

A-13:

(a)

$$l^*(p_l, p_a, D) = \frac{16p_aD}{p_l\,(16p_a + 9p_l))} \qquad a^*(p_l, p_a, D) = \frac{9p_lD}{p_a\,(16p_a + 9p_l)}$$

(b) Both of them are normal goods.

(c)

$$\frac{\partial}{\partial p_l} l^*(p_l, p_a, D) = \frac{-(16p_aD)(16p_a + 18p_l)}{(16p_ap_l + 9p_l^2)^2}$$

If the price for Lomachenko's tickets p_l decreases, then the demand for Lomachenko's tickets l^* increases.

(d)

$$a^h(p_l, p_a, U) = \left(\frac{3U^2p_l}{16p_a + 9p_l}\right)^2 \qquad l^h(p_l, p_a, U) = \left(\frac{4U^2p_a}{16p_a + 9p_l}\right)^2$$

(e)

$$\frac{\partial a^h}{\partial p_a} = \frac{-96U^2p_l}{(16p_a + 9p_l)^3}$$

As the price for Anthony Joshua's tickets p_a increases, Anthony's Hicksian demand function a^h decreases.

Answers to Exercises 3.1 — Jump to TABLE OF CONTENTS

A-1: The area is between 1.5 and 2.5 square units.

A-2: The shaded area is between 2.75 and 4.25 square units. (Other estimates are possible, but this is a reasonable estimate, using methods from this chapter.)

A-3: The area under the curve is a number in the interval $\left(\frac{3}{8}\left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right], \frac{3}{8}\left[1 + \frac{1}{\sqrt{2}}\right]\right)$.

A-4: left

A-5: Many answers are possible. One example is $f(x) = \sin x$, $[a, b] = [0, \pi]$, $n = 1$. Another example is $f(x) = \sin x$, $[a, b] = [0, 5\pi]$, $n = 5$.

A-6: Some of the possible answers are given, but more exist.

(a) $\displaystyle\sum_{i=3}^{7} i \quad ; \quad \sum_{i=1}^{5} (i + 2)$

(b) $\sum_{i=3}^{7} 2i \quad ; \quad \sum_{i=1}^{5} (2i+4)$

(c) $\sum_{i=3}^{7} (2i+1) \quad ; \quad \sum_{i=1}^{5} (2i+5)$

(d) $\sum_{i=1}^{8} (2i-1) \quad ; \quad \sum_{i=0}^{7} (2i+1)$

A-7: Some answers are below, but others are possible.

(a) $\sum_{i=1}^{4} \frac{1}{3^i} \quad ; \quad \sum_{i=1}^{4} \left(\frac{1}{3}\right)^i$

(b) $\sum_{i=1}^{4} \frac{2}{3^i} \quad ; \quad \sum_{i=1}^{4} 2\left(\frac{1}{3}\right)^i$

(c) $\sum_{i=1}^{4} (-1)^i \frac{2}{3^i} \quad ; \quad \sum_{i=1}^{4} \frac{2}{(-3)^i}$

(d) $\sum_{i=1}^{4} (-1)^{i+1} \frac{2}{3^i} \quad ; \quad \sum_{i=1}^{4} -\frac{2}{(-3)^i}$

A-8:

(a) $\sum_{i=1}^{5} \frac{2i-1}{3^i}$

(b) $\sum_{i=1}^{5} \frac{1}{3^i+2}$

(c) $\sum_{i=1}^{7} i \cdot 10^{4-i} \quad ; \quad \sum_{i=1}^{7} \frac{i}{10^{i-4}}$

A-9:

(a) $\frac{5}{2}\left[1-\left(\frac{3}{5}\right)^{101}\right]$

(b) $\frac{5}{2}\left(\frac{3}{5}\right)^{50}\left[1-\left(\frac{3}{5}\right)^{51}\right]$

(c) 270

(d) $\frac{1-\left(\frac{1}{e}\right)^b}{e-1} + \frac{e}{4}\left[b(b+1)\right]^2$

A-10:

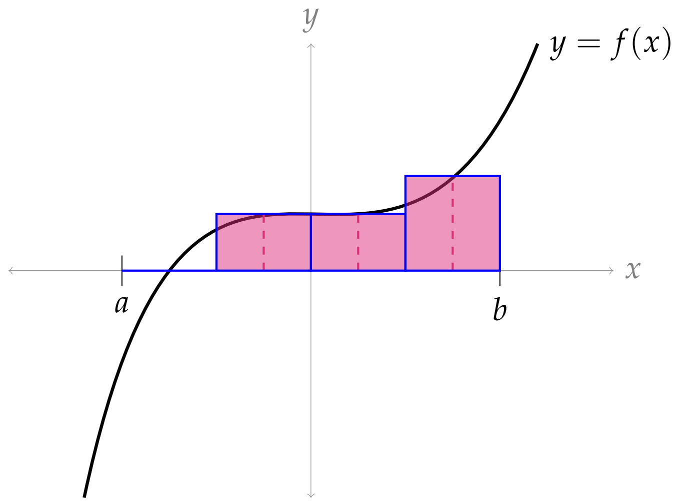
(a) $50 \cdot 51 = 2550$

(b) $\left[\frac{1}{2}(95)(96)\right]^2 - \left[\frac{1}{2}(4)(5)\right]^2$

(c) -1

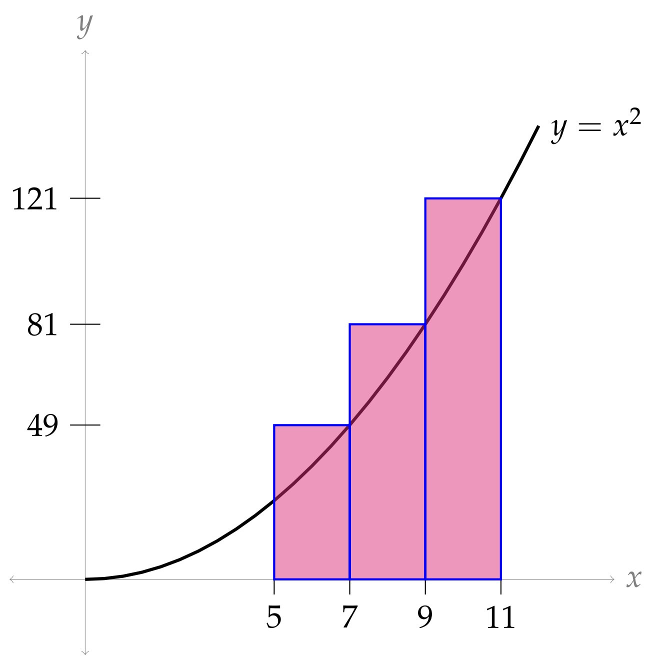
(d) -10

A-11:

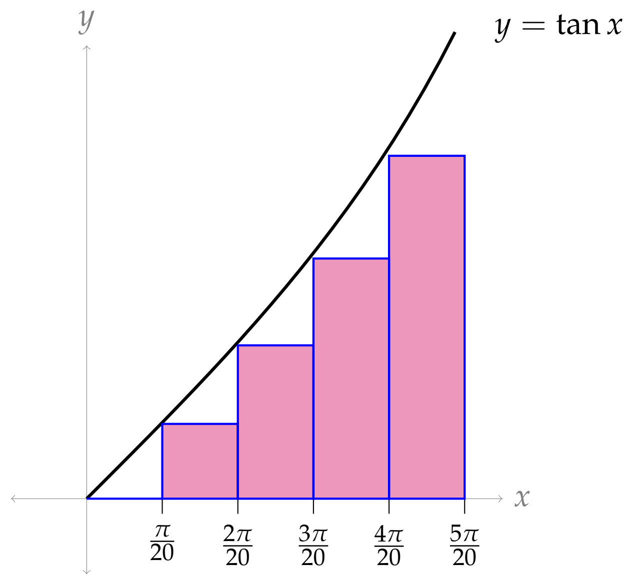


A-12: $n = 4$, $a = 2$, and $b = 6$

A-13: One answer is below, but other interpretations exist.



A-14: Many interpretations are possible–see the solution to Question 13 for a more thorough discussion–but the most obvious is given below.



A-15: Three answers are possible. It is a midpoint Riemann sum for f on the interval $[1,5]$ with $n=4$. It is also a left Riemann sum for f on the interval $[1.5,5.5]$ with $n=4$. It is also a right Riemann sum for f on the interval $[0.5,4.5]$ with $n=4$.

A-16: $\dfrac{25}{2}$

A-17: $\dfrac{21}{2}$

A-18: $\sum\limits_{i=1}^{50}\left(5+\left(i-1/2\right)\frac{1}{5}\right)^8\frac{1}{5}$

A-19: 54

A-20: $\displaystyle\int_{-1}^{7} f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{8i}{n}\right)\frac{8}{n}$

A-21: $f(x)=\sin^2(2+x)$ and $b=4$

A-22: $f(x)=x\sqrt{1-x^2}$

A-23: $\int_0^3 e^{-x/3}\cos(x)\,\mathrm{d}x$

A-24: $\displaystyle\int_0^1 xe^x\,\mathrm{d}x$

A-25: Possible answers include:

$$\int\limits_0^2 e^{-1-x}\,\mathrm{d}x,\quad \int\limits_1^3 e^{-x}\,\mathrm{d}x,\quad 2\int_{1/2}^{3/2} e^{-2x}\,\mathrm{d}x,\quad \text{and}\quad 2\int\limits_0^1 e^{-1-2x}\,\mathrm{d}x.$$

A-26: $\dfrac{r^{3n+3}-1}{r-1}$

A-27: $r^5\left(\dfrac{r^{96}-1}{r-1}\right)$

A-28: 5

A-29: 16

A-30: $\dfrac{b^2-a^2}{2}$

A-31: $\dfrac{b^2-a^2}{2}$

A-32: 4π

A-33: $\displaystyle\int_0^3 f(x)\ \mathrm{d}x = 2.5$

A-34: 53 m

A-35: true

A-36: 3200 km

A-37: (a) There are many possible answers. Two are $\int_{-2}^{0}\sqrt{4-x^2}\ \mathrm{d}x$ and $\int_0^2\sqrt{4-(-2+x)^2}\ \mathrm{d}x$. (b) π

A-38: (a) 30 (b) $41\frac{1}{4}$

A-39: $\dfrac{56}{3}$

A-40: 6

A-41: 12

A-42: $f(x) = \dfrac{3}{10}\left(\dfrac{x}{5}+8\right)^2 \sin\left(\dfrac{2x}{5}+2\right)$

A-43: $\dfrac{1}{\log 2}$

A-44: (a) $\dfrac{1}{\log 10}\left(10^b - 10^a\right)$

(b) $\dfrac{1}{\log c}\left(c^b - c^a\right)$; yes, it agrees.

A-45: $\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}$

A-46:

(a) $[f(b)-f(a)]\cdot\dfrac{b-a}{n}$

(b) Choose n to be an integer that is greater than or equal to $100\,[f(b)-f(a)]\,(b-a)$.

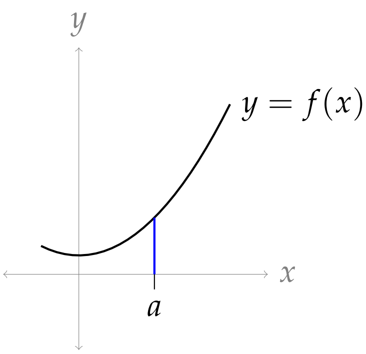
A-47: true (but note, for a non-linear function, it is possible that the midpoint Riemann sum is *not* the average of the other two)

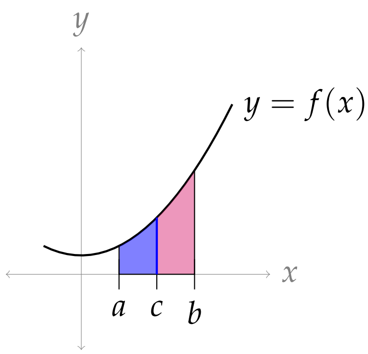
A-48:

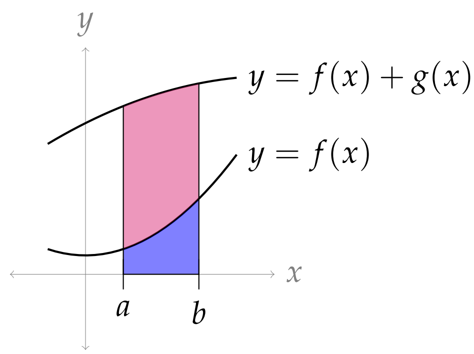
1. There are 150 rounds in the blanket.
2. There are $4 \cdot 150^2$ stitches in the blanket.
3. The crocheter is halfway finished some time during the 107th round.

Answers to Exercises 3.2 — Jump to TABLE OF CONTENTS

A-1: Possible drawings:







A-2: $\sin b - \sin a$

A-3: (a) False. For example, the function

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geqslant 0 \end{cases}$$

provides a counterexample.

(b) False. For example, the function $f(x) = x$ provides a counterexample.

(c) False. For example, the functions

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} \\ 1 & \text{for } x \geqslant \frac{1}{2} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0 & \text{for } x \geqslant \frac{1}{2} \\ 1 & \text{for } x < \frac{1}{2} \end{cases}$$

provide a counterexample.

A-4: (a) $-\dfrac{1}{20}$ (b) positive (c) negative (d) positive

A-5: $A_1 - A_2 + A_3 = A_4$

A-6: -21

A-7: -6

A-8: 20

A-9:

(a) $\frac{\pi}{4} - \frac{1}{2}\arccos(-a) - \frac{1}{2}a\sqrt{1-a^2}$

(b) $\frac{1}{2}\arccos(a) - \frac{1}{2}a\sqrt{1-a^2}$

A-10: 5

A-11: $\int_0^1 e^{x^2}\,\mathrm{d}x \leqslant e-1$

A-12: 0

A-13: 5

A-14: $20+2\pi$

A-15: $\sin^2 x \leqslant x\sin x$

A-16: 0

A-17: 0

A-18: 0

A-19: (a) $y = \dfrac{1}{b}\sqrt{1-(ax)^2}$ (b) $\dfrac{a}{b}\displaystyle\int_{-\frac{1}{a}}^{\frac{1}{a}}\sqrt{\frac{1}{a^2}-x^2}\,\mathrm{d}x$ (c) $\dfrac{\pi}{ab}$

A-20:

$\times$	even	odd
even	even	odd
odd	odd	even

A-21: $f(0)=0$; $g(0)$ can be any real number

A-22: $f(x)=0$ for every x

A-23: The derivative of an even function is odd, and the derivative of an odd function is even.

Answers to Exercises 3.3 — Jump to TABLE OF CONTENTS

A-1: $e^2 - e^{-2}$

A-2: $F(x) = \dfrac{x^4}{4} + \dfrac{1}{2}\cos 2x + \dfrac{1}{2}$.

A-3: (a) True (b) False (c) False, unless $\int_a^b f(x)\,\mathrm{d}x = \int_a^b xf(x)\,\mathrm{d}x = 0$.

A-4: false

A-5: false

A-6: $\sin(x^2)$

A-7: $\sqrt[3]{e}$

A-8: For any constant C, $F(x)+C$ is an antiderivative of $f(x)$. So, for example, $F(x)$ and $F(x)+1$ are both antiderivatives of $f(x)$.

A-9:

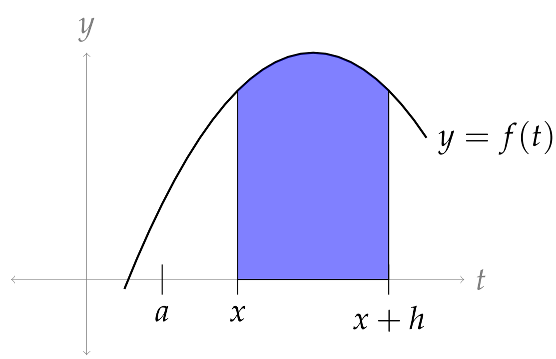
(a) We differentiate with respect to a. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$. To differentiate $\frac{1}{2}a\sqrt{1-a^2}$, we use the product and chain rules.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\left\{\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}\right\} &= 0 - \frac{1}{2}\cdot\frac{-1}{\sqrt{1-a^2}} + \left(\frac{1}{2}a\right)\cdot\frac{-2a}{2\sqrt{1-a^2}} + \frac{1}{2}\sqrt{1-a^2} \\
&= \frac{1}{2\sqrt{1-a^2}} - \frac{a^2}{2\sqrt{1-a^2}} + \frac{1-a^2}{2\sqrt{1-a^2}} \\
&= \frac{1-a^2+1-a^2}{2\sqrt{1-a^2}} \\
&= \frac{2(1-a^2)}{2\sqrt{1-a^2}} \\
&= \sqrt{1-a^2}
\end{aligned}$$

(b) $F(x) = \frac{5\pi}{4} - \frac{1}{2}\arccos(x) + \frac{1}{2}x\sqrt{1-x^2}$

A-10: (a) 0 (b),(c) The FTC does not apply, because the integrand is not continuous over the interval of integration.

A-11:



A-12: (a) zero (b) increasing when $0 < x < 1$ and $3 < x < 4$; decreasing when $1 < x < 3$

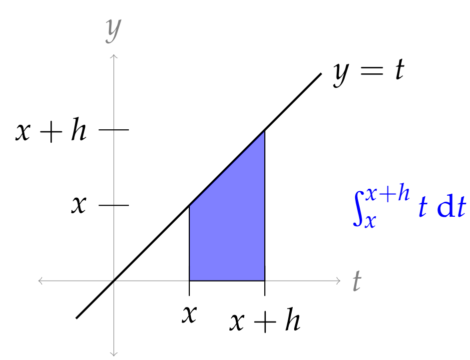
A-13: (a) zero (b) $G(x)$ is increasing when $1 < x < 3$, and it is decreasing when $0 < x < 1$ and when $3 < x < 4$.

A-14: Using the definition of the derivative,

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\frac{\int_a^{x+h} t\,\mathrm{d}t - \int_a^x t\,\mathrm{d}t}{h} \\ &= \lim_{h\to 0}\frac{\int_x^{x+h} t\,\mathrm{d}t}{h} \end{aligned}$$

The numerator describes the area of a trapezoid with base h and heights x and $x+h$.

$$\begin{aligned} &= \lim_{h\to 0}\frac{\frac{1}{2}h(x+x+h)}{h} \\ &= \lim_{h\to 0}\left(x+\frac{1}{2}h\right) \\ &= x \end{aligned}$$



So, $F'(x) = x$.

A-15: $f(t) = 0$

A-16: $\int \ln(ax)\,\mathrm{d}x = x\ln(ax) - x + C$, where a is a given constant, and C is any constant.

A-17: $\int x^3e^x\,\mathrm{d}x = e^x\left(x^3-3x^2+6x-6\right)+C$

A-18: $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2+a^2}\right| + C$ when a is a given constant. As usual, C is an arbitrary constant.

A-19: $\displaystyle\int \frac{x}{\sqrt{x(a+x)}}\,\mathrm{d}x = \sqrt{x(a+x)} - a\ln\left(\sqrt{x}+\sqrt{a+x}\right)+C$

A-20: $5-\cos 2$

A-21: 2

A-22: $\dfrac{1}{5}\arctan(5x)+C$

A-23: $\arcsin\left(\dfrac{x}{\sqrt{2}}\right)+C$

A-24: $\tan x - x + C$

A-25: $-\dfrac{3}{4}\cos(2x) + C$, or equivalently, $\dfrac{3}{2}\sin^2 x + C$

A-26: $\dfrac{1}{2}x + \dfrac{1}{4}\sin(2x) + C$

A-27: $F'\left(\frac{\pi}{2}\right) = \ln(3) \qquad G'\left(\frac{\pi}{2}\right) = -\ln(3)$

A-28: $f(x)$ is increasing when $-\infty < x < 1$ and when $2 < x < \infty$.

A-29: $F'(x) = -\dfrac{\sin x}{\cos^3 x + 6}$

A-30: $4x^3 e^{(1+x^4)^2}$

A-31: $\left(\sin^6 x + 8\right)\cos x$

A-32: $F'(1) = 3e^{-1}$

A-33: $\dfrac{\sin u}{1 + \cos^3 u}$

A-34: $f(x) = 2x$

A-35: $f(4) = 4\pi$

A-36: (a) $(2x+1)e^{-x^2}$ (b) $x = -1/2$

A-37: $e^{\sin x} - e^{\sin(x^4 - x^3)}\left(4x^3 - 3x^2\right)$

A-38: $-2x\cos\left(e^{-x^2}\right) - 5x^4\cos\left(e^{x^5}\right)$

A-39: $e^x\sqrt{\sin(e^x)} - \sqrt{\sin(x)}$

A-40: 14

A-41: $\dfrac{5}{2}$

A-42: 45 m

A-43: $f'(x) = (2-2x)\ln\left(1 + e^{2x-x^2}\right)$ and $f(x)$ achieves its absolute maximum at $x = 1$, because $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$.

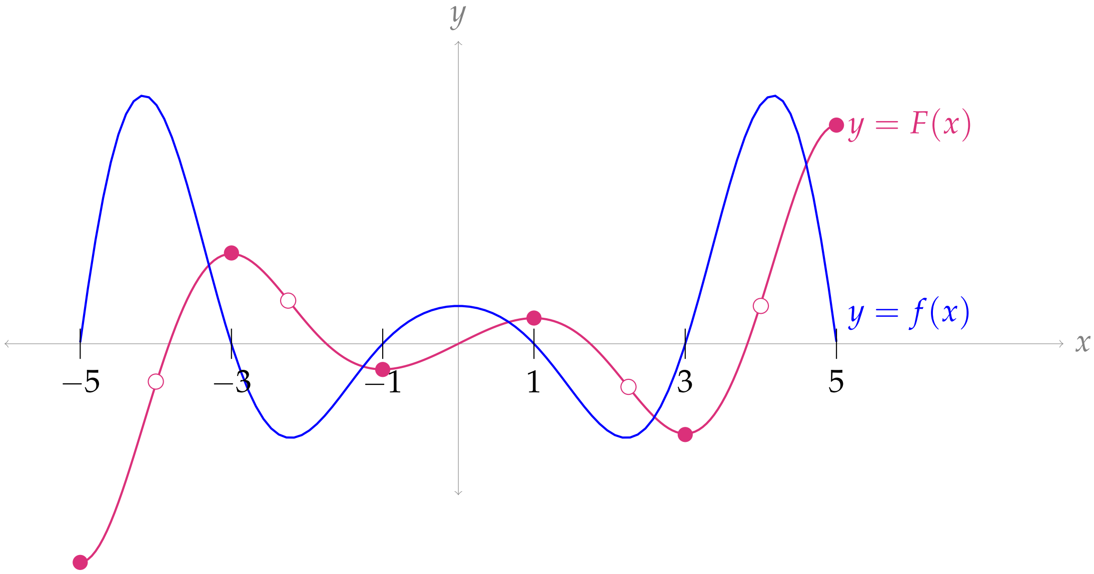
A-44: The minimum is $\int_0^{-1} \frac{\mathrm{d}t}{1+t^4}$. As x runs from $-\infty$ to ∞, the function $f(x) = \int_0^{x^2-2x} \frac{\mathrm{d}t}{1+t^4}$ decreases until x reaches 1 and then increases all $x > 1$. So the minimum is achieved for $x = 1$. At $x = 1$, $x^2 - 2x = -1$.

A-45: F achieves its maximum value at $x = \pi$.

A-46: 2

A-47: $\ln 2$

A-48: In the sketch below, open dots denote inflection points, and closed dots denote extrema.



A-49: (a) $3x^2 \int_0^{x^3+1} e^{t^3} \mathrm{d}t + 3x^5 e^{(x^3+1)^3}$ (b) $y = -3(x+1)$

A-50: Both students.

A-51: (a) $27(1 - \cos 3)$ (b) $x^3 \sin(x) + 3x^2[1 - \cos(x)]$

A-52: If $f(x) = 0$ for all x, then $F(x)$ is even and possibly also odd.

If $f(x) \neq 0$ for some x, then $F(x)$ is not even. It might be odd, and it might be neither even nor odd.

(Perhaps surprisingly, every antiderivative of an odd function is even.)

A-53:

(a) CS: $10\ln(q_e + 1) - p_e \cdot q_e$

(b) PS: $p_e \cdot q_e - e^{q_e} + q_e + 1$

(c) TS: $10\ln(q_e + 1) - e^{q_e} + q_e + 1$

A-54: (a) $1 - 2\int_0^2 L(x)\mathrm{d}x$ (b) 0 (c) $\frac{11}{21} \approx 0.52$

A-55:

(a) $TC = \ln(q+1) + \frac{1}{2}q^2 + 2q + 1000$ and $TC(2000) = \ln(2001) + 2{,}005{,}000$

(b) $TC = 40q - 5q^2 + \frac{e^q}{10} + 49{,}999.90$ and $TC(10) = 50{,}379.90 + \frac{e^{10}}{10}$

A-56:

(a) $\text{TR} = \sin q + \frac{q^2}{10} + 2q$ and $P = \frac{\sin q}{q} + \frac{q}{10} + 2$

(b) $\text{TR} = \frac{e^q - 1}{1000} + \sqrt{q}$ and $P = \frac{e^q + 1}{1000q} + \frac{1}{\sqrt{q}}$

Answers to Exercises 3.4 — Jump to TABLE OF CONTENTS

A-1: (a) true (b) false

A-2: The reasoning is not sound: when we do a substitution, we need to take care of the differential (dx). Remember the method of substitution comes from the chain rule: there should be a function *and its derivative*. Here's the way to do it:

Problem: Evaluate $\int (2x+1)^2 dx$.

Work: We use the substitution $u = 2x+1$. Then $du = 2dx$, so $dx = \frac{1}{2}du$:

$$\begin{aligned}\int (2x+1)^2 dx &= \int u^2 \cdot \frac{1}{2}\, du \\ &= \frac{1}{6}u^3 + C \\ &= \frac{1}{6}(2x+1)^3 + C\end{aligned}$$

A-3: The problem is with the limits of integration, as in Question 1. Here's how it ought to go:

Problem: Evaluate $\int_1^{\pi} \frac{\cos(\ln t)}{t} dt$.

Work: We use the substitution $u = \ln t$, so $du = \frac{1}{t}dt$. When $t = 1$, we have $u = \ln 1 = 0$ and when $t = \pi$, we have $u = \ln(\pi)$. Then:

$$\begin{aligned}\int_1^{\pi} \frac{\cos(\ln t)}{t} dt &= \int_{\ln 1}^{\ln(\pi)} \cos(u) du \\ &= \int_0^{\ln(\pi)} \cos(u) du \\ &= \sin(\ln(\pi)) - \sin(0) = \sin(\ln(\pi)).\end{aligned}$$

A-4: This one is OK.

A-5: $\int_0^1 \frac{f(u)}{\sqrt{1-u^2}}\, du$

A-6: some constant C

A-7: $\frac{1}{2}\big(\sin(e) - \sin(1)\big)$

A-8: $\frac{1}{3}$

A-9: $-\frac{1}{300(x^3+1)^{100}} + C$

A-10: $\ln 4$

A-11: $\ln 2$

A-12: $\dfrac{4}{3}$

A-13: $e^6 - 1$

A-14: $\dfrac{1}{3}(4-x^2)^{3/2} + C$

A-15: $e^{\sqrt{\ln x}} + C$

A-16: 0

A-17: $\dfrac{1}{2}[\cos 1 - \cos 2] \approx 0.478$

A-18: $\dfrac{1}{2} - \dfrac{1}{2}\ln 2$

A-19: $\frac{1}{2}\tan^2\theta - \ln|\sec\theta| + C$

A-20: $\arctan(e^x) + C$

A-21: $\dfrac{\pi}{4} - \dfrac{2}{3}$

A-22: $-\frac{1}{2}\left(\ln(\cos x)\right)^2 + C$

A-23: $\frac{1}{2}\sin(1)$

A-24: $\dfrac{1}{3}[2\sqrt{2}-1] \approx 0.609$

A-25: Using the definition of a definite integral with right Riemann sums:

$$\begin{aligned}
\int_a^b 2f(2x)\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x \cdot 2f(2(a+i\Delta x)) && \Delta x = \frac{b-a}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\cdot 2f\left(2\left(a+i\left(\frac{b-a}{n}\right)\right)\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{2b-2a}{n}\right)\cdot f\left(2a+i\left(\frac{2b-2a}{n}\right)\right)\\
\int_{2a}^{2b} f(x)\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x \cdot f(2a+i\Delta x) && \Delta x = \frac{2b-2a}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{2b-2a}{n}\right)\cdot f\left(2a+i\left(\frac{2b-2a}{n}\right)\right)
\end{aligned}$$

Since the Riemann sums are exactly the same,

$$\int_a^b 2f(2x)\mathrm{d}x = \int_{2a}^{2b} f(x)\mathrm{d}x$$

A-26:

(a) $\mathrm{TC} = 2\sqrt{2q^3 - 80q} + 2000$

(b) $\mathrm{TR} = \frac{1}{3}\left(q^2+1\right)^{3/2} - \left(q^2+1\right)^{1/2} + \frac{2}{3}$

(c) $\left(\frac{1}{3}\left(q^2+1\right)^{3/2} - \left(q^2+1\right)^{1/2} + \frac{2}{3}\right) - \left(2\sqrt{2q^3-80q}+2000\right)$

(d) For $0 \leqslant q \leqslant \sqrt{40}$, q is a number that makes sense as a quantity of production (i.e. q isn't negative), but it isn't in the domain of our cost functions.

Answers to Exercises 3.5 — Jump to TABLE OF CONTENTS

A-1: chain; product

A-2: The part chosen as u will be differentiated. The part chosen as $\mathrm{d}v$ will be antidifferentiated.

A-3: $\displaystyle\int \frac{f'(x)}{g(x)}\,\mathrm{d}x = \frac{f(x)}{g(x)} + \int \frac{f(x)g'(x)}{g^2(x)}\mathrm{d}x + C$

A-4: All the antiderivatives differ only by a constant, so we can write them all as $v(x)+C$ for some C. Then, using the formula for integration by parts,

$$\begin{aligned}\int u(x)\cdot v'(x)\mathrm{d}x &= \underbrace{u(x)}_{u}\ \underbrace{[v(x)+C]}_{v} - \int \underbrace{[v(x)+C]}_{v}\ \underbrace{u'(x)\mathrm{d}x}_{\mathrm{d}u}\\ &= u(x)v(x) + Cu(x) - \int v(x)u'(x)\mathrm{d}x - \int Cu'(x)\mathrm{d}x\\ &= u(x)v(x) + Cu(x) - \int v(x)u'(x)\mathrm{d}x - Cu(x) + D\\ &= u(x)v(x) - \int v(x)u'(x)\mathrm{d}x + D\end{aligned}$$

where D is any constant.

Since the terms with C cancel out, it didn't matter what we chose for C–all choices end up the same.

A-5: Suppose we choose $\mathrm{d}v = f(x)\mathrm{d}x$, $u = 1$. Then $v = \displaystyle\int f(x)\mathrm{d}x$, and $\mathrm{d}u = \mathrm{d}x$. So, our integral becomes:

$$\int \underbrace{(1)}_{u}\ \underbrace{f(x)\mathrm{d}x}_{\mathrm{d}v} = \underbrace{(1)}_{u}\ \underbrace{\int f(x)\mathrm{d}x}_{v} - \int \underbrace{\left(\int f(x)\mathrm{d}x\right)}_{v}\ \underbrace{\mathrm{d}x}_{\mathrm{d}u}$$

In order to figure out the first product (and the second integrand), you need to know the antiderivative of $f(x)$–but that's exactly what you're trying to figure out!

A-6: $\dfrac{x^2\ln x}{2} - \dfrac{x^2}{4} + C$

A-7: $-\dfrac{\ln x}{6x^6} - \dfrac{1}{36x^6} + C$

A-8: π

A-9: $\dfrac{\pi}{2} - 1$

A-10: $e^x \left(x^3 - 3x^2 + 6x - 6\right) + C$

A-11: $\dfrac{x^2}{2} \ln^3 x - \dfrac{3x^2}{4} \ln^2 x + \dfrac{3x^2}{4} \ln x - \dfrac{3x^2}{8} + C$

A-12: $(2 - x^2) \cos x + 2x \sin x + C$

A-13: $\left(t^3 - \frac{5}{2}t^2 + 6t\right) \ln t - \frac{1}{3}t^3 + \frac{5}{4}t^2 - 6t + C$

A-14: $e^{\sqrt{s}}\,(2s - 4\sqrt{s} + 4) + C$

A-15: $x \ln^2 x - 2x \ln x + 2x + C$

A-16: $e^{x^2+1} + C$

A-17: $y \arccos y - \sqrt{1 - y^2} + C$

A-18: $2y^2 \arctan(2y) - y + \frac{1}{2} \arctan(2y) + C$

A-19: $\dfrac{x^3}{3} \arctan x - \dfrac{1}{6}(1 + x^2) + \dfrac{1}{6} \ln(1 + x^2) + C$

A-20: $\dfrac{2}{17} e^{x/2} \cos(2x) + \dfrac{8}{17} e^{x/2} \sin(2x) + C$

A-21: $\dfrac{x}{2} \big[\sin(\ln x) - \cos(\ln x)\big] + C$

A-22: $\dfrac{2^x}{\ln 2} \left(x - \dfrac{1}{\ln 2}\right) + C$

A-23: $2e^{\cos x}[1 - \cos x] + C$

A-24: $\displaystyle\int \frac{xe^{-x}}{(1-x)^2} \mathrm{d}x = \frac{xe^{-x}}{1-x} + e^{-x} + C = \frac{e^{-x}}{1-x} + C$

A-25: (a) We integrate by parts with $u = \sin^{n-1} x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$, so that $\mathrm{d}u = (n-1) \sin^{n-2} x \cos x \mathrm{d}x$ and $v = -\cos x$.

$$\int \sin^n x\,\mathrm{d}x = \underbrace{-\sin^{n-1} x\ \cos x}_{uv} + \underbrace{(n-1) \int \cos^2 x\ \sin^{n-2} x\,\mathrm{d}x}_{-\int v \mathrm{d}u}$$

Using the identity $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned} &= -\sin^{n-1} x\ \cos x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x\,\mathrm{d}x \\ &= -\sin^{n-1} x\ \cos x + (n-1) \int \sin^{n-2} x\,\mathrm{d}x - (n-1) \int \sin^n x\,\mathrm{d}x \end{aligned}$$

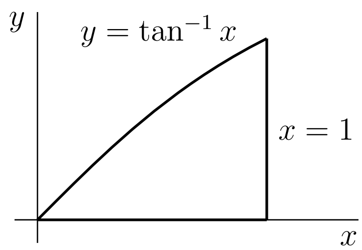
Moving the last term on the right hand side to the left hand side gives

$$n \int \sin^n x \, dx = -\sin^{n-1} x \ \cos x + (n-1) \int \sin^{n-2} x \, dx$$

Dividing across by n gives the desired reduction formula.

(b) $\frac{35}{256}\pi \approx 0.4295$

A-26: Area: $\frac{\pi}{4} - \frac{\ln 2}{2}$



A-27: $\pi \left(\frac{17e^{18} - 4373}{36} \right)$

A-28: 12

A-29: $\frac{2}{e}$

A-30: $10 - \ln 11$

A-31: (a) $\text{TC} = \left(\frac{q}{10} - 1\right) e^{q/10} - \frac{1}{2}q^2 + 30q + 1001$ (b) \$125.10

Answers to Exercises 3.6 — Jump to TABLE OF CONTENTS

A-1: (e)

A-2: $\frac{1}{n} \sec^n x + C$

A-3: We divide both sides by $\cos^2 x$, and simplify.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\frac{\sin^2 x + \cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
\frac{\sin^2 x}{\cos^2 x} + 1 &= \sec^2 x \\
\tan^2 x + 1 &= \sec^2 x
\end{aligned}$$

A-4: $\sin x - \frac{\sin^3 x}{3} + C$

A-5: $\frac{\pi}{2}$

A-6: $\dfrac{\sin^{37} t}{37} - \dfrac{\sin^{39} t}{39} + C$

A-7: $\dfrac{1}{3\cos^3 x} - \dfrac{1}{\cos x} + C$

A-8: $\dfrac{\pi}{8} - \dfrac{9\sqrt{3}}{64}$

A-9: $-\cos x + \dfrac{2}{3}\cos^3 x - \dfrac{1}{5}\cos^5 x + C$

A-10: $\dfrac{1}{2.2}\sin^{2.2} x + C$

A-11: $\dfrac{1}{2}\tan^2 x + C$, or equivalently, $\dfrac{1}{2}\sec^2 + C$

A-12: $\dfrac{1}{7}\sec^7 x - \dfrac{1}{5}\sec^5 x + C$

A-13: $\dfrac{\tan^{49} x}{49} + \dfrac{\tan^{47} x}{47} + C$

A-14: $\dfrac{1}{3.5}\sec^{3.5} x - \dfrac{1}{1.5}\sec^{1.5} x + C$

A-15: $\dfrac{1}{4}\sec^4 x - \dfrac{1}{2}\sec^2 x + C$

A-16: $\dfrac{1}{5}\tan^5 x + C$

A-17: $\dfrac{1}{1.3}\sec^{1.3} x + \dfrac{1}{0.7}\cos^{0.7} x + C$

A-18: $= \dfrac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C$

A-19: $\dfrac{41}{45\sqrt{3}} - \dfrac{\pi}{6}$

A-20: $\dfrac{1}{11} + \dfrac{1}{9}$

A-21: $2\sqrt{\sec x} + C$

A-22: $\tan^{e+1}\theta\left(\dfrac{\tan^6\theta}{7+e} + \dfrac{3\tan^4\theta}{5+e} + \dfrac{3\tan^2\theta}{3+e} + \dfrac{1}{1+e}\right) + C$

A-23: (a) Using the trig identity $\tan^2 x = \sec^2 x - 1$ and the substitution $y = \tan x$,

$\mathrm{d}y = \sec^2 x \, \mathrm{d}x$,

$$\int \tan^n x \, \mathrm{d}x = \int \tan^{n-2} x \, \tan^2 x \, \mathrm{d}x \qquad = \int \tan^{n-2} x \, \sec^2 x \, \mathrm{d}x - \int \tan^{n-2} x \, \mathrm{d}x$$
$$= \int y^{n-2} \, \mathrm{d}y - \int \tan^{n-2} x \, \mathrm{d}x = \frac{y^{n-1}}{n-1} - \int \tan^{n-2} x \, \mathrm{d}x$$
$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, \mathrm{d}x$$

(b) $\dfrac{13}{15} - \dfrac{\pi}{4} \approx 0.0813$

A-24: $\dfrac{1}{2\cos^2 x} + 2|\cos x| - \dfrac{1}{2}\cos^2 x + C$

A-25: $\tan\theta + C$

A-26: $|\sin x| + C$

A-27: $\dfrac{1}{2}\sin^2(e^x) + C$

A-28: $(\sin^2 x + 2)\cos(\cos x) + 2\cos x \sin(\cos x) + C$

A-29: $\dfrac{x}{2}\sin^2 x - \dfrac{x}{4} + \dfrac{1}{4}\sin x \cos x + C$

Answers to Exercises 3.7 — Jump to TABLE OF CONTENTS

A-1: (a) $x = \dfrac{4}{3}\sec\theta$ (b) $x = \dfrac{1}{2}\sin\theta$ (c) $x = 5\tan\theta$

A-2: (a) $x - 2 = \sqrt{3}\sec u$ (b) $x - 1 = \sqrt{5}\sin u$ (c) $\left(2x + \dfrac{3}{2}\right) = \dfrac{\sqrt{31}}{2}\tan u$
(d) $x - \dfrac{1}{2} = \dfrac{1}{2}\sec u$

A-3: (a) $\dfrac{\sqrt{399}}{20}$ (b) $\dfrac{5\sqrt{2}}{7}$ (c) $\dfrac{\sqrt{x-5}}{2}$

A-4: (a) $\dfrac{\sqrt{4-x^2}}{2}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{\sqrt{1-x}}$

A-5: $\dfrac{1}{4} \cdot \dfrac{x}{\sqrt{x^2+4}} + C$

A-6: $\dfrac{1}{2\sqrt{5}}$

A-7: $\dfrac{\pi}{6}$

A-8: $\ln\left|\sqrt{1 + \dfrac{x^2}{25}} + \dfrac{x}{5}\right| + C$

A-9: $\dfrac{1}{2}\sqrt{2x^2+4x}+C$

A-10: $-\dfrac{1}{16}\dfrac{\sqrt{x^2+16}}{x}+C$

A-11: $\dfrac{\sqrt{x^2-9}}{9x}+C$

A-12: (a) We'll use the trig identity $\cos 2\theta = 2\cos^2\theta - 1$. It implies that

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2} \implies \cos^4\theta = \frac{1}{4}\big[\cos^2 2\theta + 2\cos 2\theta + 1\big] = \frac{1}{4}\Big[\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\Big]$$
$$= \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}$$

So,

$$\begin{aligned}\int_0^{\pi/4} \cos^4\theta \, d\theta &= \int_0^{\pi/4} \Big(\frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\Big)\, d\theta \\ &= \left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3}{8}\theta\right]_0^{\pi/4} \\ &= \frac{1}{4} + \frac{3}{8}\cdot\frac{\pi}{4} \\ &= \frac{8+3\pi}{32}\end{aligned}$$

as required.
(b) $\dfrac{8+3\pi}{16}$

A-13: 0

A-14: $2\arcsin\dfrac{x}{2} + \dfrac{x}{2}\sqrt{4-x^2}+C$

A-15: $\sqrt{25x^2-4} - 2\,\mathrm{arcsec}\,\frac{5x}{2} + C$

A-16: $\dfrac{40}{3}$

A-17: $\arcsin\dfrac{x+1}{2}+C$

A-18: $\dfrac{1}{4}\left(\arccos\left(\dfrac{1}{2x-3}\right) + \dfrac{\sqrt{4x^2-12x+8}}{(2x-3)^2}\right)+C$, or equivalently,
$\dfrac{1}{4}\left(\mathrm{arcsec}\,(2x-3) + \dfrac{\sqrt{4x^2-12x+8}}{(2x-3)^2}\right)+C$

A-19: $\ln(1+\sqrt{2}) - \dfrac{1}{\sqrt{2}}$

A-20: $\dfrac{1}{2}\left(\arctan x + \dfrac{x}{x^2+1}\right)+C$

A-21: $\dfrac{3+x}{2}\sqrt{x^2-2x+2}+\dfrac{1}{2}\ln\left|\sqrt{x^2-2x+2}+x-1\right|+C$

A-22: $\dfrac{1}{\sqrt{3}}\ln\left|\left(\dfrac{6}{5}x+1\right)+\dfrac{2}{5}\sqrt{9x^2+15x}\right|+C$

A-23: $\dfrac{1}{3}\sqrt{1+x^2}(4+x^2)+\ln\left|\dfrac{1-\sqrt{1+x^2}}{x}\right|+C$

A-24: $\dfrac{8\pi}{3}+4\sqrt{3}$

A-25: Area: $\dfrac{4}{3}-\sqrt[4]{\dfrac{4}{3}}$ Volume: $\dfrac{\pi^2}{6}-\dfrac{\sqrt{3}\pi}{4}$

A-26: $2\sqrt{1+e^x}+2\ln\left|1-\sqrt{1+e^x}\right|-x+C$

A-27:

(a) $\dfrac{1}{1-x^2}$

(b) False

(c) The work in the question is not correct. The most salient problem is that when we make the substitution $x=\sin\theta$, we restrict the possible values of x to $[-1,1]$, since this is the range of the sine function. However, the original integral had no such restriction.

How can we be sure we avoid this problem in the future? In the introductory text to Section 3.7 (before Example 3.7.1), the text tells us that we are allowed to write our old variable as a function of a new variable (say $x=s(u)$) *as long as that function is invertible to recover our original variable x*. There is one very obvious reason why invertibility is necessary: after we antidifferentiate using our new variable u, we need to get it back in terms of our original variable, so we need to be able to recover x. Moreover, invertibility reconciles potential problems with domains: if an inverse function $u=s^{-1}(x)$ exists, then for any x, there exists a u with $s(u)=x$. (This was not the case in the work for the question, because we chose $x=\sin\theta$, but if $x=2$, there is no corresponding θ. Note, however, that $x=\sin\theta$ is invertible over $[-1,1]$, so the work is correct if we restrict x to those values.)

A-28: (a), (b): None. (c): $x<-a$

Answers to Exercises 3.8 — Jump to TABLE OF CONTENTS

A-1: (a) (ii) (b) (i) (c) (i) (d) (ii)

A-2: $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}+\dfrac{D}{(x+1)^2}+\dfrac{E}{(x+1)^3}$

A-3: 3

A-4: (a) $\dfrac{x^3+2x+2}{x^2+1}=x+\dfrac{x+2}{x^2+1}$

(b) $\dfrac{15x^4+6x^3+34x^2+4x+20}{5x^2+2x+8} = 3x^2+2+\dfrac{4}{5x^2+2x+8}$

(c) $\dfrac{2x^5+9x^3+12x^2+10x+30}{2x^2+5} = x^3+2x+6$

A-5: (a) $5x^3-3x^2-10x+6 = (x+\sqrt{2})(x-\sqrt{2})(5x-3)$

(b) $x^4-11x^2+30 = \left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)\left(x+\sqrt{6}\right)\left(x-\sqrt{6}\right)$

A-6:

(a) $x^4-3x^3-15x^2+15x+50 = (x+2)(x-5)(x+\sqrt{5})(x-\sqrt{5})$
(b) $2x^4+12x^3-x^2-52x+15 = (x+3)(x+5)\left(x-(2+\sqrt{2})\right)\left(x-(2-\sqrt{2})\right)$

A-7: The goal of partial fraction decomposition is to write our integrand in a form that is easy to integrate. The antiderivative of (1) can be easily determined with the substitution $u=(ax+b)$. It's less clear how to find the antiderivative of (2).

A-8: $\ln\dfrac{4}{3}$

A-9: $-2\ln|x-3|+3\ln|x+2|+C$

A-10: $-9\ln|x+2|+14\ln|x+3|+C$

A-11: $5x+\dfrac{1}{2}\ln|x-1|-\dfrac{7}{2}\ln|x+1|+C$

A-12: $\dfrac{1}{x}-\dfrac{2}{x-1}+C$

A-13: $\frac{1}{3}\ln|3x+5|-\frac{2}{3(3x+5)}+C$

A-14: $\ln|x-4|+\ln|2x-1|+\frac{5}{2(2x-1)}+C$

A-15: $3\ln|x-1|+\frac{3}{x-1}+10\ln|x+1|+\frac{10}{x+1}+C$

A-16: $3\ln|x-5|+7\ln|x-3|-\frac{6}{x-3}+C$

A-17: $-\dfrac{1}{2}\ln|x-2|+\dfrac{1}{2}\ln|x+2|+\dfrac{3}{2}\ln|2x-1|+C$

A-18: $-\dfrac{1}{x}-\arctan x+C$

A-19: $\dfrac{1}{2}\ln\left|\dfrac{1-\cos x}{1+\cos x}\right|+C$

A-20: $\dfrac{-\cos x}{2\sin^2 x}+\dfrac{1}{4}\ln\left|\dfrac{1-\cos x}{1+\cos x}\right|+C$

A-21: $\ln\left|\dfrac{\sin\theta-1}{\sin\theta-2}\right|+C$

A-22: $-\frac{t}{2}+\frac{1}{3}\ln\left(e^t+1\right)+\frac{1}{6}\ln\left|e^t+2\right|+C$

A-23: $2\sqrt{1+e^x} + \ln\left|\dfrac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$

A-24: $2\ln\dfrac{5}{3} + \dfrac{4}{\sqrt{3}}\arctan\dfrac{1}{4\sqrt{3}}$

A-25: (a) $\dfrac{1}{6}\left(\ln\left|2\cdot\dfrac{x-3}{x+3}\right|\right)$ (b) $F'(x) = \frac{1}{x^2-9}$

A-26: $\frac{1}{5}\ln|x| - \ln\left|2x^{1/5}+1\right|$

A-27: $-\ln|x| + \frac{6}{x^{1/6}} + \frac{3}{x^{1/3}} + 6\ln\left|x^{1/6}-1\right| + C$

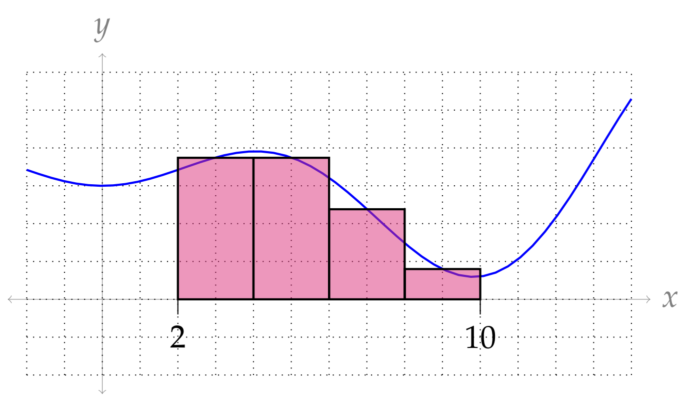
A-28: $-3\ln\left|x^{1/3}+1\right| + 6\ln\left|x^{1/3}+2\right| + C$

A-29: $2\sqrt{x-5} + \sqrt{2}\ln\left|\sqrt{x-5}-\sqrt{2}\right| - \sqrt{2}\ln\left|\sqrt{x-5}+\sqrt{2}\right| + C$

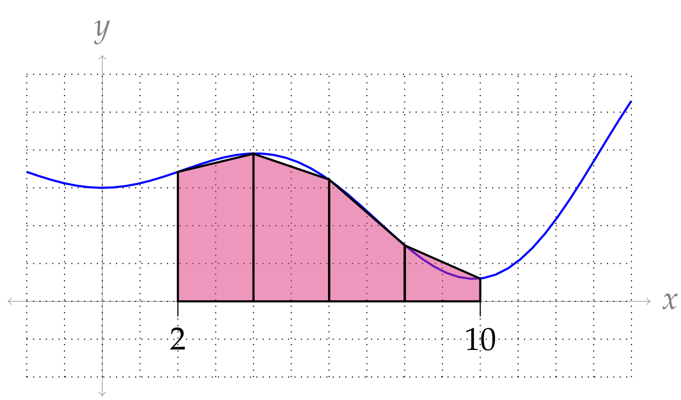
Answers to Exercises 3.9 — Jump to TABLE OF CONTENTS

A-1: Relative error: ≈ 0.08147; absolute error: 0.113; percent error: $\approx 8.147\%$.

A-2: Midpoint rule:



Trapezoidal rule:



A-3: $M = 6.25$, $L = 2$

A-4: One reasonable answer is $M = 3$.

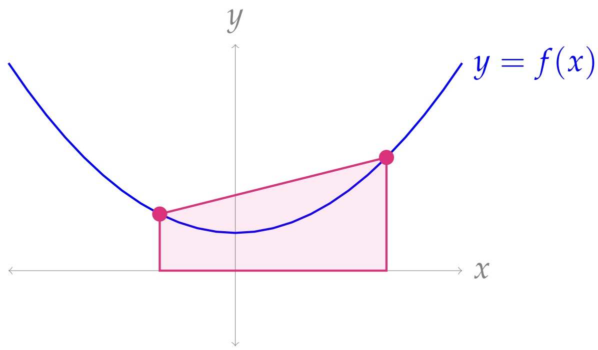
A-5: (a) $\dfrac{\pi^5}{180 \cdot 8}$ (b) 0 (c) 0

A-6: Possible answers: $f(x) = \frac{3}{2}x^2 + Cx + D$ for any constants C, D.

A-7: my mother

A-8: (a) true (b) false

A-9: True. Because $f(x)$ is positive and concave up, the graph of $f(x)$ is always below the top edges of the trapezoids used in the trapezoidal rule.



A-10: Any polynomial of degree at most 3 will do. For example, $f(x) = 5x^3 - 27$, or $f(x) = x^2$.

A-11:

Midpoint:

$$\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x \approx \left[\frac{1}{(2.5)^3+1} + \frac{1}{(7.5)^3+1} + \frac{1}{(12.5)^3+1} + \frac{1}{(17.5)^3+1} + \frac{1}{(22.5)^3+1} + \frac{1}{(27.5)^3+1}\right] 5$$

Trapezoidal:

$$\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x \approx \left[\frac{1/2}{0^3+1} + \frac{1}{5^3+1} + \frac{1}{10^3+1} + \frac{1}{15^3+1} + \frac{1}{20^3+1} + \frac{1}{25^3+1} + \frac{1/2}{30^3+1}\right] 5$$

Simpson's:

$$\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x \approx \left[\frac{1}{0^3+2} + \frac{4}{5^3+1} + \frac{2}{10^3+1} + \frac{4}{15^3+1} + \frac{2}{20^3+1} + \frac{4}{25^3+1} + \frac{1}{30^3+1}\right] \frac{5}{3}$$

A-12: $\dfrac{2\pi}{3}$

A-13: $1720\pi \approx 5403.5 \text{ cm}^3$

A-14: $\dfrac{\pi}{12}(16.72) \approx 4.377 \text{ m}^3$

A-15: $\dfrac{12.94}{6\pi} \approx 0.6865 \text{ m}^3$

A-16: (a) 363,500 (b) 367,000

A-17: (a) $\dfrac{49}{2}$ (b) $\dfrac{77}{3}$

A-18: Let $f(x) = \sin(x^2)$. Then $f'(x) = 2x\cos(x^2)$ and

$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2).$$

Since $|x^2| \leqslant 1$ when $|x| \leqslant 1$, and $|\sin\theta| \leqslant 1$ and $|\cos\theta| \leqslant 1$ for all θ, we have

$$\left|2\cos(x^2) - 4x^2\sin(x^2)\right| \leqslant 2|\cos(x^2)| + 4x^2|\sin(x^2)| \leqslant 2\times 1 + 4\times 1\times 1 = 2+4 = 6$$

We can therefore choose $M = 6$, and it follows that the error is at most

$$\frac{M[b-a]^3}{24n^2} \leqslant \frac{6\cdot[1-(-1)]^3}{24\cdot 1000^2} = \frac{2}{10^6} = 2\cdot 10^{-6}$$

A-19: $\dfrac{3}{100}$

A-20: (a)
$\dfrac{1/3}{3}\left((-3)^5 + 4\left(\frac{1}{3}-3\right)^5 + 2\left(\frac{2}{3}-3\right)^5 + 4(-2)^5 + 2\left(\frac{4}{3}-3\right)^5 + 4\left(\frac{5}{3}-3\right)^5 + (-1)^5\right)$

(b) Simpson's Rule results in a smaller error bound.

A-21: $\dfrac{8}{15}$

A-22: $\dfrac{1}{180\times 3^4} = \dfrac{1}{14580}$

A-23: (a) $T_4 = \dfrac{1}{4}\left[\left(\dfrac{1}{2}\times 1\right) + \dfrac{4}{5} + \dfrac{2}{3} + \dfrac{4}{7} + \left(\dfrac{1}{2}\times\dfrac{1}{2}\right)\right]$,

(b) $S_4 = \dfrac{1}{12}\left[1 + \left(4\times\dfrac{4}{5}\right) + \left(2\times\dfrac{2}{3}\right) + \left(4\times\dfrac{4}{7}\right) + \dfrac{1}{2}\right]$

(c) $\left|I - S_4\right| \leqslant \dfrac{24}{180\times 4^4} = \dfrac{3}{5760}$

A-24: (a) $T_4 = 8.03515$, $S_4 \approx 8.03509$

(b) $\left|\displaystyle\int_a^b f(x)\,dx - T_n\right| \leqslant \dfrac{2}{1000}\dfrac{8^3}{12(4)^2} \leqslant 0.00533$,
$\left|\displaystyle\int_a^b f(x)\,dx - S_n\right| \leqslant \dfrac{4}{1000}\dfrac{8^5}{180(4)^4} \leqslant 0.00284$

A-25: Any $n \geqslant 68$ works.

A-26: $\dfrac{472}{3} \approx 494\ \text{ft}^3$

A-27: (a) 0.025635 (b) 1.8×10^{-5}

A-28: (a) ≈ 0.6931698 (b) $n \geqslant 12$ with n even

A-29: (a) 0.01345 (b) $n \geqslant 28$ with n even

A-30: $n \geqslant 259$

A-31: (a) When $0 \leqslant x \leqslant 1$, then $x^2 \leqslant 1$ and $x+1 \geqslant 1$, so $|f''(x)| = \dfrac{x^2}{|x+1|} \leqslant \dfrac{1}{1} = 1$.

(b) $\dfrac{1}{2}$ (c) $n \geqslant 65$ (d) $n \geqslant 46$

A-32: $\dfrac{x-1}{12}\left[1 + \dfrac{16}{x+3} + \dfrac{4}{x+1} + \dfrac{16}{3x+1} + \dfrac{1}{x}\right]$

A-33: *Note: for more detail, see the solutions.*

First, we use Simpson's rule with $n = 4$ to approximate $\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x$. The choice of this method (what we're approximating, why $n = 4$, etc.) is explained in the solutions–here, we only show that it works.

$$\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x \approx \frac{1}{12}\left[\frac{1}{2} + \frac{64}{41} + \frac{8}{13} + \frac{64}{65} + \frac{1}{5}\right] \approx 0.321748$$

For ease of notation, define $A = 0.321748$.

Now, we bound the error associated with this approximation. Define $N(x) = 24(5x^4 - 10x^2 + 1)$ and $D(x) = (x^2+1)^5$, so $N(x)/D(x)$ gives the fourth derivative of $\frac{1}{1+x^2}$. When $1 \leqslant x \leqslant 2$, $|N(x)| \leqslant N(2) = 984$ (because $N(x)$ is increasing over that interval) and $|D(x)| \geqslant D(1) = 2^5$ (because $D(x)$ is also increasing over that interval), so $\left|\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{\frac{1}{1+x^2}\right\}\right| = \left|\frac{N(x)}{D(x)}\right| \leqslant \frac{984}{2^5} = 30.75$. Now we find the error bound for Simpson's rule with $L = 30.75$, $b = 2$, $a = 1$, and $n = 4$.

$$\left|\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x - A\right| = |\text{error}| \leqslant \frac{L(b-a)^5}{180 \cdot n^4} = \frac{30.75}{180 \cdot 4^4} < 0.00067$$

So,

$$\begin{array}{rcccr}
-0.00067 & < & \displaystyle\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x - A & < & 0.00067 \\
A - 0.0067 & < & \displaystyle\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x & < & A + 0.00067 \\
A - 0.00067 & < & \arctan(2) - \arctan(1) & < & A + 0.00067 \\
A - 0.00067 & < & \arctan(2) - \dfrac{\pi}{4} & < & A + 0.00067 \\
\dfrac{\pi}{4} + A - 0.00067 & < & \arctan(2) & < & \dfrac{\pi}{4} + A + 0.00067 \\
\dfrac{\pi}{4} + 0.321748 - 0.00067 & < & \arctan(2) & < & \dfrac{\pi}{4} + 0.321748 + 0.00067 \\
\dfrac{\pi}{4} + 0.321078 & < & \arctan(2) & < & \dfrac{\pi}{4} + 0.322418 \\
\dfrac{\pi}{4} + 0.321 & < & \arctan(2) & < & \dfrac{\pi}{4} + 0.323
\end{array}$$

This was the desired bound.

Answers to Exercises 3.10 — Jump to TABLE OF CONTENTS

A-1: Any real number in $[1, \infty)$ or $(-\infty, -1]$, and $b = \pm\infty$.

A-2: $b = \pm\infty$

A-3: The red dotted function is $f(x)$, and the blue solid function is $g(x)$.

A-4: False. For example, the functions $f(x) = e^{-x}$ and $g(x) = 1$ provide a counterexample.

A-5:

(a) Not enough information to decide. For example, consider $h(x) = 0$ versus $h(x) = -1$.

(b) Not enough information to decide. For example, consider $h(x) = f(x)$ versus $h(x) = g(x)$.

(c) $\displaystyle\int_0^\infty h(x)\,\mathrm{d}x$ converges by the comparison test, since $|h(x)| \leqslant 2f(x)$ and $\displaystyle\int_0^\infty 2f(x)\,\mathrm{d}x$ converges.

A-6: The integral diverges.

A-7: The integral diverges.

A-8: The integral does not converge.

A-9: The integral converges.

A-10: The integral diverges.

A-11: The integral diverges.

A-12: The integral diverges.

A-13: The integral diverges.

A-14: The integral diverges.

A-15: The integral converges.

A-16: $q = \frac{1}{5}$

A-17: $p > 1$

A-18: The integral converges.

A-19: $\dfrac{1}{2}$

A-20: The integral converges.

A-21: The integral converges.

A-22: $t = 10$ and $n = 2042$ will do the job. There are many other correct answers.

A-23: (a) The integral converges. (b) The interval converges.

A-24: false

Answers to Exercises 3.11 — Jump to TABLE OF CONTENTS

A-1: (A)–(I), (B)–(IV), (C)–(II), (D)–(III)

A-2: $\dfrac{1}{5} - \dfrac{2}{7} + \dfrac{1}{9} = \dfrac{8}{315}$

A-3: $\dfrac{3}{2\sqrt{5}} \arcsin\left(x\sqrt{\dfrac{5}{3}}\right) + \dfrac{x}{2}\sqrt{3-5x^2} + C$

A-4: 0

A-5: $\ln\left|\dfrac{x+1}{3x+1}\right| + C$

A-6: $\dfrac{8}{3}\ln 2 - \dfrac{7}{9}$

A-7: $\dfrac{1}{2}\ln\left|x^2-3\right| + C$

A-8: (a) 2 (b) $\dfrac{2}{15}$ (c) $\dfrac{3e^4}{16} + \dfrac{1}{16}$

A-9: (a) 1 (b) $\dfrac{8}{15}$

A-10: (a) e^2+1 (b) $\ln(\sqrt{2}+1)$ (c) $\ln\frac{15}{13} \approx 0.1431$

A-11: (a) $\dfrac{9}{4}\pi$ (b) $\ln 2 - 2 + \dfrac{\pi}{2} \approx 0.264$ (b) $2\ln 2 - \frac{1}{2} \approx 0.886$

A-12: $\dfrac{1}{3}\sin^3\theta - 2\sin\theta + 12\ln\left|\dfrac{\sin\theta-3}{\sin\theta-2}\right| + C$

A-13: (a) $\dfrac{1}{15}$ (b) $\dfrac{1}{9}\cdot\dfrac{x}{\sqrt{x^2+9}} + C$ (c) $\dfrac{1}{2}\ln|x-1| - \dfrac{1}{4}\ln(x^2+1) - \dfrac{1}{2}\arctan x + C$

(d) $\dfrac{1}{2}\left[x^2\arctan x - x + \arctan x\right] + C$

A-14: (a) $\dfrac{1}{12}$ (b) $2\sin^{-1}\dfrac{x}{2} + x\sqrt{1-\dfrac{x^2}{4}} + C$

(c) $-2\ln|x| + \dfrac{1}{x} + 2\ln|x-1| + C$

A-15: (a) $\dfrac{2}{5}$ (b) $\dfrac{1}{2\sqrt{2}}$ (c) $\ln 2 - \dfrac{1}{2} \approx 0.193$ (d) $\ln 2 - \dfrac{1}{2} \approx 0.193$

A-16: (a) $\dfrac{1}{2}x^2\ln x - \dfrac{1}{4}x^2 + C$ (b) $\dfrac{1}{2}\ln[x^2+4x+5] - 3\arctan(x+2) + C$

(c) $\dfrac{1}{2}\ln|x-3| - \dfrac{1}{2}\ln|x-1| + C$ (d) $\dfrac{1}{3}\arctan x^3 + C$

A-17: (a) $\frac{\pi}{4} - \frac{1}{2}\ln 2$ (b) $\ln|x^2-2x+5| + \frac{1}{2}\arctan\frac{x-1}{2} + C$

A-18: (a) $-\frac{1}{300(x^3+1)^{100}} + C$ (b) $\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

A-19: -2

A-20: (a) $-\frac{1}{4}\ln|e^x+1| + \frac{1}{4}\ln|e^x-3| + C$ (b) $\frac{4\pi}{3} - 2\sqrt{3}$

A-21: (a) $\frac{1}{2}\sec^2 x + \ln|\cos x| + C$ (b) $\frac{1}{10}\arctan 8 \approx 0.1446$

A-22: $\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$

A-23: $\ln\left|x+\sqrt{x^2-2}\right| - \frac{\sqrt{x^2-2}}{x} + C$

A-24: $\frac{7}{24}$

A-25: $3\ln|x+1| + \frac{2}{x+1} - \frac{5}{2(x+1)^2} + C$

A-26: $\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$

A-27: $\frac{1}{2}(x - \sin x\cos x) + C$

A-28: $\frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$

A-29: $3x^3\arcsin x + 3\sqrt{1-x^2} - (1-x^2)^{3/2} + C$

A-30: 2

A-31: $\frac{1}{4}$

A-32: $\ln\left(\frac{\ln(\cos(0.1))}{\ln(\cos(0.2))}\right)$

A-33: (a) $\frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right] + C$ (b) $2\ln 2 - \ln 3 = \ln\frac{4}{3}$

A-34: (a) $\frac{9}{4}\pi + 9$ (b) $2\ln|x-2| - \ln(x^2+4) + C$ (c) $\frac{\pi}{2}$

A-35: $-\arcsin(\sqrt{1-x}) - \sqrt{1-x}\sqrt{x} + C$

A-36: $e^e(e-1)$

A-37: $\frac{e^x}{x+1} + C$

A-38: $x\sec x - \ln|\sec x + \tan x| + C$

A-39: $\displaystyle\int x(x+a)^n\,\mathrm{d}x = \begin{cases} \frac{(x+a)^{(n+2)}}{n+2} - a\frac{(x+a)^{n+1}}{n+1} + C & \text{if } n \neq -1, -2 \\ (x+a) - a\ln|x+a| + C & \text{if } n = -1 \\ \ln|x+a| + \frac{a}{x+a} + C & \text{if } n = -2 \end{cases}$

A-40: $x\arctan(x^2) - \frac{1}{\sqrt{2}}\left(\frac{1}{2}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + \arctan\left(\sqrt{2}x+1\right) + \arctan\left(\sqrt{2}x-1\right)\right) + C$

Answers to Exercises 3.12 — Jump to TABLE OF CONTENTS

A-1: (a) yes (b) yes (c) no

A-2:

(a) One possible answer: $f(x) = x$, $g(y) = \dfrac{3y}{\sin y}$.

(b) One possible answer: $f(x) = e^x$, $g(y) = e^{-y}$.

(c) One possible answer: $f(x) = x + 1$, $g(y) = 1$.

(d) The given equation is equivalent to the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x$, which fits the form of a separable equation with $f(x) = x$, $g(y) = 1$.

A-3: The mnemonic allows us to skip from the separable differential equation we want to solve (very first line) to the equation

$$\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

We also generally skip the explanation about C_1 and C_2 being replaced with C.

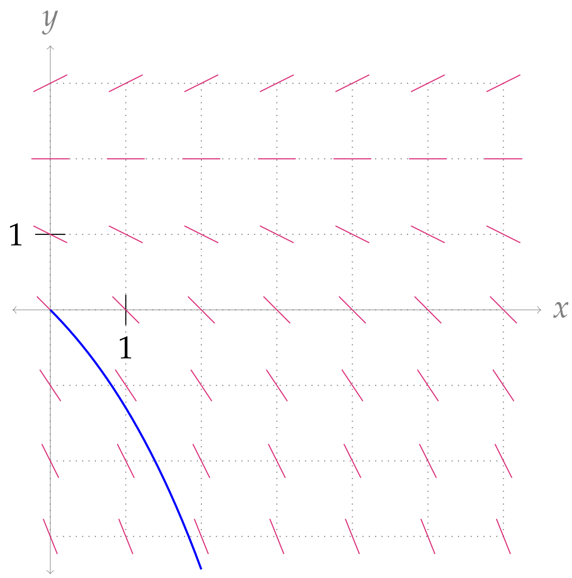
A-4: false

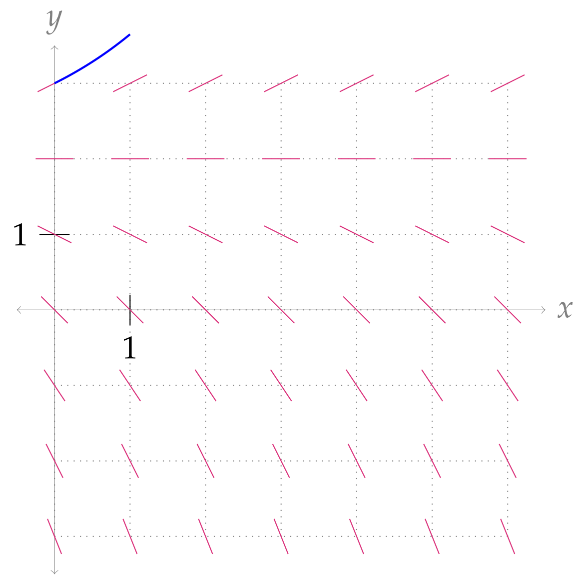
A-5: (a) $[0, \infty)$
(b) No such function exists. If $|f(x)| = Cx$ and $f(x)$ switches from $f(x) = Cx$ to $f(x) = -Cx$ at some point, then that point is a jump discontinuity. Where $f(x)$ contains a discontinuity, $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not exist.

A-6: $\dfrac{\mathrm{d}Q}{\mathrm{d}t} = -0.003Q(t)$

A-7: $\frac{\mathrm{d}p}{\mathrm{d}t} = \alpha p(t)\big(1 - p(t)\big)$, for some constant α.

A-8: (a) -1 (b) 0 (c) 0.5
(d) Two possible answers are shown below:





Another possible answer is the constant function $y = 2$.

A-9: $y(t) = \frac{7}{5} - \frac{22}{5}e^{5t}$

A-10: $y(t) = \frac{1}{2}e^{2t} - \frac{1}{2}$

A-11: $y(t) = \frac{5}{2}e^{2(t-1)} - \frac{3}{2}$

A-12: $y = \frac{7}{3}$

A-13: $y = 0, y = 1, y = -1$

A-14: $y = \ln(x^2 + 2)$

A-15: $y(x) = 3\sqrt{1 + x^2}$

A-16: $y(t) = 3\ln\left(\dfrac{-3}{C + \sin t}\right)$

A-17: $y = \sqrt[3]{\frac{3}{2}e^{x^2} + C}$.

A-18: $y = -\ln\left(C - \dfrac{x^2}{2}\right)$

The solution only exists for $C - \frac{x^2}{2} > 0$, i.e. $C > 0$ and the function has domain $\{x : |x| < \sqrt{2C}\}$.

A-19: $y = (3e^x - 3x^2 + 24)^{1/3}$

A-20: $y = f(x) = -\dfrac{1}{\sqrt{x^2 + 16}}$

A-21: $y = \sqrt{10x^3 + 4x^2 + 6x - 4}$

A-22: $y(x) = e^{x^4/4}$

A-23: $y = \frac{1}{1-2x}$

A-24: $f(x) = e \cdot e^{x^2/2}$

A-25: $y(x) = \sqrt{4 + 2\ln \frac{2x}{x+1}}$. Note that, to satisfy $y(1) = 2$, we need the positive square root.

A-26: $y^2 + \frac{2}{3}(y^2 - 4)^{3/2} = 2\sec x + 2$

A-27: 12 weeks

A-28: $t = \sqrt{\frac{m}{kg}} \arctan\left(\sqrt{\frac{k}{mg}}\, v_0\right)$

A-29: (a) $k = \frac{1}{400}$ (b) $t = 70$sec

A-30: (a) $x(t) = \dfrac{3 - 4e^{kt}}{1 - 2e^{kt}}$ (b) As $t \to \infty$, $x \to 2$.

A-31: (a) $P = \dfrac{4}{1 + e^{-4t}}$ (b) At $t = \frac{1}{2}$, $P \approx 3.523$. As $t \to \infty$, $P \to 4$.

A-32: (a) $\dfrac{\mathrm{d}v}{\mathrm{d}t} = -kv^2$ (b) $v = \dfrac{400}{t+1}$ (c) $t = 7$

A-33: (a) $f_{rr}(1,0) = m(m-1)$, $f_{r\theta}(1,0) = 0$, $f_{\theta\theta}(1,0) = -m^2$ (b) $\lambda = 1$

A-34: (a) $B(t) = C\, e^{0.06t - 0.02\cos t}$ with the arbitrary constant $C \geqslant 0$. (b) \$1159.89

A-35: $y(x) = \dfrac{4 - e^{1-\cos x}}{2 - e^{1-\cos x}}$. The largest allowed interval is

$$-\arccos(1 - \ln 2) < x < \arccos(1 - \ln 2)$$

or, roughly, $-1.259 < x < 1.259$.

A-36: (a) 3 (b) $y' = (y-1)(y-2)$ (c) $f(x) = \dfrac{4 - e^x}{2 - e^x}$

A-37:

(a) One possible answer: $f(t) = 0$

(b) $\dfrac{1}{\sqrt{x-a}}\left[f(x) - \dfrac{1}{2(x-a)}\displaystyle\int_a^x f(t)\,\mathrm{d}t\right] = \dfrac{f^2(x)}{2\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}}$

(c) $\dfrac{2}{x-a}\displaystyle\int_a^x f(t)\,\mathrm{d}t\left[f(x) - \dfrac{1}{2(x-a)}\displaystyle\int_a^x f(t)\,\mathrm{d}t\right] = f^2(x)$

(d) $Y(x) = D(x-a)$, where D is any constant

(e) $f(t) = D$, for any nonnegative constant D

A-38: $x = \dfrac{1}{4}\left(y - 1 + \dfrac{1}{4}\ln\left|\dfrac{2y-1}{2y+1}\right|\right)$

A-39:

(a) $-\frac{a}{r}, -\frac{b}{r}$

(b) $r < 0$, but the sign of a is not restricted

(c) $W(t) = \left(W(0) + \frac{a}{r+c}\right) e^{(r+c)t} - \frac{a}{r+c}$

A-40:

(a) $\frac{1}{2} - \frac{1}{2}e^{3/4} + \sum_{i=1}^{300} e^{i/400}$

(b) $|\text{error}| \leqslant \frac{e^{3/4}}{6400}$

(c) $\frac{1}{2} - \frac{1}{2}e^{3/4}$

(d) $[315342.95, 315144.07]$

(e) absolute error of less than \$938, and a relative error of less than 0.24%.

A-41:

(a) $\frac{r}{100}P(t)$

(b) $-P'(t)$ is the amount paid towards the loan

(c) $\frac{r}{100}P(t) - P'(t) = C$

(d) $P(t) = \left(P_0 - \frac{100}{r}C\right) e^{\frac{r}{100}t} + \frac{100}{r}C$

(e) $C = \frac{r}{100}\left(\frac{P_0 e^{rN/100}}{e^{rN/100}-1}\right)$

(f) $\frac{r}{100}P(t) - P'(t) = C_0 \cdot 1.001^t$

Answers to Exercises 4.1 — Jump to TABLE OF CONTENTS

A-1: $Pr(X = 5) = 0.1$

A-2: Yes

A-3: X

A-4: $\frac{1}{100}$

A-5: $\frac{15}{93}$

A-6: $\frac{2}{3}$

A-7: $\frac{7}{10}$

Answers to Exercises 4.2 — Jump to TABLE OF CONTENTS

A-1: Probability Mass Function

A-2: Discrete

A-3:

(a) $\mathcal{S} = \{5, 6, 7, 8\}$

(b) No

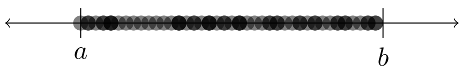
A-4: 0.2

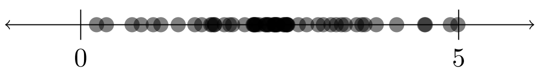
A-5: 8.2

A-6: $\mathcal{S} = \{1, 2, 3, \ldots\}$ (all whole numbers); $Pr(X \leqslant 3) = \frac{7}{8}$.

Answers to Exercises 4.3 — Jump to TABLE OF CONTENTS

A-1: $F(0) = \frac{1}{10}$, $F(20) = \frac{2}{5}$, and $F(30) = \frac{9}{10}$.

A-2: 

A-3: 

A-4: yes

A-5:

(a) True, Corollary 4.3.10, part 3.

(b) False, for examples a variable X that takes the value $1,000,001$ with probability 1.

(c) True, Corollary 4.3.10, part 2.

(d) True, Corollary 4.3.10, part 1.

A-6: $F(11) = 1$

A-7:

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leqslant x < 1 \\ 1 & 1 \leqslant x \end{cases}$$

A-8:

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{6} & 1 \leqslant x < 2 \\ \frac{1}{3} & 2 \leqslant x < 3 \\ \frac{1}{2} & 3 \leqslant x < 4 \\ \frac{2}{3} & 4 \leqslant x < 5 \\ \frac{5}{6} & 5 \leqslant x < 6 \\ 1 & x \geqslant 6 \end{cases}$$

A-9:

$$F(x) = \begin{cases} 0 & x < -4 \\ \frac{1}{2} & -4 \leqslant x < -2 \\ \frac{5}{6} & -2 \leqslant x < -1 \\ 1 & -1 \leqslant x \end{cases}$$

A-10: $A = 1$; B can be any nonnegative number.

A-11: $0 \leqslant A \leqslant 1$, $A = C = D$, and $B = 1 - A$

A-12:

x	$Pr(W = x)$
5	$\frac{1}{4}$
6	$\frac{1}{12}$
8	$\frac{1}{6}$
12	$\frac{1}{2}$

A-13: $\mathcal{S} = \{1, 2, 4\}$, and $Pr(x = 2) > Pr(x = 4) > Pr(x = 1)$.

Answers to Exercises 4.4 — Jump to TABLE OF CONTENTS

A-1:

(a) 

(b) 

(c) 

A-2:

(a) limit at negative infinity is 1: neither

(b) never negative: both

(c) nondecreasing: CDFs

(d) never more than 1: CDFs

(e) area under the curve gives a probability: PDFs

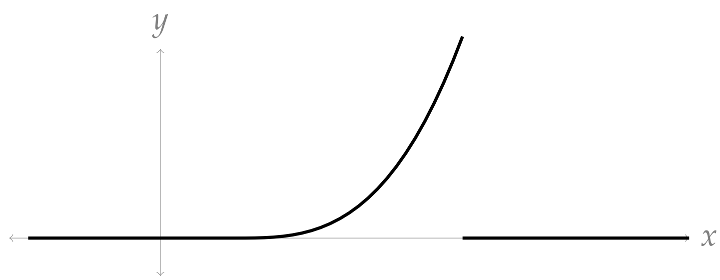
(f) value of function gives a probability: CDFs

(g) area under the curve from $-\infty$ to ∞ is 1: PDFs

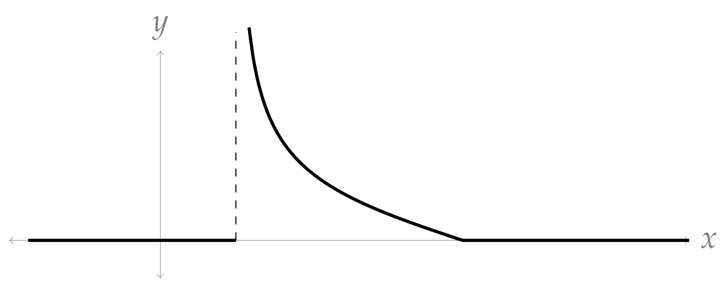
A-3: b, c, and e: CDF
a, d, and f: PDF

A-4:

(a)

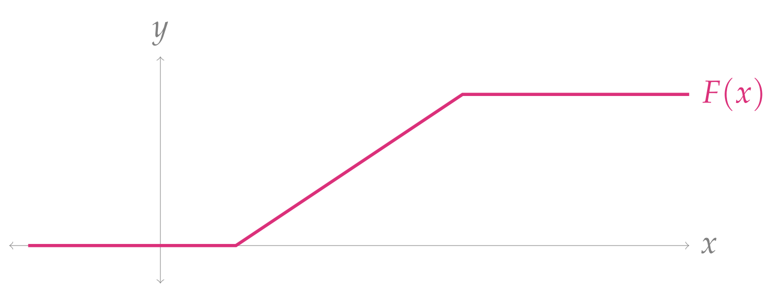


(b)

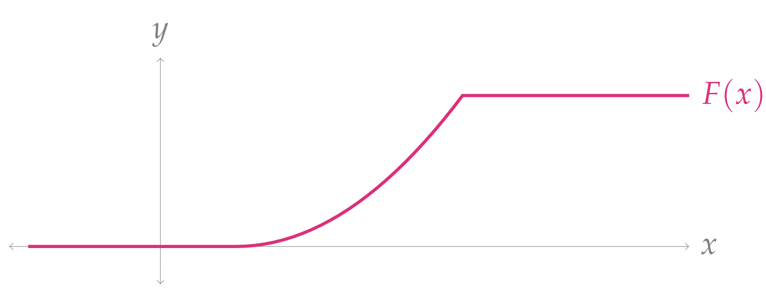


(c) The CDF of this random variable is not a continuous function, so the variable is not continuous.

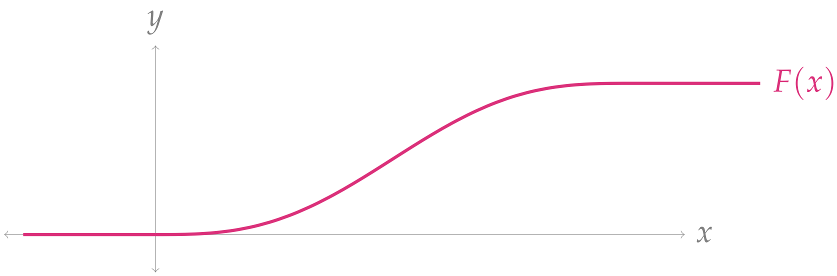
A-5:



(a)



(b)



(c)

A-6: $\frac{\arctan(170)-\arctan(40)}{\pi} \approx 0.006$

A-7: 0.6

A-8: $\frac{2}{10}$

A-9: $F(x) = \begin{cases} 0 & x \leqslant 0 \\ \frac{x^2}{100} & 0 < x < 10 \\ 1 & x \geqslant 10 \end{cases}$

A-10: $F(x) = \begin{cases} 0 & x \leqslant 0 \\ \frac{x}{10} & 0 < x \leqslant 3 \\ \frac{3}{10} & 3 < x \leqslant 4 \\ \frac{x}{5} - \frac{1}{2} & 4 \leqslant x < 6 \\ \frac{7}{10} & 6 \leqslant x < 7 \\ \frac{3x}{10} - \frac{7}{5} & 7 \leqslant x < 8 \\ 1 & 8 \leqslant x \end{cases}$

A-11: X is a continuous random variable, and its PDF is

$$f(x) = \begin{cases} e^x & x < 0 \\ 0 & x > 0 \end{cases}$$

A-12: Yes, X is continuous. It has PDF

$$f(x) = \begin{cases} \frac{1}{(x+1)^2} & x > 0 \\ 0 & x < 0 \end{cases}$$

A-13: $\ln\left(1 + \frac{1}{e}\right)$

A-14: $\frac{2}{\pi}$

A-15: $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^4}{2} & 0 \leqslant x \leqslant 1 \\ \frac{1}{2} & 1 < x < 2 \\ \frac{1}{2} + \frac{(x-2)^4}{2} & 2 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$

A-16: $F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} - \frac{x^2}{2} & -1 \leqslant x \leqslant 0 \\ \frac{1}{2} + \frac{x^2}{2} & 0 < x \leqslant 1 \\ 1 & x > 1 \end{cases}$

A-17:

(a) $c = \frac{12}{200^4}$

(b) $\mathcal{S}_1 = \{0, 10, 20, \ldots, 200\}$

(c) $Pr(M_1 = m) = \begin{cases} \frac{157}{40^4} & m = 0 \\ \frac{12}{200^4}\left[-10m^3 + 2,000m^2 - 250m + \frac{50,000}{3}\right] & m = 10, 20, \ldots, 180 \\ \frac{3595}{40^4} & m = 200 \end{cases}$

(d) They are nearly identical, except for y-scale

A-18: $\frac{1}{3}\left[\sqrt{\frac{2}{\pi}} + 4\left(\sqrt{\frac{2}{\pi}}e^{-1/8}\right) + 2\left(\sqrt{\frac{2}{\pi}}e^{-1/2}\right) + 4\left(\sqrt{\frac{2}{\pi}}e^{-9/8}\right) + \left(\sqrt{\frac{2}{\pi}}e^{-2}\right)\right]$

Answers to Exercises 4.5 — Jump to TABLE OF CONTENTS

A-1: false

A-2: I don't ... but I guess I can't speak for you.

A-3: a

A-4: false

A-5: 1.1

A-6: 2.32

A-7: If you play N times, where N is a large number, you expect to make about $\frac{N}{4}$ dollars.

A-8: You expect to *lose* $\frac{11}{5}N$ dollars after N games. Equivalently, you expect to win $-\frac{11}{5}N$ dollars.

A-9: $\mathbb{E}(M) = 66 + \frac{2}{3}$. From Section 4.5.3, we see $50 < \mathbb{E}(M) \leqslant 100$, and this is true of $66 + \frac{2}{3}$, so it passes the checks.

A-10: $\mathbb{E}(N) = 0$. From Section 4.5.3, we see $-1 \leqslant \mathbb{E}(M) \leqslant 1$, and this is true of 0, so it passes the check.

A-11: $\mathbb{E}(P) = \frac{17}{18}$. This accords with $0 \leqslant \mathbb{E}(P) \leqslant 3$ from Theorem 4.5.7

A-12: $\mathbb{E}(Q) = \frac{1}{2}\left(\frac{e}{e-2}\right)$. As a check, we expect $1 \leqslant \mathbb{E}(Q) \leqslant e$, and this is true.

A-13: $\frac{b+a}{2}$

A-14:

(a) $p > 1$

(b) $a_p = p - 1$

(c) $p > 2$

A-15: $e - 2$

A-16: $\frac{2}{\ln^2 2}(2\ln 2 - 1)$

A-17: $(b-2) + \frac{1}{2}\ln\left(4\frac{b-1}{b+2}\right)$

A-18: $\frac{1}{4-\pi}\left(\pi - 2\ln 2 - \frac{\pi^2}{8}\right)$

A-19: 0

A-20: \$985

A-21: (a) 15% (b) \$2700

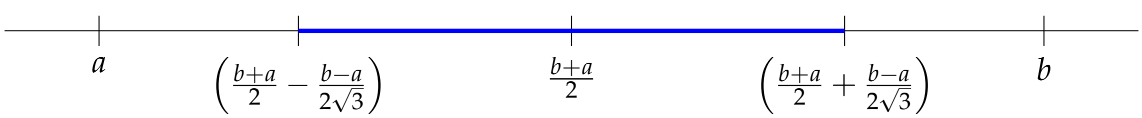
A-22: (a) $1.2x + 1.6y$ (b) \$250 in Asset A, \$50 in Asset B

Answers to Exercises 4.6 — Jump to TABLE OF CONTENTS

A-1: A - 4 B -6 C -5 D -1 E -3 F -2

A-2: X has sample space $\mathbb{S} = \{-100, 100\}$ and is equally likely to take either value.

A-3: $\text{Var}(X) = \frac{(b-a)^2}{12}$, $\sigma(X) = \frac{b-a}{2\sqrt{3}}$



A-4: 0 and 0

A-5: 0

A-6:

(a) $F(x) = \begin{cases} 0 & x < 0 \\ 1 - \cos x & 0 \leqslant x \leqslant \frac{\pi}{2} \\ 1 & x > \frac{\pi}{2} \end{cases}$

(b) $\text{Var}(X) = \pi - 3$

(c) $\sigma(X) = \sqrt{\pi - 3}$

A-7: $\mathbb{E}(X) = 0$, $\text{Var}(X) = \frac{1}{6}$, $\sigma(X) = \frac{1}{\sqrt{6}}$

A-8:

(a) Yes, X is continuous

(b) $\mathbb{E}(X) = \frac{1}{12}$, $\text{Var}(V) = \frac{67}{720}$, $\sigma(X) = \frac{\sqrt{67}}{12\sqrt{5}}$

A-9: $\mathbb{E}(T) = \frac{7}{4}$, $\mathrm{Var}(T) = \frac{11}{16}$, $\sigma(T) = \frac{\sqrt{11}}{4}$

A-10: $\mathbb{E}(S) = \frac{8}{3}$, $\mathrm{Var}(S) = \frac{242}{9}$, $\sigma(S) = \frac{11\sqrt{2}}{3}$

A-11:

A-12: $\mathrm{Var}(Z) = \frac{2\pi}{3}a - \left(\frac{4}{3}a\right)^2$, $\sigma(Z) = \sqrt{\frac{2\pi}{3}a - \left(\frac{4}{3}a\right)^2}$

Answers to Exercises 5.1 — Jump to TABLE OF CONTENTS

A-1: (a) -2 (b) 0 (c) the limit does not exist

A-2: true

A-3: (a) $\dfrac{A-B}{C}$ (b) 0 (c) $\dfrac{A}{B}$

A-4: Two possible answers, of many:

- $a_n = \begin{cases} 3000 - n & \text{if } n \leqslant 1000 \\ -2 + \frac{1}{n} & \text{if } n > 1000 \end{cases}$
- $a_n = \dfrac{1,002,001}{n} - 2$

A-5: One possible answer is $a_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, -1, \ldots\}$.
Another is $a_n = n(-1)^n = \{-1, 2, -3, 4, -5, 6, -7, \ldots\}$.

A-6: One sequence of many possible is $a_n = \dfrac{(-1)^n}{n} = \left\{-1, \dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, -\dfrac{1}{5}, \dfrac{1}{6}, \cdots\right\}$.

A-7: Some possible answers:

(a) $\dfrac{-1}{n} \leqslant \dfrac{\sin n}{n} \leqslant \dfrac{1}{n}$

(b) $\dfrac{n^2}{13e^n} \leqslant \dfrac{n^2}{e^n(7 + \sin n - 5\cos n)} \leqslant \dfrac{n^2}{e^n}$

(c) $\dfrac{-1}{n^n} \leqslant (-n)^{-n} \leqslant \dfrac{1}{n^n}$

A-8: (a) $a_n = b_n = h(n) = i(n)$, $c_n = j(n)$, $d_n = f(n)$, $e_n = g(n)$
(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lim\limits_{x\to\infty} h(x) = 1$, $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} e_n = \lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} j(x) = 0$, $\lim\limits_{n\to\infty} d_n$, $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} i(x)$ do not exist.

A-9: (a) Some possible answers: $a_{22} \approx -0.99996$, $a_{66} \approx -0.99965$, and $a_{110} \approx -0.99902$.

(b) Some possible answers: $a_{11} \approx 0.0044$, $a_{33} \approx -0.0133$, and $a_{55} \approx 0.0221$.

The integers 11, 33, and 55 were found by approximating π by $\dfrac{22}{7}$ and finding when an odd multiple of $\dfrac{11}{7}$ (which is the corresponding approximation of $\dfrac{\pi}{2}$) is an integer.

A-10:

(a) $\{4, 34, 334, 3334, 33334, \ldots\}$

(b) $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$

(c) $\{0, 0, 0, 0, 0, \ldots\}$

(d) $\{1, -1, 2, -3, 5, \ldots\}$

A-11:

(a) $\{1, 1, 1, 1, 1, \ldots\}$

(b) $\{1, 2, 3, 4, 5, \ldots\}$

(c) $\{0, 0, 0, 0, 0, \ldots\}$

(d) $\{1, -1, 1, -1, 1, \ldots\}$

A-12:

(a) $a_n = 2^{2^n}$

(b) $b_n = (-1)^n \cdot 5$

(c) $c_n = 8$

A-13: Here are possible answers, assuming our indices start from $n = 0$.

(a) n^2

(b) $(-2)^n$

(c) $\frac{n+1}{n+2}$

(d) $1.5 + \frac{n}{2}$

A-14: (a) ∞ (b) $\dfrac{3}{4}$ (c) 0

A-15: ∞

A-16: 0

A-17: 0

A-18: 0

A-19: 1

A-20: 0

A-21: ∞

A-22: $\lim\limits_{k\to\infty} a_k = 0$.

A-23: The sequence converges to 0.

A-24: 9

A-25: There are infinitely many potential answers to these questions. Only several are given below.

(a) $a_n = \frac{9}{2}n^2 - \frac{3}{2}n + 1$: $\quad b_n = 4^n$;

(b) $c_n = n^3 - 3n^2 + 5n + 3$: $\quad d_n = 3(n+1)$

(c) $e_n = n(n-1)(n-2)$: $\quad f_n = 0$

A-26: any negative value

A-27: $\ln 2$

A-28: 5

A-29: $-\infty$

A-30: $100 \cdot 2^{99}$.

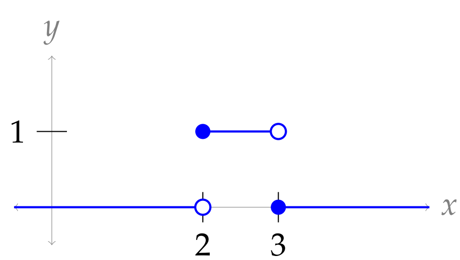
A-31: Possible answers are $\{a_n\} = \left\{ n \left[f\left(a + \frac{1}{n}\right) - f(a) \right] \right\}$ or $\{a_n\} = \left\{ n \left[f(a) - f\left(a - \frac{1}{n}\right) \right] \right\}$.

A-32: (a) $A_n = \frac{n}{2} \sin\left(\frac{2\pi}{n}\right)$ $\qquad$ (b) π

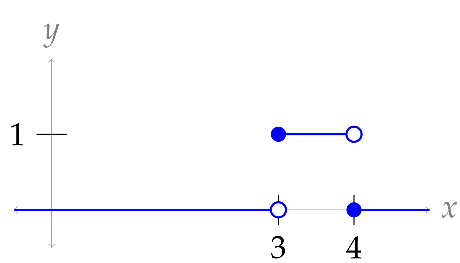
A-33: There are 32 or 33 equation in the text, and around 212 million to 219 million remaining readers under the assumptions of the second paragraph.

A-34:

(a)



(b)



(c) $A_n = 1$ for all n

(d) $\lim_{n\to\infty} A_n = 1$.

(e) $g(x) = 0$

(f) $\int_0^\infty g(x)\,\mathrm{d}x = 0$.

A-35: e^3

A-36: (a) 4 (b) $x = 4$ (c) see solution

A-37: (a) decreasing (b)$f_n = \frac{1}{n} f_1$ (c) 2% (d) 0.18%
(e) "be": 11,019,308; "and": 7,346,205

A-38:

(a) I: $P(1+r)^2$ II: $P(1+r)^2 + r \cdot P(1+r)^2$ II: $P(1+r)^3$

(b) $121

(c) about $225.40

Answers to Exercises 5.1.1 — Jump to TABLE OF CONTENTS

A-1: 165, 225, and 300

A-2: It's not even-tempered. The octave is divided into 5 intervals.

A-3:

0. 444
1. $444 \cdot 2^{1/12} \approx 470.40$
2. $444 \cdot 2^{2/12} \approx 498.37$
3. $444 \cdot 2^{3/12} \approx 528.01$
4. $444 \cdot 2^{4/12} \approx 559.40$
5. $444 \cdot 2^{5/12} \approx 592.67$
6. $444 \cdot 2^{6/12} \approx 627.91$
7. $444 \cdot 2^{7/12} \approx 665.25$
8. $444 \cdot 2^{8/12} \approx 704.81$
9. $444 \cdot 2^{9/12} \approx 746.72$
10. $444 \cdot 2^{10/12} \approx 791.12$
11. $444 \cdot 2^{11/12} \approx 838.16$
12. $444 \cdot 2^{12/12} = 888$

A-4:

0. 100
1. $100 \cdot 2^{1/10} \approx 107.12$
2. $100 \cdot 2^{2/10} \approx 114.87$
3. $100 \cdot 2^{3/10} \approx 1213.11$

4. $100 \cdot 2^{4/10} \approx 131.95$
5. $100 \cdot 2^{5/10} \approx 141.42$
6. $100 \cdot 2^{6/10} \approx 151.57$
7. $100 \cdot 2^{7/10} \approx 162.45$
8. $100 \cdot 2^{8/10} \approx 174.11$
9. $100 \cdot 2^{9/10} \approx 186.61$
10. $100 \cdot 2^{10/10} = 200$

A-5:

1. yes
2. no

A-6: The transposition can be played for *any* lowest note a_k.

A-7: none

A-8: In each question, we specified subdivisions of an octave. An octave has ratio 2. So, in both cases, we had a geometric series

$$a_n = ar^n$$

where $a_k = 2a_0$ was specified. That's where the 2 comes from.

A-9: 360, 660, 990

Answers to Exercises 5.2 — Jump to TABLE OF CONTENTS

A-1:

N	$\mathbf{S_N}$
1	1
2	$1 + \frac{1}{2}$
3	$1 + \frac{1}{2} + \frac{1}{3}$
4	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
5	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

A-2: 3

A-3: (a) $a_n = \begin{cases} \frac{1}{2} & \text{if } n = 1 \\ \dfrac{1}{n(n+1)} & \text{else} \end{cases}$ (b) 0 (c) 1

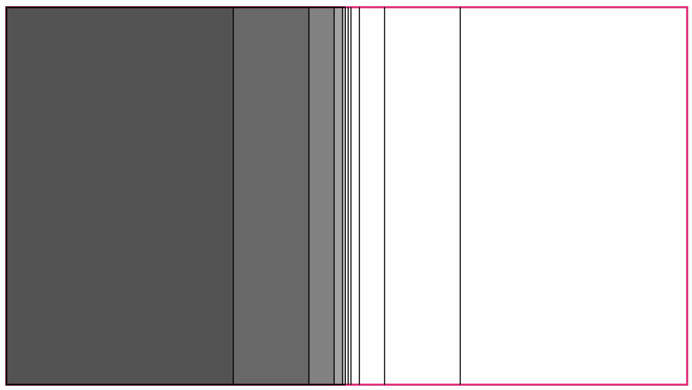
A-4: $a_n = \begin{cases} 0 & \text{if } n = 1 \\ 2(-1)^n - \frac{1}{n(n-1)} & \text{else} \end{cases}$

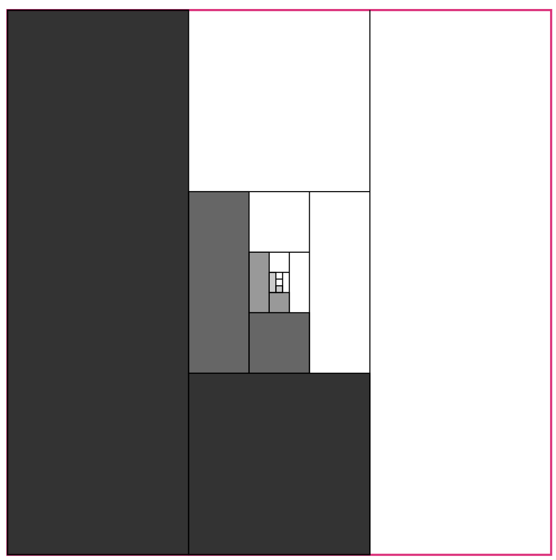
A-5: $a_n < 0$ for all $n \geqslant 2$

A-6: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4^n}$ (b) $\dfrac{2}{3}$

A-7: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9^n}$ (b) $\dfrac{1}{8}$

A-8: Two possible pictures:





A-9: $\dfrac{5^{101}-1}{4 \cdot 5^{100}}$

A-10: All together, there were 36 cookies brought by Student 11 through Student 20.

A-11: $\dfrac{5^{51}-1}{4 \cdot 5^{100}}$

A-12: (a) As time passes, your gains increase, approaching \$1. (b) 1
(c) As time passes, you *lose* more and more money, without bound. (d) $-\infty$

A-13: $A + B + C - c_1$

A-14: in general, false

A-15: $\dfrac{3}{2}$

A-16: $\dfrac{1}{7\times 8^6}$

A-17: 6

A-18: $\cos\left(\dfrac{\pi}{3}\right)-\cos(0)=-\dfrac{1}{2}$

A-19: (a) $a_n=\dfrac{11}{16n^2+24n+5}$ (b) $\dfrac{3}{4}$

A-20: $\dfrac{24}{5}$

A-21: $\dfrac{7}{30}$

A-22: $\dfrac{263}{99}$

A-23: $\dfrac{321}{999}=\dfrac{107}{333}$

A-24: 3

A-25: $\dfrac{1}{2}+\dfrac{5}{7}=\dfrac{17}{14}$

A-26: $\dfrac{40}{3}$

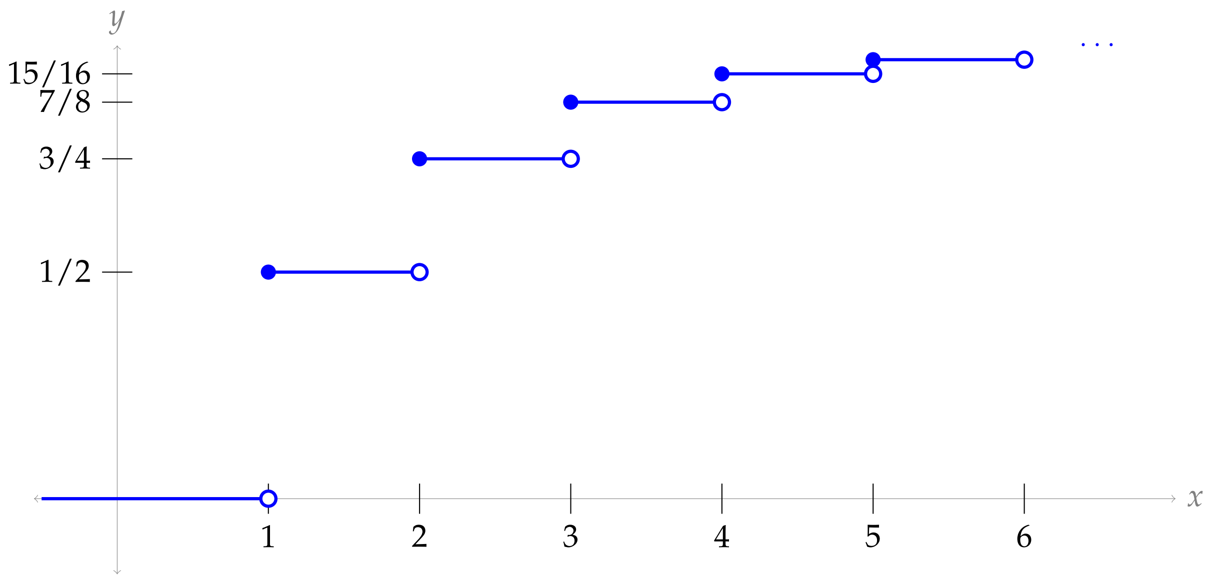
A-27: The series diverges to $-\infty$.

A-28: $-\dfrac{1}{2}$

A-29:

$$F(x)=\begin{cases}0 & x<1\\ 1-\frac{1}{2^{\lfloor x\rfloor}} & x\geqslant 1\end{cases}$$

where $\lfloor x\rfloor$ is the integer obtained from x by rounding down. Sketched:



A-30: $\dfrac{4\pi}{3\left(\pi^3-1\right)}$

A-31: $\dfrac{\sin^2 3}{8}+32 \approx 32.0025$

A-32:

(a)

$$\begin{aligned} i(i+1)(i+2)-(i-1)i(i+1) &= i(i^2+3i+2)-i(i^2-1) \\ &= i(i^2-i^2+3i+2+1) \\ &= i(3i+3) = 3i^2+3i \end{aligned}$$

(b) We'll start by evaluating the telescoping sum

$$\sum\big(i(i+1)(i+2)-(i-1)i(i+1)\big)$$

We can use tables, but this is actually a simpler relationship than some of the other examples we've seen.

$\mathbf{n}$	$\mathbf{i(i+1)(i+2)}$	$\mathbf{-(i-1)i(i+1)}$	$\mathbf{s_n}$
1	$1\cdot 2\cdot 3$	$-0\cdot 1\cdot 2$	$1\cdot 2\cdot 3$
2	$2\cdot 3\cdot 4$	$-1\cdot 2\cdot 3$	$2\cdot 3\cdot 4$
3	$3\cdot 4\cdot 5$	$-2\cdot 3\cdot 4$	$3\cdot 4\cdot 5$
4	$4\cdot 5\cdot 6$	$-3\cdot 4\cdot 5$	$4\cdot 5\cdot 6$
$\vdots$			
n			$n\cdot(n+1)\cdot(n+2)$

So:

$$\begin{aligned}
\sum_{i=1}^{n}(3i^2+3i) &= \sum\big(i(i+1)(i+2)-(i-1)i(i+1)\big) = n(n+1)(n+2) \\
3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i &= n(n+1)(n+2) \\
\sum_{i=1}^{n} i^2 &= \frac{1}{3}n(n+1)(n+2) - \sum_{i=1}^{n} \\
&= \frac{1}{3}n(n+1)(n+2) - \frac{n(n+1)}{2} \\
&= n(n+1)\left(\frac{n+2}{3}-\frac{1}{2}\right) \\
&= n(n+1)\left(\frac{2(n+2)-3}{6}\right) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

(c) First, note:

$$\begin{aligned}
i^2(i+1)^2-(i-1)^2i^2 &= i^2\Big(i^2+2i+1-(i^2-2i+1)\Big) \\
&= i^2\Big(4i\Big) = 4i^3
\end{aligned}$$

So,

$$\sum_{i=1}^{n} 4i^3 = \sum_{i=1}^{n} \left(i^2(i+1)^2 - (i-1)^2 i^2\right)$$
$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\sum_{i=1}^{n} \left(i^2(i+1)^2 - (i-1)^2 i^2\right)$$

This is a telescoping series.

n	$\mathbf{i^2(i+1)^2}$	$\mathbf{-(i-1)^2 i^2}$	$\mathbf{s_n}$
1	$1^2 \cdot 2^2$	$-0^2 \cdot 1^2$	$1^2 \cdot 2^2$
2	$2^2 \cdot 3^2$	$-1^2 \cdot 2^2$	$2^2 \cdot 3^2$
3	$3^2 \cdot 4^2$	$-2^2 \cdot 3^2$	$3^2 \cdot 4^2$
4	$4^2 \cdot 5^2$	$-3^2 \cdot 4^2$	$4^2 \cdot 5^2$
$\vdots$			
n			$n^2 \cdot (n+1)^2$

All together,

$$\sum_{i=1}^{n} i^3 = \frac{1}{4} \cdot n^2 \cdot (n+1)^2$$

A-33: $a_n = \begin{cases} \frac{2}{n(n-1)(n-2)} & \text{if } n \geqslant 3, \\ -\frac{5}{2} & \text{if } n = 2, \\ 2 & \text{if } n = 1 \end{cases}$

A-34: Yes, $f(x)$ could be a PDF.

A-35: $\frac{1}{2} - \frac{1}{1001} + \frac{1}{1002}$

A-36: $\dfrac{5}{8}$

Answers to Exercises 5.3 — Jump to TABLE OF CONTENTS

A-1: (B), (C)

A-2: (A)

A-3: (a) I am old (b) not enough information to tell
(c) not enough information to tell (d) I am young

A-4:

	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$ CONVERGES	inconclusive
and if $\{a_n\}$ is the blue series	inconclusive	then $\sum b_n$ DIVERGES

A-5: (a) both direct comparison and limit comparison (b) direct comparison (c) limit comparison (d) neither

A-6: It diverges by the divergence test, because $\lim\limits_{n\to\infty} a_n \neq 0$.

A-7: We cannot use the divergence test to show that a series converges. It is inconclusive in this case.

A-8: The integral test does not apply because $f(x)$ is not decreasing.

A-9: The inequality goes the wrong way, so the direct comparison test (with this comparison series) is inconclusive.

A-10: One possible answer: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

A-11: By the divergence test, for a series $\sum a_n$ to *converge*, we need $\lim\limits_{n\to\infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller. If $\lim\limits_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right| < 1$ or (equivalently) $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1$, then $|a_{n+1}| > |a_n|$ for sufficiently large n, so the terms are actually *growing* in magnitude. That means the series diverges, by the divergence test.

A-12: One possible answer: $f(x) = \sin(\pi x)$, $a_n = 0$ for every n.

By the integral test, any answer will use a function $f(x)$ that is *not* both positive and decreasing.

A-13: One possible answer: $b_n = \dfrac{2^n}{3^n}$

A-14: (a) In general false. The harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ provides a counterexample.

(b) In general false. If $a_n = (-1)^n \frac{1}{n}$, then $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ is again the harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, which diverges.

(c) In general false. Take, for example, $a_n = 0$ and $b_n = 1$.

A-15: No. It diverges.

A-16: It diverges.

A-17: The series diverges.

A-18: It diverges.

A-19: This is a geometric series with $r = 1.001$. Since $|r| > 1$, it is divergent.

A-20: The series converges to $-\dfrac{1}{50}$.

A-21: The series converges.

A-22: It diverges.

A-23: The series converges.

A-24: The series converges to $\dfrac{1}{3}$.

A-25: The series converges.

A-26: It converges.

A-27: Let $f(x) = \dfrac{5}{x(\ln x)^{3/2}}$. Then $f(x)$ is positive and decreases as x increases. So the sum $\displaystyle\sum_{3}^{\infty} f(n)$ and the integral $\displaystyle\int_3^\infty f(x)\,\mathrm{d}x$ either both converge or both diverge, by the integral test, which is Theorem 5.3.5 in the text. For the integral, we use the substitution $u = \ln x$, $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ to get

$$\int_3^\infty \frac{5\,\mathrm{d}x}{x(\ln x)^{3/2}} = \int_{\ln 3}^\infty \frac{5\,\mathrm{d}u}{u^{3/2}}$$

which converges by the p–test (which is Example 3.10.8 in the text) with $p = \frac{3}{2} > 1$.

A-28: $p > 1$

A-29: It converges.

A-30: The series $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{3}}{n^2}$ converges by the p–test with $p = 2$.

Note that

$$0 < a_n = \frac{\sqrt{3n^2 - 7}}{n^3} < \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2}$$

for all $n \geqslant 2$. As the series $\sum_{n=2}^{\infty} \frac{\sqrt{3}}{n^2}$ converges, the comparison test says that $\sum_{n=2}^{\infty} \frac{\sqrt{3n^2-7}}{n^3}$ converges too.

A-31: The series converges.

A-32: It diverges.

A-33: The series diverges.

A-34: (a) converges (b) diverges

A-35: $\dfrac{1}{e^5 - e^4}$

A-36: $\frac{1}{7}$

A-37: (a) diverges by limit comparison with the harmonic series

(b) converges by the ratio test

A-38: (a) Converges by the limit comparison test with $b = \frac{1}{k^{5/3}}$.

(b) Diverges by the ratio test.

(c) Diverges by the integral test.

A-39: It converges.

A-40: (a) converges (b) converges

A-41: (a) See the solution.

(b) $f(x) = \dfrac{x + \sin x}{1 + x^2}$ is not a decreasing function.

(c) See the solution.

A-42: The sum is between 0.9035 and 0.9535.

A-43: Since $\lim_{n\to\infty} a_n = 0$, there must be some integer N such that $\frac{1}{2} > a_n \geqslant 0$ for all $n > N$. Then, for $n > N$,

$$\frac{a_n}{1 - a_n} \leqslant \frac{a_n}{1 - 1/2} = 2a_n$$

From the information in the problem statement, we know

$$\sum_{n=N+1}^{\infty} 2a_n = 2 \sum_{n=N+1}^{\infty} a_n \qquad \text{converges.}$$

So, by the direct comparison test,

$$\sum_{n=N+1}^{\infty} \frac{a_n}{1 - a_n} \qquad \text{converges as well.}$$

Since the convergence of a series is not affected by its first N terms, as long as N is finite, we conclude

$$\sum_{n=1}^{\infty} \frac{a_n}{1 - a_n} \qquad \text{converges.}$$

A-44: It diverges.

A-45: It converges to $-\ln 2 = \ln \frac{1}{2}$,

A-46: See the solution.

A-47: About 9% to 10%

A-48: The total population is between 29,820,091 and 29,244,727 people.

Answers to Exercises 5.4 — Jump to TABLE OF CONTENTS

A-1: False. For example, $b_n = \frac{1}{n}$ provides a counterexample.

A-2:

	$\sum a_n$ **converges**	$\sum a_n$ **diverges**
$\sum \lvert a_n \rvert$ **converges**	converges absolutely	not possible
$\sum \lvert a_n \rvert$ **diverges**	converges conditionally	diverges

A-3: conditionally convergent

A-4: The series diverges.

A-5: It diverges.

A-6: It converges absolutely.

A-7: It converges absolutely.

A-8: It diverges.

A-9: It converges absolutely.

A-10: See solution.

A-11: See solution.

A-12: See solution.

A-13: (a) See the solution. (b) $|S - S_5| \leqslant 24 \times 36e^{-6^3}$

A-14: $\cos 1 \approx \frac{389}{720}$; the actual associated error (using a calculator) is about 0.000025.

A-15: See solution.

Answers to Exercises 5.5 — Jump to TABLE OF CONTENTS

A-1: 2

A-2: $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{n(x-5)^{n-1}}{n!+2}$

A-3: only $x = c$

A-4: $R = 6$

A-5: (a) $R = \dfrac{1}{2}$ (b) $\dfrac{2}{1+2x}$ for all $|x| < \dfrac{1}{2}$

A-6: $R = \infty$

A-7: 1

A-8: The radius of convergence is 1

A-9: The radius of convergence is 3.

A-10: 5

A-11: $(-3, -1)$

A-12: The interval of convergence is from $\frac{3}{4}$ to $\frac{5}{4}$; we don't need to find out whether the endpoints are included or not.

A-13: The radius of convergence is 2. The interval of convergence is from -1 to 3; we don't know whether or not the endpoints are included.

A-14: The interval of convergence is from $a-1$ to $a+1$. We don't know which endpoints are included.

A-15: (a) from -10 to 8 (b) This series converges only for $x = 1$.

A-16: $\displaystyle\sum_{n=0}^{\infty} x^{n+3} = \sum_{n=3}^{\infty} x^n$

A-17: $\displaystyle f(x) = 3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)}$

A-18: The series converges absolutely for $|x| < 9$, converges conditionally for $x = -9$ and diverges otherwise.

A-19: (a) See the solution.

(b) $\displaystyle\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$. The series converges for $-1 < x < 1$.

A-20: See the solution.

A-21: (a) 1. (b) The series converges for x between -1 and 1.

A-22: The point $x = c$ corresponds to a local maximum if $A_2 < 0$ and a local minimum if $A_2 > 0$.

A-23: $\dfrac{13}{80}$

Answers to Exercises 5.6 — Jump to TABLE OF CONTENTS

A-1: A: linear B: constant C: quadratic

A-2: $T(5) = \arctan^3\left(e^5 + 7\right)$

A-3: A - V B - I C - IV D - VI E - II F - III

A-4: (a) $f^{(20)}(3) = 20^2\left(\dfrac{20!}{20!+1}\right)$ (b) $g^{(20)}(3) = 10^2\left(\dfrac{20!}{10!+1}\right)$

(c) $h^{(20)}(0) = 0; \quad h^{(22)}(0) = \dfrac{22!\cdot 5^{13}}{13}$

A-5: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n$

A-6: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$

A-7: $\displaystyle\frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{10-x}{10}\right)^n$ with interval of convergence $(0, 20)$.

A-8: $\displaystyle\sum_{n=0}^{\infty}\frac{3^n e^{3a}}{n!}(x-a)^n$, with infinite radius of convergence

A-9: $\displaystyle-\sum_{n=0}^{\infty}2^n x^n$

A-10: $b_n = 3(-1)^n + 2^n$

A-11: $c_5 = \dfrac{3^5}{5!}$

A-12: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^{n+1}x^{n+1}}{n+1}$ for all $|x| < \frac{1}{2}$

A-13: $a = 1, b = -\dfrac{1}{3!} = -\dfrac{1}{6}$.

A-14: $\displaystyle\int\frac{e^{-x^2}-1}{x}\,\mathrm{d}x = C - \frac{x^2}{2} + \frac{x^4}{8} + \cdots$.

It is not clear from the wording of the question whether or not the arbitrary constant C is to be counted as one of the "first two nonzero terms".

A-15: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+6}}{(2n+1)(2n+6)} + C = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n+6}}{(2n+1)(n+3)} + C$

A-16: $\displaystyle f(x) = 1 + \sum_{n=0}^{\infty}(-1)^n\frac{3^n}{3n+2}x^{3n+2}$

A-17: $\dfrac{\pi}{2\sqrt{3}}$

A-18: $\dfrac{1}{e}$

A-19: $e^{1/e}$

A-20: $e^{1/\pi} - 1$

A-21: $\ln(3/2)$

A-22: $(e+2)e^e - 2$

A-23: The sum diverges–see the solution.

A-24: $\dfrac{1+\sqrt{2}}{\sqrt{2}}$

A-25: (a) See the solution. (b) $\dfrac{1}{2}\left(e+\dfrac{1}{e}\right)$

A-26: S_{13} or higher

A-27: S_9 or higher

A-28: S_{18} or higher

A-29: The error is in the interval $\left(\dfrac{-5^7}{14\cdot 3^7}\left[1+\dfrac{1}{3^7}\right] \quad , \quad \dfrac{-5^7}{7\cdot 6^7}\right) \approx (-0.199, -0.040)$

A-30: -1

A-31: $\dfrac{1}{5!} = \dfrac{1}{120}$

A-32: e^2

A-33: $\sqrt{e}$

A-34: $\dfrac{2}{(6/7)^3} = \dfrac{343}{108}$

A-35: $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+4}}{(2n+1)(2n+2)} = x^3 \arctan x - \frac{x^2}{2}\ln(1+x^2)$

A-36: (a) the Maclaurin series for $f(x)$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}\,(n!)^2}x^n$, and its radius of convergence is $R=1$.

(b) the Maclaurin series for $\arcsin x$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}\,(n!)^2(2n+1)}x^{2n+1}$, and its radius of convergence is $R=1$.

A-37: $\ln(x) = \ln 2 + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,2^n}(x-2)^n$. It converges when $0 < x \leqslant 4$.

A-38: $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{4n+1}$

A-39: $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n\,n!}$

A-40: (a) $\Sigma(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)(2n+1)!}$ (b) $x = \pi$ (c) 1.8525

A-41: $I(x) = \sum_{n=1}^{\infty} (-1)^n \dfrac{x^{2n-1}}{(2n)!(2n-1)}$

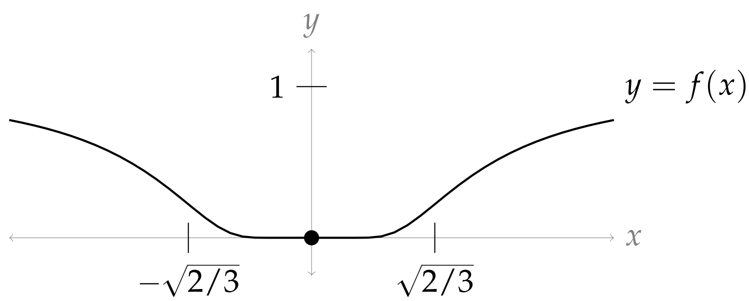
A-42: $I(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{2n-1}}{(2n)!} = \dfrac{1}{2!}x - \dfrac{1}{4!}x^3 + \dfrac{1}{6!}x^5 - \dfrac{1}{8!}x^8 + \cdots$

A-43: (a) See the solution. (b) The series converges for all x.

A-44: See the solution.

A-45: (a) $\cosh(x) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \dfrac{x^n}{n!} = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$ for all x.

A-46: (a)



(b) the constant function 0 (c) everywhere (d) only at $x = 0$

A-47: 0

Part IV

SOLUTIONS TO QUESTIONS

Solutions to Exercises 1.1 — Jump to TABLE OF CONTENTS

S-1:

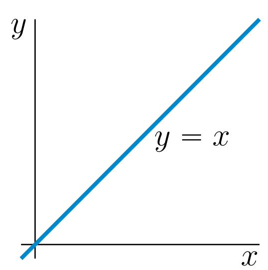
The xz plane is filled with vertical lines; the yz plane is crosshatched; and the xy plane is solid.

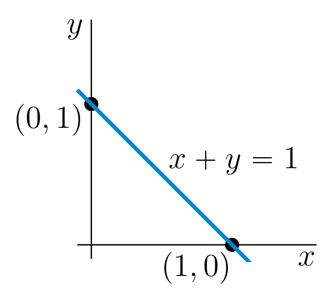
The left bottom triangle vertex is $(1,0,0)$; the right bottom triangle vertex is $(0,1,0)$; the top triangle vertex is $(0,0,1)$.

S-2: (a) The point (x,y,z) satisfies $x^2+y^2+z^2 = 2x-4y+4$ if and only if it satisfies $x^2-2x+y^2+4y+z^2 = 4$, or equivalently $(x-1)^2+(y+2)^2+z^2 = 9$. Since $\sqrt{(x-1)^2+(y+2)^2+z^2}$ is the distance from $(1,-2,0)$ to (x,y,z), our point satisfies the given equation if and only if its distance from $(1,-2,0)$ is three. So the set is the sphere of radius 3 centered on $(1,-2,0)$.

(b) As in part (a), $x^2+y^2+z^2 < 2x-4y+4$ if and only if $(x-1)^2+(y+2)^2+z^2 < 9$. Hence our point satifies the given inequality if and only if its distance from $(1,-2,0)$ is strictly smaller than three. The set is the interior of the sphere of radius 3 centered on $(1,-2,0)$.

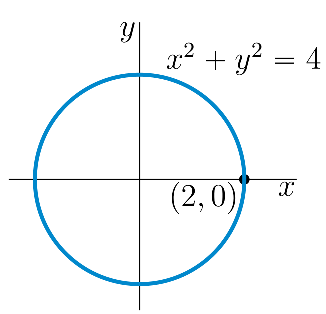
S-3: (a) $x = y$ is a straight line and passes through the points $(0,0)$ and $(1,1)$. So it is the straight line through the origin that makes an angle $45°$ with the x– and y–axes. It is sketched in the figure on the left below.

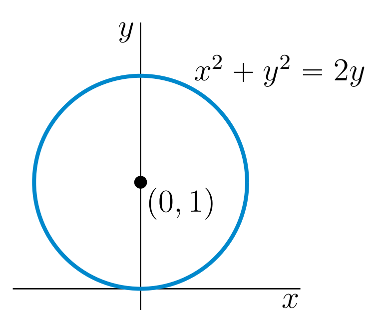




(b) $x+y = 1$ is the straight line through the points $(1,0)$ and $(0,1)$. It is sketched in the figure on the right above.

(c) x^2+y^2 is the square of the distance from $(0,0)$ to (x,y). So $x^2+y^2 = 4$ is the circle with centre $(0,0)$ and radius 2. It is sketched in the figure on the left below.



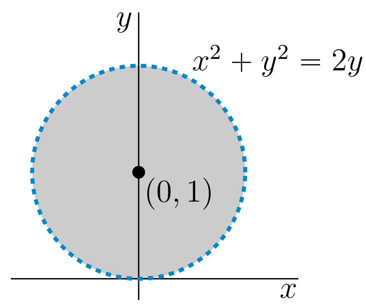


(d) The equation $x^2 + y^2 = 2y$ is equivalent to $x^2 + (y-1)^2 = 1$. As $x^2 + (y-1)^2$ is the square of the distance from $(0,1)$ to (x,y), $x^2 + (y-1)^2 = 1$ is the circle with centre $(0,1)$ and radius 1. It is sketched in the figure on the right above.

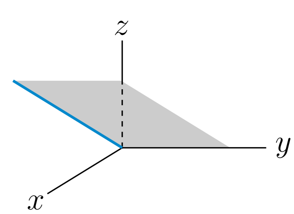
(e) As in part (d),

$$x^2 + y^2 < 2y \iff x^2 + y^2 - 2y < 0 \iff x^2 + y^2 - 2y + 1 < 1 \iff x^2 + (y-1)^2 < 1$$

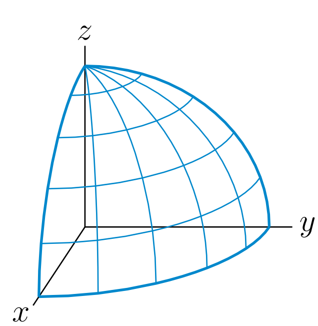
As $x^2 + (y-1)^2$ is the square of the distance from $(0,1)$ to (x,y), $x^2 + (y-1)^2 < 1$ is the set of points whose distance from $(0,1)$ is strictly less than 1. That is, it is the set of points strictly inside the circle with centre $(0,1)$ and radius 1. That set is the shaded region (not including the dashed circle) in the sketch below.



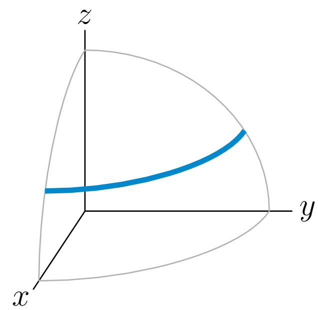
S-4: (a) For each fixed y_0, $z = x,\ y = y_0$ is a straight line that lies in the plane, $y = y_0$ (which is parallel to the plane containing the x and z axes and is a distance y_0 from it). This line passes through $x = z = 0$ and makes an angle 45° with the xy–plane. Such a line (with $y_0 = 0$) is sketched in the figure below. The set $z = x$ is the union of all the lines $z = x,\ y = y_0$ with all values of y_0. As y_0 varies $z = x,\ y = y_0$ sweeps out the plane which contains the y–axis and which makes an angle 45° with the xy–plane. Here is a sketch of the part of the plane that is in the first octant.



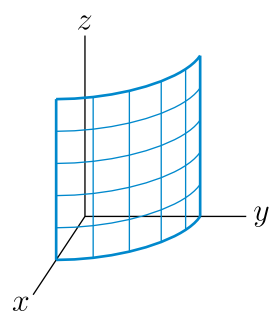
(b) $x^2 + y^2 + z^2$ is the square of the distance from $(0,0,0)$ to (x,y,z). So $x^2 + y^2 + z^2 = 4$ is the set of points whose distance from $(0,0,0)$ is 2. It is the sphere with centre $(0,0,0)$ and radius 2. Here is a sketch of the part of the sphere that is in the first octant.



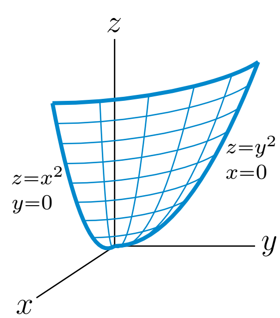
(c) $x^2 + y^2 + z^2 = 4$, $z = 1$ or equivalently $x^2 + y^2 = 3$, $z = 1$, is the intersection of the plane $z = 1$ with the sphere of centre $(0, 0, 0)$ and radius 2. It is a circle in the plane $z = 1$ that has centre $(0, 0, 1)$ and radius $\sqrt{3}$. The part of the circle in the first octant is the heavy quarter circle in the sketch



(d) For each fixed z_0, $x^2 + y^2 = 4$, $z = z_0$ is a circle in the plane $z = z_0$ with centre $(0, 0, z_0)$ and radius 2. So $x^2 + y^2 = 4$ is the union of $x^2 + y^2 = 4$, $z = z_0$ for all possible values of z_0. It is a vertical stack of horizontal circles. It is the cylinder of radius 2 centered on the z–axis. Here is a sketch of the part of the cylinder that is in the first octant.



(e) For each fixed $z_0 \geqslant 0$, the curve $z = x^2 + y^2$, $z = z_0$ is the circle in the plane $z = z_0$ with centre $(0, 0, z_0)$ and radius $\sqrt{z_0}$. As $z = x^2 + y^2$ is the union of $z = x^2 + y^2$, $z = z_0$ for all possible values of $z_0 \geqslant 0$, it is a vertical stack of horizontal circles. The intersection of the surface with the yz–plane is the parabola $z = y^2$. Here is a sketch of the part of the paraboloid that is in the first octant.



S-5: From the text, the distance from the point (x, y, z) to the point (x', y', z') is

$$\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

So, our distance is

$$\sqrt{(1-4)^2+(2-(-5))^2+(3-6)^2} = \sqrt{9+49+9} = \sqrt{67}$$

S-6: From the text, the distance from the point (x,y,z) to the xy-plane is $|z|$. In this case, 9.

S-7: From the text, the distance from the point (x,y,z) to the xy-plane is $|z|$. Let the nest be the origin $(0,0,0)$ with the z-axis pointing north, the x-axis pointing south, and the y-axis pointing east. Then the bird's coordinates after flying are $(-1,2,0.1)$. So, its distance from its nest is

$$\sqrt{(-1-0)^2+(2-0)^2+(0.1-0)^2} = \sqrt{1+4+0.01} = \sqrt{5.01}\text{ km}$$

S-8: Let the nest be the origin $(0,0,0)$ with the z-axis pointing north, the x-axis pointing south, and the y-axis pointing east. From the text, the distance from the point (x,y,z) to the xy-plane (which, in this case, is the ground) is $|z|$. Then the bird's coordinates after flying are $(-2,2,z)$. So,

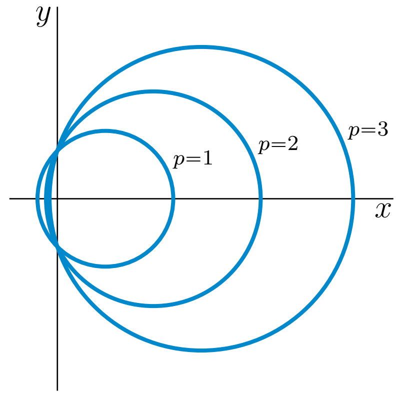
$$\begin{aligned} 3 &= \sqrt{(-2-0)^2+(2-0)^2+(z-0)^2} = \sqrt{4+4+z^2} \\ 9 &= 8+z^2 \\ |z| &= 1 \end{aligned}$$

So, the bird is 1 km above the ground. (Or, possibly, 1 km below it.)

S-9: The first 2 km of the journey bring you 2 km away from the wall. Walking parallel to the wall neither increases nor decreases your distance to the wall. Similarly, moving vertically neither increases nor decreases your distance to the wall. So, the murder hornets are 2 km from the wall.

If we wanted to impose a coordinate system, we could place the wall as the xz axis, with z being the vertical direction, and the origin the place where you started walking. Then the murder hornets are at the point $(1,2,0.003)$. The distance from (x,y,z) to the xz axis is $|y|$. In this case, 2 km.

S-10: For each fixed c, the isobar $p(x,y) = c$ is the curve $x^2-2cx+y^2 = 1$, or equivalently, $(x-c)^2+y^2 = 1+c^2$. This is a circle with centre $(c,0)$ and radius $\sqrt{1+c^2}$, which for large c is just a bit bigger than c.



S-11: Let (x, y, z) be a point in P. The distances from (x, y, z) to $(3, -2, 3)$ and to $(3/2, 1, 0)$ are

$$\sqrt{(x-3)^2 + (y+2)^2 + (z-3)^2} \quad \text{and} \quad \sqrt{(x-3/2)^2 + (y-1)^2 + z^2}$$

respectively. To be in P, (x, y, z) must obey

$$\begin{aligned}
\sqrt{(x-3)^2 + (y+2)^2 + (z-3)^2} &= 2\sqrt{(x-3/2)^2 + (y-1)^2 + z^2} \\
(x-3)^2 + (y+2)^2 + (z-3)^2 &= 4(x-3/2)^2 + 4(y-1)^2 + 4z^2 \\
x^2 - 6x + 9 + y^2 + 4y + 4 + z^2 - 6z + 9 &= 4x^2 - 12x + 9 + 4y^2 - 8y + 4 + 4z^2 \\
3x^2 - 6x + 3y^2 - 12y + 3z^2 + 6z - 9 &= 0 \\
x^2 - 2x + y^2 - 4y + z^2 + 2z - 3 &= 0 \\
(x-1)^2 + (y-2)^2 + (z+1)^2 &= 9
\end{aligned}$$

This is a sphere of radius 3 centered on $(1, 2, -1)$.

S-12: Call the centre of the circumscribing circle $(\bar{x}, \bar{y})$. This centre must be equidistant from the three vertices. So

$$\bar{x}^2 + \bar{y}^2 = (\bar{x} - a)^2 + \bar{y}^2 = (\bar{x} - b)^2 + (\bar{y} - c)^2$$

or, subtracting $\bar{x}^2 + \bar{y}^2$ from the three equal expressions,

$$0 = a^2 - 2a\bar{x} = b^2 - 2b\bar{x} + c^2 - 2c\bar{y}$$

which implies

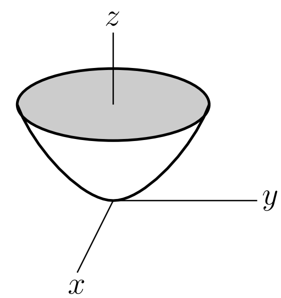
$$\bar{x} = \frac{a}{2} \qquad \bar{y} = \frac{b^2 + c^2 - 2b\bar{x}}{2c} = \frac{b^2 + c^2 - ab}{2c}$$

The radius is the distance from the vertex $(0, 0)$ to the centre $(\bar{x}, \bar{y})$, which is $\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b^2+c^2-ab}{2c}\right)^2}$.

S-13: The distance from P to the point $(0, 0, 1)$ is $\sqrt{x^2 + y^2 + (z-1)^2}$. The distance from P to the specified plane is $|z+1|$. Hence the equation of the surface is

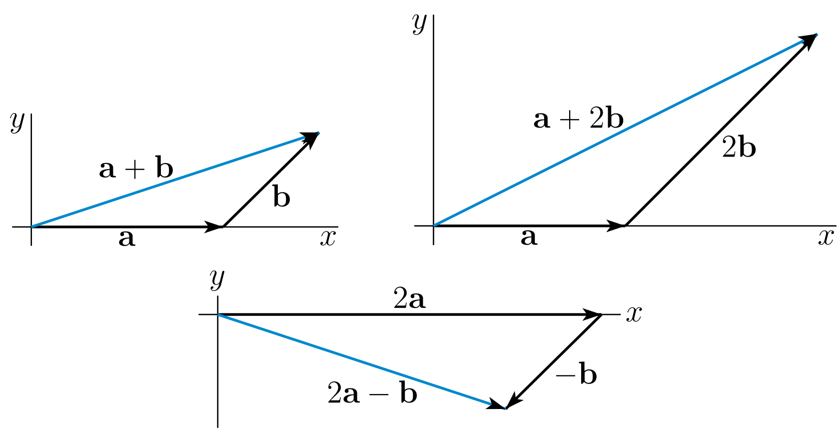
$$x^2 + y^2 + (z-1)^2 = (z+1)^2 \text{ or } x^2 + y^2 = 4z$$

All points on this surface have $z \geqslant 0$. The set of points on the surface that have any fixed value, $z_0 \geqslant 0$, of z consists of a circle that is centred on the z–axis, is parallel to the xy-plane and has radius $2\sqrt{z_0}$. The surface consists of a stack of these circles, starting with a point at the origin and with radius increasing vertically. The surface is a paraboloid and is sketched below.

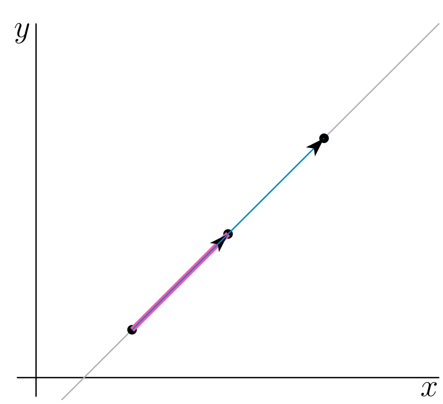


Solutions to Exercises 1.2 — Jump to TABLE OF CONTENTS

S-1: $\mathbf{a}+\mathbf{b} = \langle 3,1\rangle$, $\mathbf{a}+2\mathbf{b} = \langle 4,2\rangle$, $2\mathbf{a}-\mathbf{b} = \langle 3,-1\rangle$



S-2: If three points are collinear, then the vector from the first point to the second point, and the vector from the first point to the third point must both be parallel to the line, and hence must be parallel to each other (i.e. must be multiples of each other).



(a) The vectors $\langle 0,3,7\rangle - \langle 1,2,3\rangle = \langle -1,1,4\rangle$ and $\langle 3,5,11\rangle - \langle 1,2,3\rangle = \langle 2,3,8\rangle$ are not parallel (i.e. are not multiples of each other), so the three points are not on the same line.

(b) The vectors $\langle 1,2,-2\rangle - \langle 0,3,-5\rangle = \langle 1,-1,3\rangle$ and $\langle 3,0,4\rangle - \langle 0,3,-5\rangle = \langle 3,-3,9\rangle$ are parallel (i.e. are multiples of each other), so the three points are on the same line.

S-3: By property 7 of Theorem 1.2.9 in the text,

$$\begin{array}{lrcl}
\text{(a)} & \langle 1,3,2\rangle\cdot\langle 2,-2,2\rangle = 1\times 2-3\times 2+2\times 2 = 0 & \Longrightarrow & \text{perpendicular}\\
\text{(b)} & \langle -3,1,7\rangle\cdot\langle 2,-1,1\rangle = -3\times 2-1\times 1+7\times 1 = 0 & \Longrightarrow & \text{perpendicular}\\
\text{(c)} & \langle 2,1,1\rangle\cdot\langle -1,4,2\rangle = -2\times 1+1\times 4+1\times 2 = 4\neq 0 & \Longrightarrow & \text{not perpendicular}
\end{array}$$

S-4: The vector from $(1,2,3)$ to $(4,0,5)$ is $\langle 3,-2,2\rangle$. The vector from $(1,2,3)$ to $(3,6,4)$ is $\langle 2,4,1\rangle$. The dot product between these two vectors is $\langle 3,-2,2\rangle\cdot\langle 2,4,1\rangle = 0$, so the vectors are perpendicular and the triangle does contain a right angle.

S-5: This statement is false. The two numbers $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\cdot\mathbf{c}$ are equal if and only if $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = 0$. This in turn is the case if and only if $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$ (under the convention that $\mathbf{0}$ is perpendicular to all vectors). For example, if $\mathbf{a} = \langle 1,0,0\rangle$, $\mathbf{b} = \langle 0,1,0\rangle$, $\mathbf{c} = \langle 0,0,1\rangle$, then $\mathbf{b}-\mathbf{c} = \langle 0,1,-1\rangle$ is perpendicular to $\mathbf{a}$ so that $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$.

S-6: You can think of the name of a vector as directions from its tail to its head. The vector $\langle 1,2\rangle$ has a head that is one unit to the right, and two units up, from its tail. So, the vector with head at $\mathbf{a}$ and tail at $\mathbf{b}$ is $\mathbf{a}-\mathbf{b}$.

(a) $\langle 0,1\rangle - \langle 1,0\rangle = \langle -1,1\rangle$

(b) $\langle 1,2,3\rangle - \langle 4,5,4\rangle = \langle -3,-3,-1\rangle$.

(c) Since the tail is at the origin, the location of the head is the same as the name of the vector: $\langle -2,2\rangle$.

(d) $\langle -2,2\rangle - \langle 0,1\rangle = \langle -2,1\rangle$

S-7:

(a) $|\langle 1,2\rangle| = \sqrt{1^2+2^2} = \sqrt{5}$

(b) $\langle -2,1\rangle = \sqrt{(-2)^2+(-1)^2} = \sqrt{5}$

(c) $\langle 2,4\rangle = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$

S-8: [(a)] If a vector $\mathbf{a}$ has length ℓ, then $\frac{1}{\ell}\mathbf{a}$ is its unit vector.

1. $|\langle 3,4\rangle| = \sqrt{3^2+4^2} = 5$, so the unit vector is $\frac{1}{5}\langle 3,4\rangle$, or $\langle \frac{3}{5},\frac{4}{5}\rangle$
2. $|\langle 1,1,1\rangle| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$, so the unit vector is $\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$, or $\left\langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$

3. $|\langle 0,1,0\rangle| = 1$, so this is already a unit vector.

4. $|\langle 7,7,8\rangle| = \sqrt{7^2+7^2+8^2} = \sqrt{162}$, so the unit vector is $\left\langle \frac{7}{\sqrt{162}}, \frac{7}{\sqrt{162}}, \frac{8}{\sqrt{162}} \right\rangle$

5. $2\hat{\boldsymbol{\imath}}$ is twice the unit vector $\hat{\boldsymbol{\imath}}$.

S-9: The magnitude of $\langle 9,0,7\rangle$ is $\sqrt{9^2+0^2+7^2} = \sqrt{130}$. We want a vector of magnitude 783, so we can multiply our original vector by $\pm\frac{783}{\sqrt{130}}$:

$$\pm\frac{783}{\sqrt{130}}\langle 9,0,7\rangle = \pm\left\langle \frac{7047}{\sqrt{130}}, 0, \frac{5481}{\sqrt{130}} \right\rangle$$

S-10:

$$\begin{aligned}
\mathbf{a}+3\mathbf{b} &= \langle 1,2,1\rangle + 3\langle 3,4,5\rangle \\
&= \langle 1+3\cdot 3, 2+4\cdot 3, 1+3\cdot 5\rangle \\
&= \langle 10,14,16\rangle \\
|\mathbf{a}-\mathbf{b}| &= |\langle 1,2,1\rangle - \langle 3,4,5\rangle| \\
&= |\langle -2,-2,-4\rangle| \\
&= \sqrt{(-2)^2+(-2)^2+(-4)^2} \\
&= \sqrt{24}
\end{aligned}$$

S-11: The center of the sphere is $\frac{1}{2}\{(2,1,4)+(4,3,10)\} = (3,2,7)$. The diameter (i.e. twice the radius) is $|(2,1,4)-(4,3,10)| = |(-2,-2,-6)| = 2|(1,1,3)| = 2\sqrt{11}$. The equation of the sphere is

$$(x-3)^2+(y-2)^2+(z-7)^2 = 11$$

S-12: Parallel vectors are nonzero scalar multiples of one another. Perpendicular vectors have a dot product of 0.

(a) $\langle 1,2\rangle\cdot\langle -2,3\rangle = (1)(-2)+(2)(3) = 4$. Since the dot product is nonzero, this pair is not perpendicular. Since they are not scalar multiples of one another, they are also not parallel.

(b) $\langle -1,1\rangle\cdot\langle 1,1\rangle = (-1)(1)+(1)(1) = 0$. Since the dot product is zero, this pair is perpendicular.

(c) $\langle 1,1\rangle\cdot\langle 2,2\rangle = (1)(2)+(1)(2) = 4$. Since $\mathbf{b} = 2\mathbf{a}$, this pair is parallel.

(d) $\langle 1,2,1\rangle\cdot\langle -1,1,1\rangle = (1)(-1)+(2)(1)+(1)(1) = 2$. Since the dot product is nonzero, this pair is not perpendicular. Since they are not scalar multiples of one another, they are also not parallel.

(e) $\langle -1,2,3\rangle \cdot \langle 3,0,1\rangle = (-1)(3)+(2)(0)+(3)(1) = 0$. Since the dot product is zero, this pair is perpendicular.

S-13:

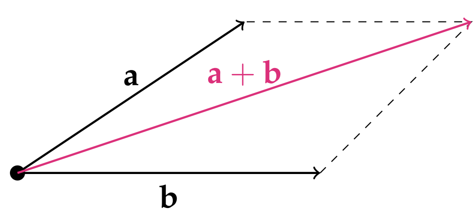
$$\begin{array}{lll} \text{(a)} & \langle 2,4\rangle \cdot \langle 2,y\rangle = 2\times 2 + 4\times y = 4+4y = 0 & \iff \ y=-1 \\ \text{(b)} & \langle 4,-1\rangle \cdot \left\langle y,y^2\right\rangle = 4\times y - 1\times y^2 = 4y-y^2 = 0 & \iff \ y=0,4 \\ \text{(c)} & \langle 3,1,1\rangle \cdot \left\langle 2,5y,y^2\right\rangle = 6+5y+y^2 = 0 & \iff \ y=-2,-3 \end{array}$$

S-14: (a) We want $0 = \mathbf{u}\cdot\mathbf{v} = -2\alpha - 10$ or $\alpha = -5$.

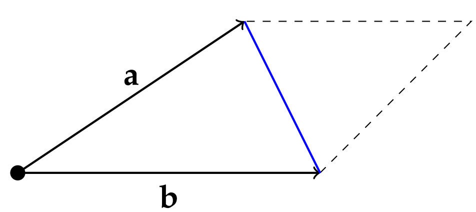
(b) We want $-2/\alpha = 5/(-2)$ or $\alpha = 0.8$.

S-15: Note: we'll gone back and forth treating $\mathbf{a}$ and $\mathbf{b}$ as points and as vectors. This is common! Make sure you know, each time they're used, which meaning is appropriate.

The parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$ has vertices $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$.



The middle point of the diagonal shown above is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.



The midpoint of the diagonal shown above is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

So, the two diagonals of the parallelogram meet in their respective midpoints, at the point $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

S-16: We may choose our coordinate axes so that $A=(0,0,0)$, $B=(s,0,0)$, $C=(s,s,0)$, $D=(0,s,0)$ and $A'=(0,0,s)$, $B'=(s,0,s)$, $C'=(s,s,s)$, $D'=(0,s,s)$.

Then

$$\begin{aligned}
|A'C'| &= |\langle s,s,s\rangle - \langle 0,0,s\rangle| = |\langle s,s,0\rangle| &&= \sqrt{2}\,s\\
|A'B| &= |\langle s,0,0\rangle - \langle 0,0,s\rangle| = |\langle s,0,-s\rangle| &&= \sqrt{2}\,s\\
|A'D| &= |\langle 0,s,0\rangle - \langle 0,0,s\rangle| = |\langle 0,s,-s\rangle| &&= \sqrt{2}\,s\\
|C'B| &= |\langle s,0,0\rangle - \langle s,s,s\rangle| = |\langle 0,-s,-s\rangle| &&= \sqrt{2}\,s\\
|C'D| &= |\langle 0,s,0\rangle - \langle s,s,s\rangle| = |\langle -s,0,-s\rangle| &&= \sqrt{2}\,s\\
|BD| &= |\langle 0,s,0\rangle - \langle s,0,0\rangle| = |\langle -s,s,0\rangle| &&= \sqrt{2}\,s
\end{aligned}$$

S-17: (a) $AA' = \langle 4,0,1\rangle$ and $BB' = \langle 4,0,1\rangle$ are opposite sides of the quadrilateral $AA'B'B$. They have the same length and direction. The same is true for $AB = \langle -1,3,0\rangle$ and $A'B' = \langle -1,3,0\rangle$. So $AA'B'B$ is a parallelogram. Because, $AA' \cdot AB = \langle 4,0,1\rangle \cdot \langle -1,3,0\rangle = -4 \neq 0$, the neighbouring edges of $AA'B'B$ are not perpendicular and so $AA'B'B$ is *not* a rectangle.

Similarly, the quadilateral $ACC'A'$ has opposing sides $AA' = \langle 4,0,1\rangle = CC' = \langle 4,0,1\rangle$ and $AC = \langle -1,0,4\rangle = A'C' = \langle -1,0,4\rangle$ and so is a parallelogram. Because $AA' \cdot AC = \langle 4,0,1\rangle \cdot \langle -1,0,4\rangle = 0$, the neighbouring edges of $ACC'A'$ are perpendicular, so $ACC'A'$ *is* a rectangle.

Finally, the quadilateral $BCC'B'$ has opposing sides $BB' = \langle 4,0,1\rangle = CC' = \langle 4,0,1\rangle$ and $BC = \langle 0,-3,4\rangle = B'C' = \langle 0,-3,4\rangle$ and so is a parallelogram. Because $BB' \cdot BC = \langle 4,0,1\rangle \cdot \langle 0,-3,4\rangle = 4 \neq 0$, the neighbouring edges of $BCC'B'$ are not perpendicular, so $BCC'B'$ is *not* a rectangle.

(b) The length of AA' is $|\langle 4,0,1\rangle| = \sqrt{16+1} = \sqrt{17}$.

Solutions to Exercises 1.3 — Jump to TABLE OF CONTENTS

S-1: We are looking for a vector that is perpendicular to $z = 0$ and hence is parallel to $\hat{\mathbf{k}}$. To be parallel of $\hat{\mathbf{k}}$, the vector has to be of the form $c\,\hat{\mathbf{k}}$ for some real number c. For the vector to be nonzero, we need $c \neq 0$ and for the vector to be different from $\hat{\mathbf{k}}$, we need $c \neq 1$. So three possible choices (out of infinitely many) are $-\hat{\mathbf{k}}$, $2\,\hat{\mathbf{k}}$, $7.12345\,\hat{\mathbf{k}}$.

S-2: Example 1.3.2 in the text shows how to use Equation 1.3.1 in the text.

(a) Normal vector $\langle 1,4,1\rangle$, point $(0,0,0)$:

$$x + 4y + z = 0$$

(b) Normal vector $\langle 7,8,9\rangle$, point (1,4,1):

$$7x + 8y + 9z = 7(1) + 8(4) + 9(1) = 48$$

(c) A plane parallel to Q has a parallel normal vector, so we can re-use the normal vector $\langle 7,8,9\rangle$. Plane R passes through the point (2,2,5), so its equation is

$$7x + 8y + 9z = 7(2) + 8(2) + 9(5) = 75$$

S-3: *Solution 1:* That's too easy. We just guess. The plane $x + y + z = 1$ contains all three given points.

Solution 2: The plane does not pass through the origin. (You can see this by just making a quick sketch.) So the plane has an equation of the form $ax + by + cz = 1$.

- For $(1,0,0)$ to be on the plane we need that

$$a(1) + b(0) + c(0) = 1 \implies a = 1$$

- For $(0,1,0)$ to be on the plane we need that

$$a(0) + b(1) + c(0) = 1 \implies b = 1$$

- For $(0,0,1)$ to be on the plane we need that

$$a(0) + b(0) + c(1) = 1 \implies c = 1$$

So the plane is $x + y + z = 1$.

S-4: That's too easy. We just guess. The plane $x + y + z = 2$ contains all three given points.

S-5: The vector from $(1,2,3)$ to $(2,3,4)$, namely $\langle 1,1,1\rangle$ is parallel to the vector from $(1,2,3)$ to $(3,4,5)$, namely $\langle 2,2,2\rangle$. So the three given points are collinear. Precisely, all three points $(1,2,3)$, $(2,3,4)$ and $(3,4,5)$ are on the line $\langle x-1, y-2, z-3\rangle = t\,\langle 1,1,1\rangle$. There are many planes through that line.

S-6: All vectors normal to a given plane are parallel to one another. Parallel vectors are nonzero scalar multiples of one another. So, there exist some constant t such that

$$t\langle 1,4,7\rangle = \langle 2,a,b\rangle$$

From the first coordinate, we see $t = 2$, so

$$\langle 2,a,b\rangle = 2\langle 1,4,7\rangle = \langle 2,8,14\rangle$$

That is, $a = 8$ and $b = 14$.

S-7: The plane has equation

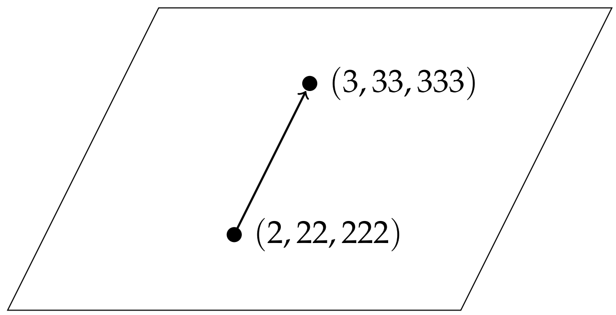
$$2x - 3y + z = d$$

where $d = 2(100) - 3(100) + 1(100) = 0$. So, the equation of the plane is

$$2x - 3y + z = 0$$

The point $(0, 0, 0)$ satisfies this equation, so that point is indeed on the plane.

S-8: Given two points in the plane, we can find a vector in the plane:



The vector $\langle 3, 33, 333\rangle - \langle 2, 22, 222\rangle = \langle 1, 11, 111\rangle$ is in the plane. So, it is orthogonal to the normal vector. That is, their dot product is equal to 0. So:

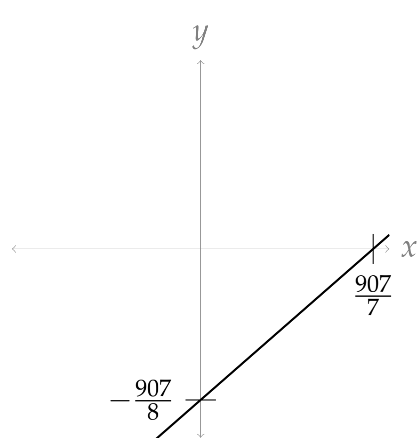
$$\begin{aligned} 0 &= \langle 2, -3, a\rangle \cdot \langle 1, 11, 111\rangle = 2 - 33 + 111a \\ 31 &= 111a \\ a &= \frac{31}{111} \end{aligned}$$

S-9:

(a) The xy plane consists of all points in $\mathbb{R}^3$ with z-coordinate equal to 0. So, the intersection of P with the xy plane are all points $(x, y, 0)$ such that

$$7x - 8y = 907$$

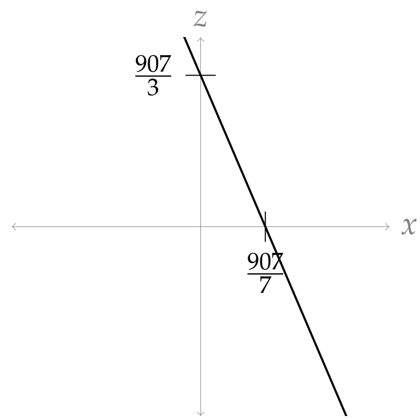
This describes a line in the xy-plane. The intercepts of this line are $\left(x = 0, y = -\frac{907}{8}\right)$ and $\left(x = \frac{907}{7}, y = 0\right)$.



(b) The xz plane consists of all points in $\mathbb{R}^3$ with y-coordinate equal to 0. So, the intersection of P with the xy plane are all points $(x,0,z)$ such that

$$7x+3z=907$$

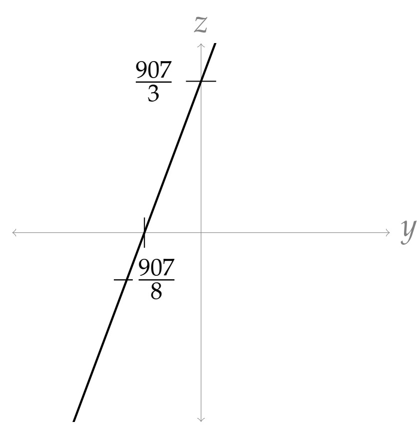
This describes a line in the xz-plane. The intercepts of this line are $\left(x=0, z=\frac{907}{3}\right)$ and $\left(x=\frac{907}{7}, z=0\right)$.



(c) The yz plane consists of all points in $\mathbb{R}^3$ with x-coordinate equal to 0. So, the intersection of P with the yz plane are all points $(0,y,z)$ such that

$$-8y+3z=907$$

This describes a line in the yz-plane. The intercepts of this line are $\left(y=0, z=\frac{907}{3}\right)$ and $\left(y=-\frac{907}{8}, z=0\right)$.



S-10: The first things we notice is that the equations for P and Q are equivalent. (To get the equation of Q, multiply both sides of the equation for P by 2.) So P and Q are identical.

The normal vector of P is $\langle 1,-1,1\rangle$, and the normal vector of R is $\langle 1,2,1\rangle$. Since $\langle 1,-1,1\rangle \cdot \langle 1,2,1\rangle = 1-2+1=0$, these two vectors are orthogonal, so the planes P and R (and so also Q and R) are perpendicular.

The normal vector of S is $\langle 1,1,1\rangle$. This isn't a scalar multiple of the other normal vectors, so S is not parallel to any of the other planes.

Also $\langle 1,-1,1\rangle \cdot \langle 1,1,1\rangle \neq 0$ and $\langle 1,2,1\rangle \cdot \langle 1,1,1\rangle \neq 0$. The normal vector of S is not perpendicular to the normal vectors of the other planes, so S is not perpendicular to the other planes.

S-11: The distance from the point (x,y,z) to $(1,2,3)$ is $\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$ and the distance from $(x,y,z)$to $(5,2,7)$ is $\sqrt{(x-5)^2+(y-2)^2+(z-7)^2}$. Hence (x,y,z) is equidistant from $(1,2,3)$ and $(5,2,7)$ if and only if

$$\begin{aligned}
& (x-1)^2+(y-2)^2+(z-3)^2 = (x-5)^2+(y-2)^2+(z-7)^2 \\
\iff\quad & x^2-2x+1+z^2-6z+9 = x^2-10x+25+z^2-14z+49 \\
\iff\quad & 8x+8z = 64 \\
\iff\quad & x+z = 8
\end{aligned}$$

This is the plane through $(3,2,5) = \frac{1}{2}(1,2,3)+\frac{1}{2}(5,2,7)$ with normal vector $\langle 1,0,1\rangle = \frac{1}{4}\big(\langle 5,2,7\rangle - \langle 1,2,3\rangle\big)$.

S-12: The equation of a generic plane in $\mathbb{R}^3$ is

$$ax+by+cz=d$$

for constants a, b, c, and d. Following Example 1.3.6 in the text, we set up a system of linear equations, which we can solve with substitution. Notice there are actually infinitely many ways of describing the same plane by taking multiples of the equation, e.g.

$$2ax+2by+2cz=2d$$

So, there will be infinitely many solutions to our systems of equations. That's fine.

(a) If $ax+by+cz=d$ is the equation of the plane containing $(1,0,1)$, $(2,4,6)$, $(1,2,-1)$, then:

$$\left\{\begin{array}{ll} a+0b+c & =d \\ 2a+4b+6c & =d \\ a+2b-c & =d \end{array}\right\}$$

Solving the first equation for a, we get $\boxed{a=d-c}$. Plugging this into the other two equations:

$$\left\{\begin{array}{ll} 2(d-c)+4b+6c & =d \\ (d-c)+2b-c & =d \end{array}\right\} \implies \left\{\begin{array}{ll} 4b+4c & =-d \\ 2b-2c & =0 \end{array}\right\}$$

Solving the second equation for b, we get $\boxed{b=c}$. Plugging this into the remaining equation:

$$\left\{\begin{array}{ll} 4b+4b & =-d \end{array}\right\} \implies \left\{\begin{array}{ll} b & =-\frac{1}{8}d \end{array}\right\}$$

Now, using the relationships shown in boxes above,

$$\left\{\begin{array}{lllll} b & & & = & -\frac{1}{8}d \\ c & = & b & = & -\frac{1}{8}d \\ a & = & d-c & = & \frac{9}{8}d \end{array}\right\}$$

So, the equation for the plane is

$$\left(\frac{9}{8}d\right)x + \left(-\frac{1}{8}d\right)y + \left(-\frac{1}{8}d\right)z = d$$

Any nonzero choice of d will do; choosing $d = 8$ gives us integer coefficients:

$$9x - y - z = 8$$

(b) If $ax + by + cz = d$ is the equation of the plane containing $(1, -2, -3),\ (4, -4, 4),\ (3, 2, -3)$, then:

$$\left\{\begin{array}{ll} a - 2b - 3c & = d \\ 4a - 4b + 4c & = d \\ 3a + 2b - 3c & = d \end{array}\right\}$$

Solving the first equation for a, we get $\boxed{a = d + 2b + 3c}$. Plugging this into the other two equations:

$$\left\{\begin{array}{ll} 4(d+2b+3c) - 4b + 4c & = d \\ 3(d+2b+3c) + 2b - 3c & = d \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} 4b + 16c & = -3d \\ 8b + 6c & = -2d \end{array}\right\}$$

Solving the second equation for $4b$, we see $\boxed{4b = -d - 3c}$. Plugging this into the remaining equation:

$$\left\{ (-d-3c) + 16c \ = -3d \right\} \Longrightarrow \left\{ c \ = -\tfrac{2}{13}d \right\}$$

Now, using the relationships shown in boxes above,

$$\left\{\begin{array}{lllllll} c & & & & & = & -\frac{2}{13}d \\ b & = & \frac{-d-3c}{4} & = & \frac{-d-3\left(-\frac{2}{13}d\right)}{4} & = & -\frac{7}{52}d \\ a & = & d+2b+3c & = & d+2\left(-\frac{7}{52}d\right)+3\left(-\frac{2}{13}d\right) & = & \frac{14}{52}d \end{array}\right\}$$

So, the equation for the plane is

$$\left(\frac{14}{52}d\right)x + \left(-\frac{7}{52}d\right)y + \left(-\frac{2}{13}d\right)z = d$$

Any nonzero choice of d will do; choosing $d = 52$ gives us integer coefficients:

$$14x - 7y - 8z = 52$$

(c) If $ax + by + cz = d$ is the equation of the plane containing $(1, -2, -3)$, $(5, 2, 1)$, $(-1, -4, -5)$, then:

$$\left\{ \begin{array}{ll} a - 2b - 3c & = d \\ 5a + 2b + c & = d \\ -a - 4b - 5c & = d \end{array} \right\}$$

Solving the first equation for a, we get $\boxed{a = d + 2b + 3c}$. Plugging this into the other two equations:

$$\left\{ \begin{array}{ll} 5(d + 2b + 3c) + 2b + c & = d \\ -(d + 2b + 3c) - 4b - 5c & = d \end{array} \right\} \implies \left\{ \begin{array}{ll} 12b + 16c & = -4d \\ -6b - 8c & = 2d \end{array} \right\}$$
$$\implies \left\{ \begin{array}{ll} 3b + 4c & = -d \\ -3b - 4c & = d \end{array} \right\}$$
$$\implies \left\{ \begin{array}{ll} 3b + 4c & = -d \\ 3b + 4c & = -d \end{array} \right\}$$

At this point, we see that there will be more than one plane containing the three points! Indeed:

- $\langle 1, -2, -3 \rangle - \langle -1, -4, -5 \rangle = \langle 2, 2, 2 \rangle$
- $\langle 5, 2, 1 \rangle - \langle -1, -4, -5 \rangle = \langle 6, 6, 6 \rangle$

So all three points lie on a line with normal vector $\langle 1, 1, 1 \rangle$. That is, they all lie on the line

$$\langle 1, -2, -3 \rangle + t\langle 1, 1, 1 \rangle$$

The question asks us to find *the* plane containing the three points, but actually there are infinitely many such planes. We can still describe the family of planes containing the points.

The relationship between b and c is $\boxed{b = \dfrac{-d - 4c}{3}}$. So,
$a = d + 2b + 3c = d + 2\left(\frac{-d-4c}{3}\right) + 3c =$

$$\left\{ \begin{array}{lllll} b & = & \frac{-d-4c}{3} & = & -\frac{1}{3}d - \frac{4}{3}c \\ a & = & d + 2b + 3c & = & d + 2\left(\frac{-d-4c}{3}\right) + 3c \; = \frac{1}{3}c + \frac{1}{3}d \end{array} \right\}$$

So, the equation for the plane is

$$\left(\frac{1}{3}c + \frac{1}{3}d\right) x + \left(-\frac{4}{3}c - \frac{1}{3}d\right) y + cz = d$$

Any nonzero choice of d will do; choosing $d = 3$ gives us the equation

$$\left(\frac{1}{3}c + 1\right) x + \left(-\frac{4}{3}c - 1\right) y + cz = 3$$

There are lots of ways to describe the collection of planes we found above. The way we just did it is this:

For any real number c, the plane with equation

$$\left(\frac{1}{3}c+1\right)x+\left(-\frac{4}{3}c-1\right)y+cz=3$$

contains the three given points

If we let a be the coefficient of x, then using the conversion $a=\frac{1}{3}c+1$, we get the following characterization:

For any real number a, the plane with equation

$$ax+(3-4a)\,y+(3a-3)\,z=3$$

contains the three given points

S-13: The distance from the point $\mathbf{x}$ to $\mathbf{a}$ is $\sqrt{(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})}$ and the distance from $\mathbf{x}$ to $\mathbf{b}$ is $\sqrt{(\mathbf{x}-\mathbf{b})\cdot(\mathbf{x}-\mathbf{b})}$. Hence $\mathbf{x}$ is equidistant from $\mathbf{a}$ and $\mathbf{b}$ if and only if

$$\begin{aligned} & (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})=(\mathbf{x}-\mathbf{b})\cdot(\mathbf{x}-\mathbf{b}) \\ \iff\quad & |\mathbf{x}|^2-2\mathbf{a}\cdot\mathbf{x}+|\mathbf{a}|^2=|\mathbf{x}|^2-2\mathbf{b}\cdot\mathbf{x}+|\mathbf{b}|^2 \\ \iff\quad & 2(\mathbf{b}-\mathbf{a})\cdot\mathbf{x}=|\mathbf{b}|^2-|\mathbf{a}|^2 \end{aligned}$$

This is the plane through $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$ with normal vector $\mathbf{b}-\mathbf{a}$.

S-14: (a) The equation of a generic plane in $\mathbb{R}^3$ is

$$ax+by+cz=d$$

for constants a, b, c, and d. Following Example 1.3.6 in the text, we set up a system of linear equations, which we can solve with substitution.

$$\left\{\begin{array}{ll} a+b+3c & =d \\ 2a+0b+2c & =d \\ 2a+b+0c & =d \end{array}\right\}$$

Solving the third equation for b, we get $\boxed{b=d-2a}$. Plugging this into the other two equations:

$$\left\{\begin{array}{ll} a+(d-2a)+3c & =d \\ 2a+2c & =d \end{array}\right\} \implies \left\{\begin{array}{ll} -a+3c & =0 \\ 2a+2c & =d \end{array}\right\}$$

Solving the first equation for a, we see $\boxed{a = 3c}$. Plugging this into the remaining equation:

$$\left\{\ 2(3c) + 2c \ = d\ \right\} \implies \left\{\ c \ = \tfrac{1}{8}d\ \right\}$$

Now, using the relationships shown in boxes above,

$$\left\{\begin{array}{lcccl} c & & & = & \frac{1}{8}d \\ a & = & 3c & = & \frac{3}{8}d \\ b & = & d - 2a & = & \frac{1}{4}d \end{array}\right\}$$

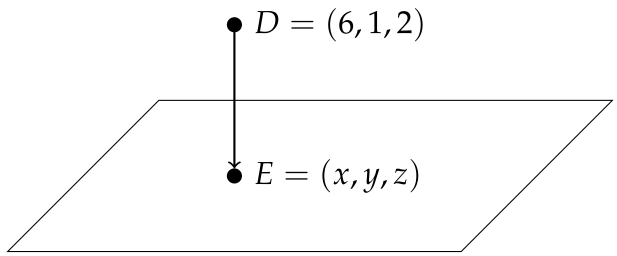
So, the equation for the plane is

$$\left(\frac{3}{8}d\right) x + \left(\frac{1}{4}d\right) y + \left(\frac{1}{8}d\right) z = d$$

Any nonzero choice of d will do; choosing $d = 8$ gives us integer coefficients:

$$3x + 2y + z = 8$$

(b) Let E be (x, y, z). Then the vector from D to E, namely $\langle x - 6\,,\ y - 1\,,\ z - 2\rangle$ has to be parallel to the vector $\langle 3\,,\ 2\,,\ 1\rangle$, which is perpendicular to Π.



That is, there must be a number t such that

$$\langle x - 6\,,\ y - 1\,,\ z - 2\rangle = t\,\langle 3\,,\ 2\,,\ 1\rangle$$
$$\text{or } x = 6 + 3t,\ y = 1 + 2t,\ z = 2 + t$$

As (x, y, z) must be in Π,

$$8 = 3x + 2y + z = 3(6 + 3t) + 2(1 + 2t) + (2 + t) = 22 + 14t \implies t = -1$$

So $(x, y, z) = \big(6 + 3(-1)\,,\ 1 + 2(-1)\,,\ 2 + (-1)\big) = \big(3\,,\ -1\,,\ 1\big)$.

S-15: The two planes $x + y + z = 3$ and $x + y + z = 9$ are parallel. The centre must be on the plane $x + y + z = 6$ half way between them. So, the center is on $x + y + z = 6$, $2x - y = 0$ and $3x - z = 0$. Solving these three equations, or equvalently,

$$y = 2x,\ z = 3x,\ x + y + z = 6x = 6$$

gives $(1, 2, 3)$ as the centre. $(1, 1, 1)$ is a point on $x + y + z = 3$. $(3, 3, 3)$ is a point on $x + y + z = 9$. So $\langle 2, 2, 2\rangle$ is a vector with tail on $x + y + z = 3$ and head on $x + y + z = 9$.

Furthermore $\langle 2,2,2 \rangle$ is perpendicular to the two planes. So the distance between the planes is $|\langle 2,2,2 \rangle| = 2\sqrt{3}$ and the radius of the sphere is $\sqrt{3}$. The sphere is

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 3$$

Solutions to Exercises 1.4 — Jump to TABLE OF CONTENTS

S-1: Any constant function will do. For example, $f(x,y) = 0$ or $f(x,y) = 1$.

S-2:

(a) The range of $f(x)$ is $[-10, 10]$, since these are the y-values in the sketch.

(b) The range of $g(x)$ is $[0, 1]$, since these are the y-values in the sketch.

(c) In order for $f(g(x))$ to be defined, we require $-1 \leqslant g(x) \leqslant 1$. That is, the range of g must be in the domain of f. This is true for all values of $g(x)$, so there is no extra domain restriction. The domain of $f(g(x))$ is $[-1, 1]$.

(d) Since the range of $g(x)$ is $[0, 1]$, the numbers that get plugged into f in the compound function $f(g(x))$ are only the numbers $[0, 1]$. So, the range of this function is $[0, 10]$. $g(x)$ never spits out any negative values, so $f(x)$ is restricted to the nonnegative part of its domain.

Remark: because we're going off imprecise sketches, it wouldn't be wrong to give open intervals, rather than closed intervals, as your answers.

S-3: If $x = 1 = y$, and (x,y,z) is a point on the function, then:

$$\begin{aligned}
1 &= z^2(1^3) + z(1^3) + (1)(1) \\
0 &= z^2 + z \\
0 &= z \text{ or } -1 = z
\end{aligned}$$

So yes, $(1,1)$ is in the domain.

There's some fine print here. There are two different values of z corresponding to the input $(x,y) = (1,1)$. That means that globally, z isn't a function of x and y, because a function should only ever have at most one output for any one input. Implicitly-defined functions often have this characteristic: it's not possible to write $z = f(x,y)$ for any single function f of x and y.

S-4: The only part of the function that could possibly limit the domain is the square root: we must not try to take the square root of a negative number.

The expression $4x^2 + y^2$ gives nonnegative numbers for any real values of x and y. So no matter what (x,y) we input, there is no danger of taking the square root of a negative number. So, the domain is all of $\mathbb{R}^2$.

We've already noted that $4x^2 + y^2$ will give us numbers from $[0, \infty)$, but we should check whether it gives us *all* of those numbers. Indeed, if we set $x = 0$, we see

$$f(0, y) = \sqrt{y^2} = |y|$$

the range of which is $[0, \infty)$.

So by choosing $x = 0$ and the appropriate y, we can indeed get $f(x, y)$ to be any nonnegative number we desire. So, the range of f is $[0, \infty)$.

S-5: The only restriction on our domain is that we can't divide by 0, and $1 + y^2$ is never 0. So, our domain is all of $\mathbb{R}^2$.

Since $x^2 \geqslant 0$ and $1 + y^2 \geqslant 0$, we see first that $h(x, y)$ is never negative. The question now is whether it can actually achieve all nonnegative real values. If we set $y = 0$, then $h(x, 0) = x^2$, which has range $[0, \infty)$. So we can indeed find a point $h(x, y) = h(x, 0)$ equal to any nonnegative number our hearts desire. That is, the range of $h(x, y)$ is $[0, \infty)$.

S-6: Recall the domain of the function $\arcsin(x)$ is $[-1, 1]$, and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

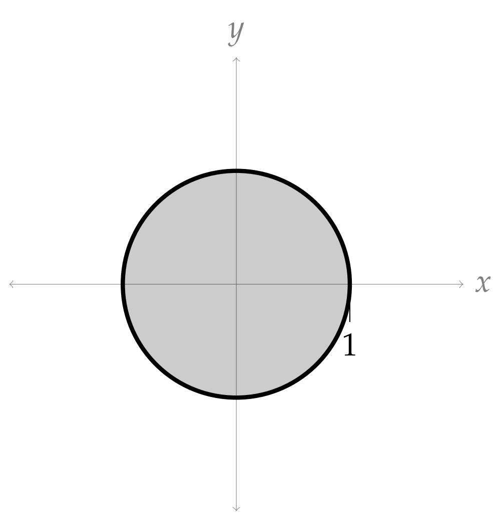
Since we can only put numbers from $[-1, 1]$ into arcsine, we require for our domain

$$-1 \leqslant x^2 + y^2 \leqslant 1$$

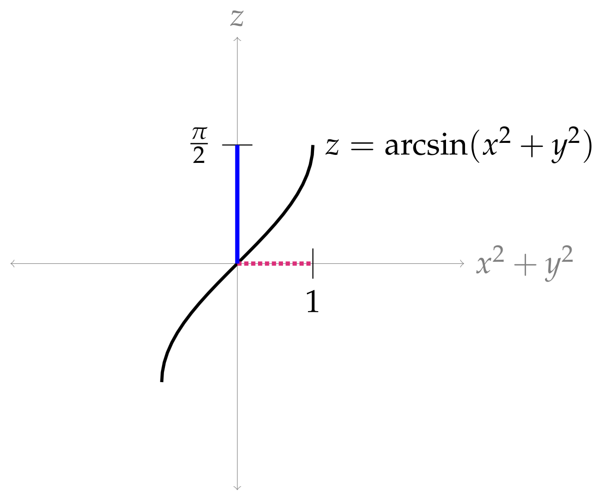
The left part of the inequality isn't hard, since $x^2 + y^2$ is never negative. The right side tells us

$$x^2 + y^2 \leqslant 1$$

i.e. (x, y) is inside (or on) the unit circle.



Subject to the constraint $x^2 + y^2 \leqslant 1$, the domain of $x^2 + y^2$ is $[0, 1]$. The range of $\arcsin x$ subject to the constraint $0 \leqslant x \leqslant 1$ is $\left[0, \frac{\pi}{2}\right]$.

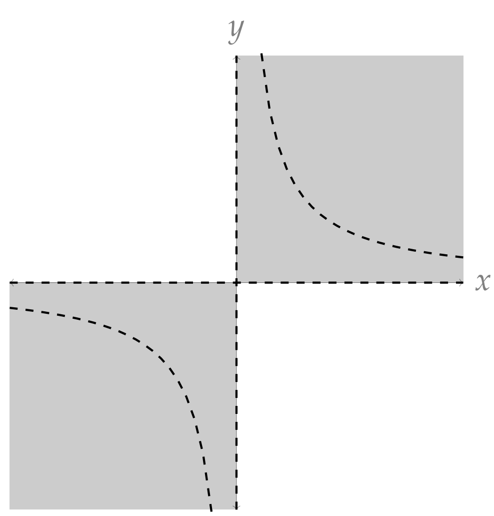


Red dotted line: range of $x^2 + y^2$ subject to restrictions.
Blue solid line: range of $\arcsin(x^2 + y^2)$.

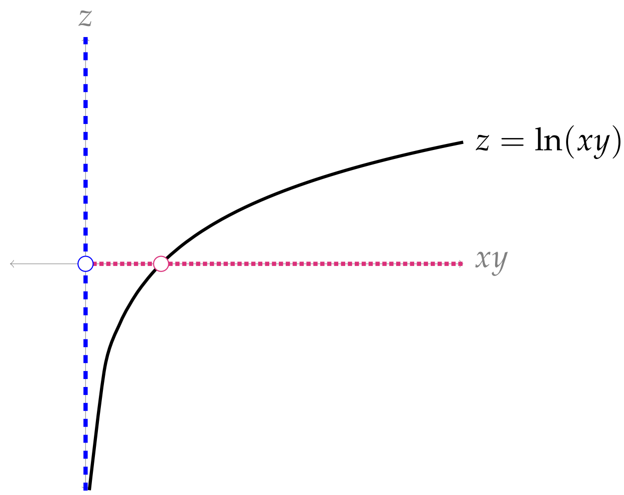
S-7: To find the domain of g, there are two potential limiting issues: we can't divide by 0, and we can't take the logarithm of a nonpositive number.

- Since we can't divide by 0, $\ln(xy) \neq 0$, which means $xy \neq 1$, or (equivalently) $y \neq \frac{1}{x}$.
- Since we can't take the logarithm of a nonpositive number, we need $xy > 0$. That is, x and y must be both negative, or both positive.

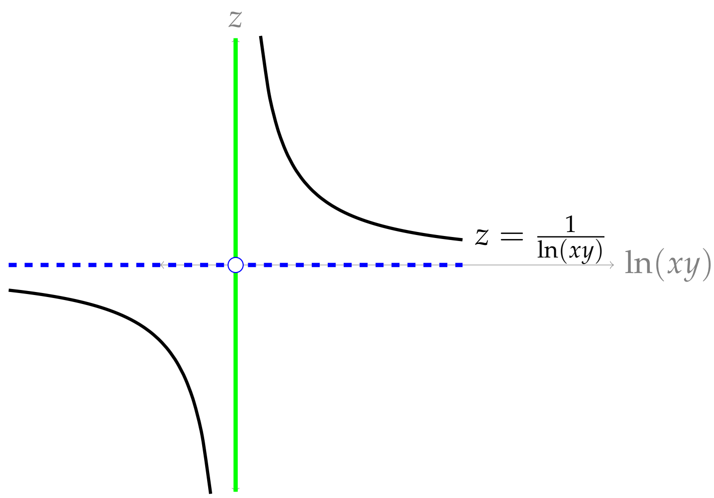
Combining these two restrictions, the domain of $g(x, y)$ is all points (x, y) such that x and y have the same sign; they are nonzero; and $y \neq \frac{1}{x}$. These points are graphed below. Dashed lines indicate points that are *not* in the domain.



With these restrictions, xy can be *any* nonnegative number except 1; which means $\ln(xy)$ can be any real number except 0; and finally the range of the entire function is $(-\infty, 0) \cup (0, \infty)$. (This is illustrated in graphs below.)



Red dotted line: values of xy. Blue dashed line: values of $\ln(xy)$



Blue dashed line: values of $\ln(xy)$. Green solid line: values of $\frac{1}{\ln(xy)}$

S-8: The only thing that might limit the domain of this function is dividing by zero; but since $x^2 + 1 > 0$ for all real values of x, we see the domain of f is the entire plane $\mathbb{R}^2$.

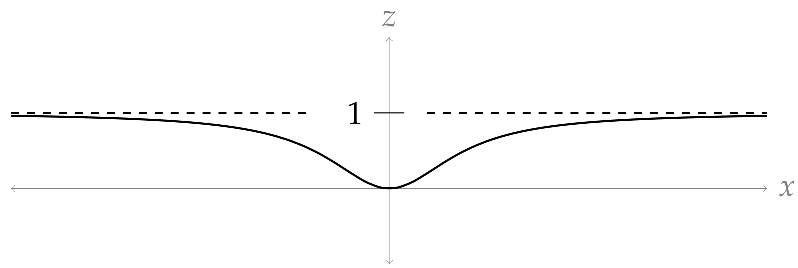
Since y doesn't impact the value of f, we can consider the single-variable function

$$g(x) = \frac{x^2}{x^2+1}$$

Since $f(x, y) = g(x)$ for any (x, y), the range of g will be the same as the range of f. Note $g(x)$ is continuous over all real numbers. So, its range will be (global min)$\leqslant g(x) \leqslant$ (global max). To help picture how $g(x)$ behaves, note further that $z = g(x)$ has a horizontal asymptote at $z = 1$, and $g(x)$ is an even function. Let's find the critical points of $g(x)$.

$$g'(x) = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

The only CP of this function is $x = 0$. Its horizontal asymptotes are 1 in both directions. So, the basic shape of the function is:



So, its range is $[0, 1)$.

S-9: The domain of $f(x, y)$ is all of $\mathbb{R}^2$: the only possible restriction is dividing by zero, but $x^2 + 1 > 0$ for all values of x.

We can write $f(x, y)$ as

$$f(x, y) = f_1(x) + f_2(y)$$

where $f_1(x) = \frac{x}{x^2+1}$ and $f_2(y) = \sin y$. Since there is not term depending on both x and y, the maximum value of f will occur when x maximizes f_1 and y maximizes f_2. Similarly, the minimum value of f will occur when x minimizes f_1 and y minimizes f_2. Since these two functions are both continuous, we see that the range of f will be

$$(\text{min of } f_1 + \text{min of } f_2) \leqslant f(x, y) \leqslant (\text{max of } f_1 + \text{max of } f_2)$$

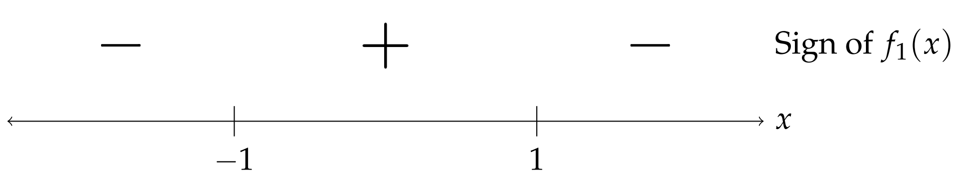
The range of $f_2(y) = \sin y$ is easy: it's $[-1, 1]$. Let's consider $f_1(x) = \frac{x}{x^2+1}$. Note its horizontal asymptotes are 0 in both directions, and it's an odd function. To find its extrema, let's sketch it, starting by finding its critical points.

$$f_1'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2} = \frac{(1+x)(1-x)}{(x^2+1)^2}$$

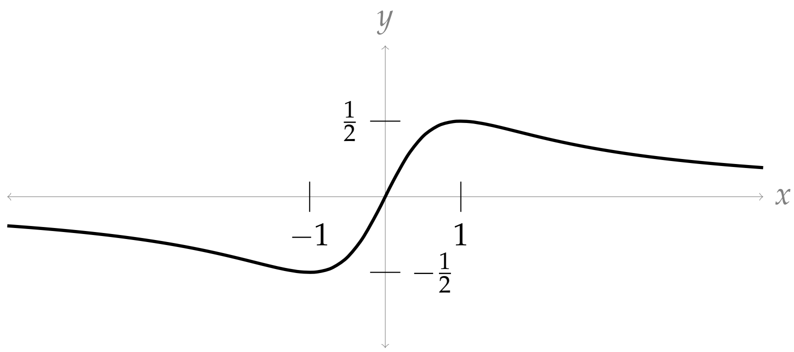
The CPs of f_1 are $x = 1$ and $x = -1$.

$$\begin{aligned} f_1(1) &= \frac{1}{1^2+1} = \frac{1}{2} \\ f_1(-1) &= \frac{-1}{(-1)^2+1} = -\frac{1}{2} \end{aligned}$$

To sketch f_1, let's find the sign of its first derivative on the intervals between its critical points.



Now we have enough information to sketch $z = f_1(x)$:



So, the range of $f_1(x)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

All together, the range of $f(x, y)$ is $\left[-\frac{3}{2}, \frac{3}{2}\right]$.

S-10: Some general assumptions might be that the amount of money spend on advertisements shoudn't be negative, so we should have $a \geqslant 0$. Similarly, it's reasonable to assume that the company is not giving away its product, nor paying people to take it, so $p > 0$. Finally, people won't demand a negative number of goods, so the range should be nonnegative.

That is one way of thinking about the problem, but different models might have different restrictions. For example, from time to time (including a time in 2020) oil futures trade at negative values: people were paying to give them away. So for certain models, negative prices and negative demands do make sense.

For other models, also an upper bound of some sort probable makes sense. Maybe you aren't able to sell more than one million of your product, because you don't have the capacity to manufacture more. Maybe demand will never exceed one product per person in your area. Such restrictions would further impact the domain and range that make sense for your model.

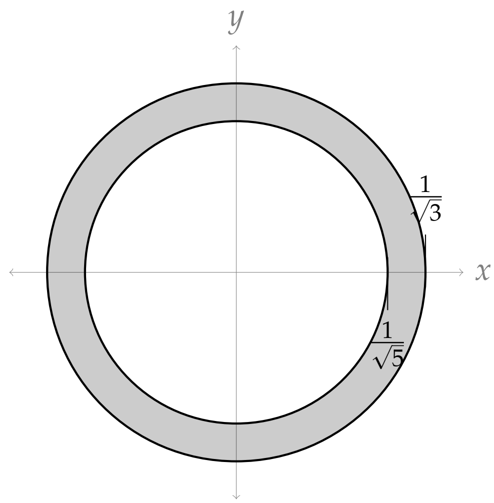
S-11: For this question, we solve two inequalities.

$$\begin{aligned} 3 &\leqslant \frac{1}{x^2+y^2} \\ \implies \frac{1}{3} &\geqslant x^2+y^2 \end{aligned}$$

$$\begin{aligned} 5 &\geqslant \frac{1}{x^2+y^2} \\ \implies \frac{1}{5} &\leqslant x^2+y^2 \end{aligned}$$

So, the points (x, y) must be both:

- inside or on the circle centred at the origin with radius $\frac{1}{\sqrt{3}}$, and
- not inside the circle centred at the origin with radius $\frac{1}{\sqrt{5}}$.



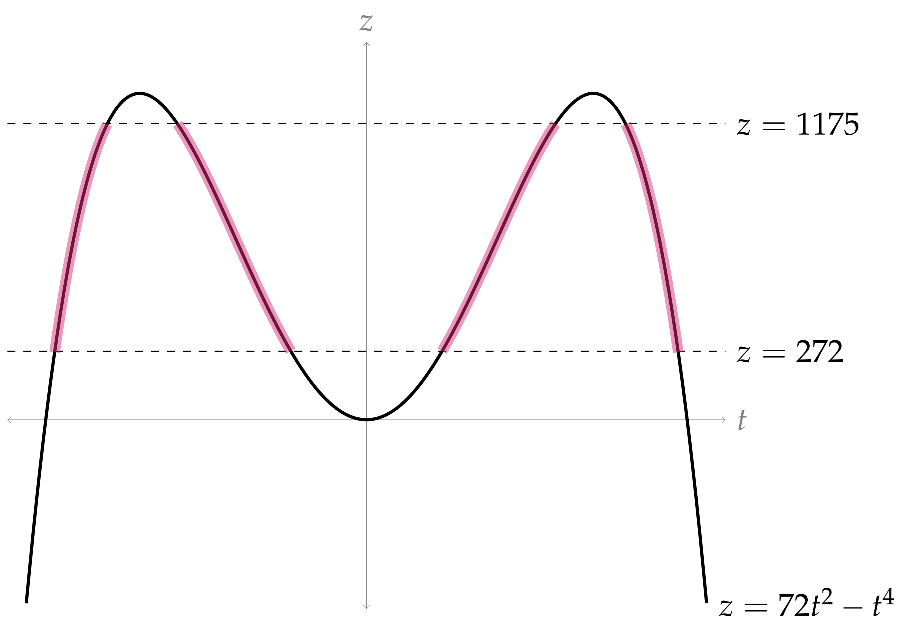
S-12: The bracketing in the definition of $g(x, y)$ is suggestive. If we define $t = x^2 - y$, then we get the function

$$h(t) = 72t^2 - t^4$$

This is easy enough to graph using tools from last semester.

- h is an even function
- $\lim_{t\to\infty} h(t) = -\infty$
- $h'(t) = 144t - 4t^3 = 4t(36 - t^2) = 5t(6 + t)(6 - t)$, so critical points are at $t = 0$ and $t = \pm 6$
- $h'(t)$ is negative on $(-6, 0) \cup (6, \infty)$ and positive on $(-\infty, -6) \cup (0, 6)$.
- The absolute maximum of $h(t)$ is $h(-6) = h(6) = 6^4 = 1296$, and $h(0) = 0$ is a local minimum.

Sketched below is $z = 72t^2 - t^4$, with parts in the model range highlighted.



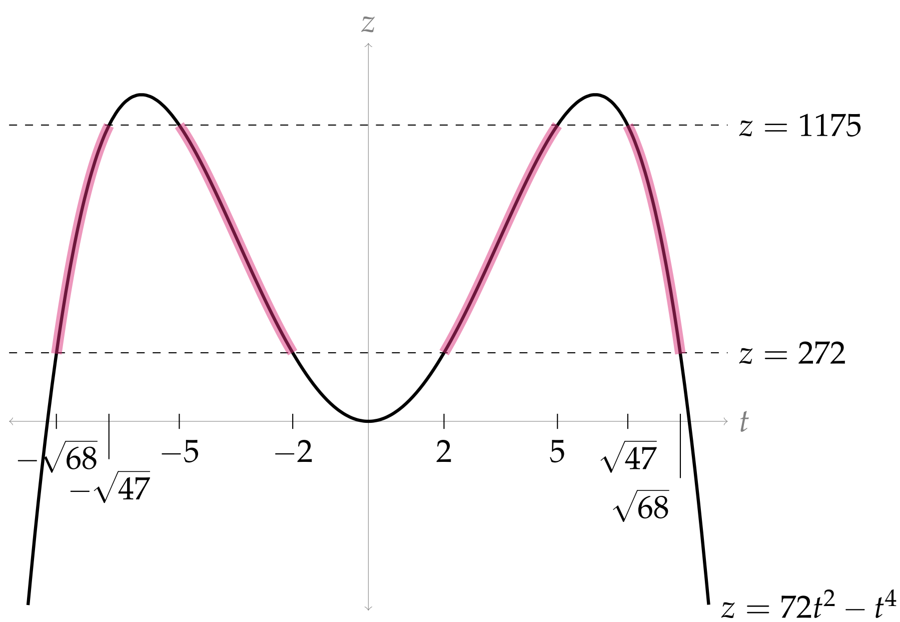
To find the t-values that correspond to the model range, we solve:

$$
\begin{aligned}
72t^2 - t^4 &= 1175 \\
0 &= t^4 - 72t^2 + 1175 \\
t^2 &= \frac{72 \pm \sqrt{72^2 - 4(1)(1175)}}{2} \\
&= \frac{72 \pm \sqrt{4(36^2) - 4(1175)}}{2} \\
&= \frac{72 \pm 2\sqrt{36^2 - 1175}}{2} \\
&= 36 \pm \sqrt{36^2 - 1175} \\
&= 36 \pm \sqrt{121} \\
&= 36 \pm 11 \\
&= 25 \text{ or } 47 \\
t &= \pm 5 \text{ or } \pm \sqrt{47}
\end{aligned}
$$

Similarly,

$$\begin{aligned}
72t^2 - t^4 &= 272 \\
0 &= t^4 - 72t^2 + 272 \\
t^2 &= \frac{72 \pm \sqrt{72^2 - 4(1)(272)}}{2} \\
&= \frac{72 \pm \sqrt{4(36^2) - 4(272)}}{2} \\
&= \frac{72 \pm 2\sqrt{36^2 - 272}}{2} \\
&= 36 \pm \sqrt{36^2 - 272} \\
&= 36 \pm \sqrt{1024} \\
&= 36 \pm 32 \\
&= 4 \text{ or } 68 \\
t &= \pm 2 \text{ or } \pm\sqrt{68}
\end{aligned}$$

So, now we can fill in our sketch with t-values:



So we need to have t in $[-\sqrt{68}, -\sqrt{47}] \cup [-5, -2] \cup [2, 5] \cup [\sqrt{47}, \sqrt{68}]$.

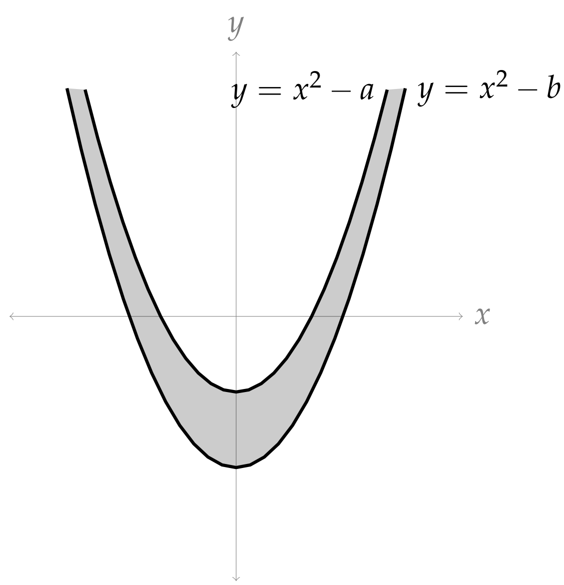
Now, recall we used $t = x^2 - y$. So if we have $a \leqslant t \leqslant b$, then this gives us two inequalities:

$$\begin{aligned}
& t \leqslant b \\
\implies & x^2 - y \leqslant b \\
\implies & x^2 - b \leqslant y
\end{aligned}$$

and

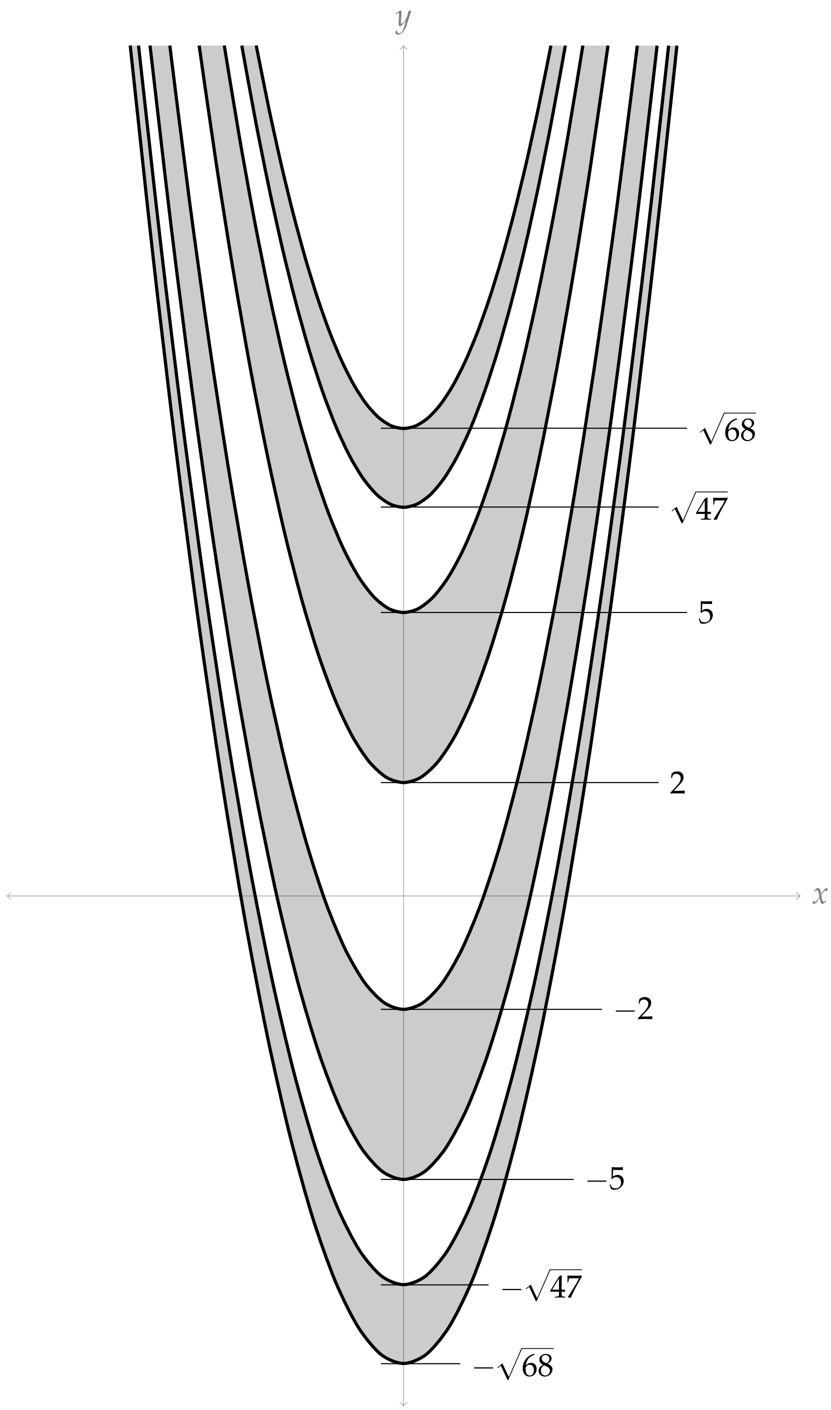
$$\begin{aligned} t &\geqslant a \\ \implies x^2 - y &\geqslant a \\ \implies x^2 - a &\geqslant y \end{aligned}$$

So, t in the interval $[a, b]$ implies that (x, y) must satisfy $x^2 - b \leqslant y \leqslant x^2 - a$:



We have four such possible intervals. All together, the point (x, y) must be in one of the following regions:

- $x^2 - \sqrt{68} \leqslant y \leqslant x^2 - \sqrt{47}$
- $x^2 - 5 \leqslant y \leqslant x^2 - 2$
- $x^2 + 2 \leqslant y \leqslant x^2 + 5$
- $x^2 + \sqrt{47} \leqslant y \leqslant x^2 + \sqrt{68}$



Solutions to Exercises 1.5 — Jump to TABLE OF CONTENTS

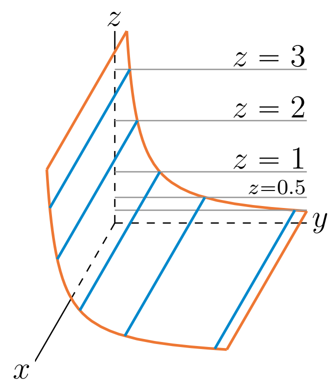
S-1:

(a) Each constant z cross–section of $x^2 + y^2 = z^2 + 1$ is a (horizontal) circle centred on the z–axis. The radius of the circle is 1 when $z = 0$ and grows as z moves away from $z = 0$. So $x^2 + y^2 = z^2 + 1$ consists of a bunch of (horizontal) circles stacked on top of each other, with the radius increasing with $|z|$. It is a hyperboloid of one sheet. The picture that corresponds to (a) is (B).

(b) Every point of $y = x^2 + z^2$ has $y \geqslant 0$. Only (A) has that property. We can also observe that every constant y cross–section is a circle centred on $x = z = 0$. The radius of the circle is zero when $y = 0$ and increases as y increases. The surface $y = x^2 + z^2$ is a paraboloid. The picture that corresponds to (b) is (A).

(c) The only possibility left is that the picture that corresponds to (c) is (C).

S-2: We first add into the sketch of the graph the horizontal planes $z = C$, for $C = 3, 2, 1,$ $0.5, 0.25$.



To reduce clutter, for each C, we have drawn in only

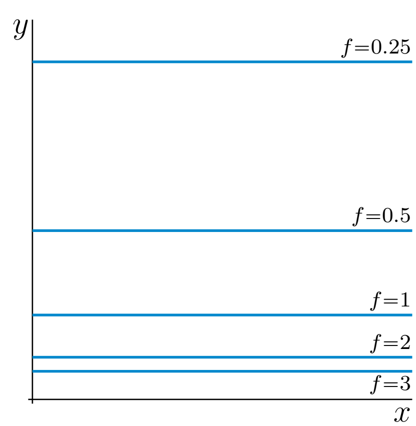
- the (gray) intersection of the horizontal plane $z = C$ with the yz–plane, i.e. with the vertical plane $x = 0$, and
- the (blue) intersection of the horizontal plane $z = C$ with the graph $z = f(x, y)$.

We have also omitted the label for the plane $z = 0.25$.

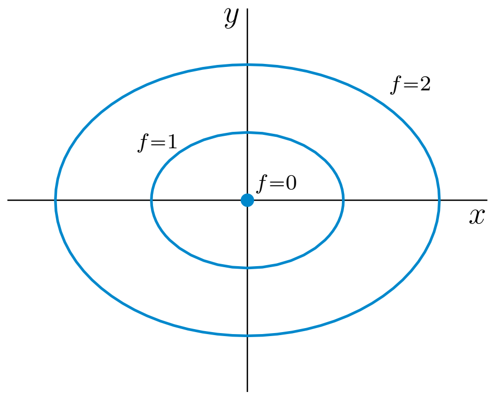
The intersection of the plane $z = C$ with the graph $z = f(x, y)$ is line

$$\{ (x, y, z) \mid z = f(x, y),\ z = C \} = \{ (x, y, z) \mid f(x, y) = C,\ z = C \}$$

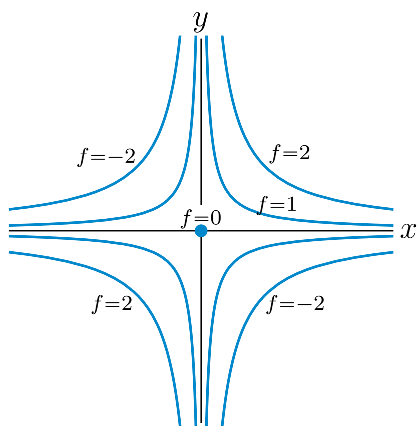
Drawing this line (which is parallel to the x-axis) in the xy-plane, rather than in the plane $z = C$, gives a level curve. Doing this for each of $C = 3, 2, 1, 0.5, 0.25$ gives five level curves.



S-3: (a) For each fixed $c > 0$, the level curve $x^2 + 2y^2 = c$ is the ellipse centred on the origin with x semi axis $\sqrt{c}$ and y semi axis $\sqrt{c/2}$. If $c = 0$, the level curve $x^2 + 2y^2 = c = 0$ is the single point $(0, 0)$.



(b) For each fixed $c \neq 0$, the level curve $xy = c$ is a hyperbola centred on the origin with asymptotes the x- and y-axes. If $c > 0$, any x and y obeying $xy = c > 0$ are of the same sign. So the hyperbola is contained in the first and third quadrants. If $c < 0$, any x and y obeying $xy = c > 0$ are of opposite sign. So the hyperbola is contained in the second and fourth quadrants. If $c = 0$, the level curve $xy = c = 0$ is the single point $(0, 0)$.



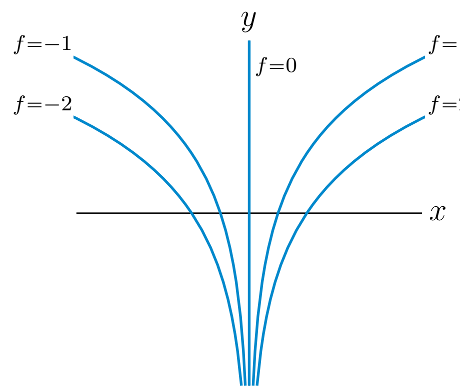
(c) For each fixed $c \neq 0$, the level curve $xe^{-y} = c$ is the logarithmic curve $y = -\ln\frac{c}{x}$. Note that, for $c > 0$, the curve

- is restricted to $x > 0$, so that $\frac{c}{x} > 0$ and $\ln\frac{c}{x}$ is defined, and that
- as $x \to 0^+$, y goes to $-\infty$, while
- as $x \to +\infty$, y goes to $+\infty$, and
- the curve crosses the x-axis (i.e. has $y = 0$) when $x = c$.

and for $c < 0$, the curve

- is restricted to $x < 0$, so that $\frac{c}{x} > 0$ and $\ln\frac{c}{x}$ is defined, and that
- as $x \to 0^-$, y goes to $-\infty$, while
- as $x \to -\infty$, y goes to $+\infty$, and
- the curve crosses the x-axis (i.e. has $y = 0$) when $x = c$.

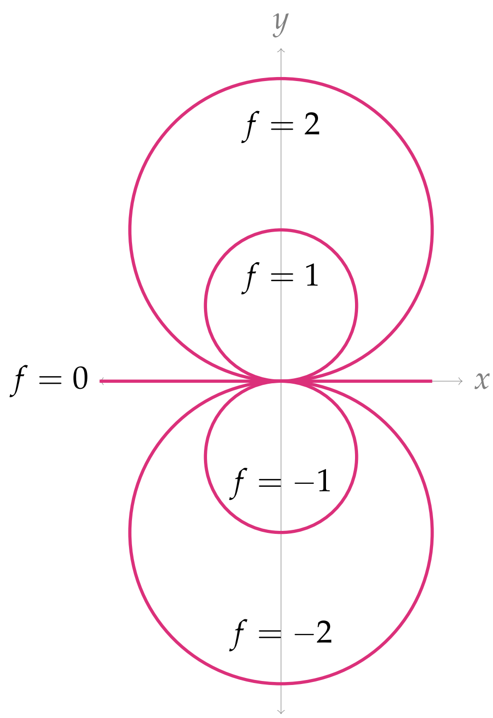
If $c = 0$, the level curve $xe^{-y} = c = 0$ is the y-axis, $x = 0$.



S-4: If $C = 0$, the level curve $f = C = 0$ is just the line $y = 0$. If $C \neq 0$ (of either sign), we may rewrite the equation, $f(x, y) = \frac{2y}{x^2+y^2} = C$, of the level curve $f = C$ as

$$x^2 - \frac{2}{C}y + y^2 = 0 \iff x^2 + \left(y - \frac{1}{C}\right)^2 = \frac{1}{C^2}$$

which is the equation of the circle of radius $\frac{1}{|C|}$ centred on $\left(0\,,\,\frac{1}{C}\right)$.

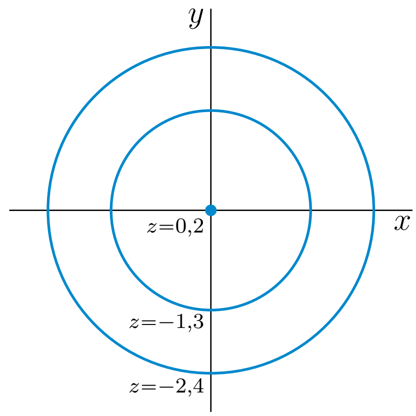


Remark. To be picky, the function $f(x, y) = \frac{2y}{x^2+y^2}$ is not defined at $(x, y) = (0, 0)$. The question should have either specified that the domain of f excludes $(0, 0)$ or have specified a value for $f(0, 0)$. In fact, it is impossible to assign a value to $f(0, 0)$ in such a way that $f(x, y)$ is continuous at $(0, 0)$, because $\lim_{x\to 0} f(x, 0) = 0$ while $\lim_{y\to 0} f(0, |y|) = \infty$. So it makes more sense to have the domain of f being $\mathbb{R}^2$ with the point $(0, 0)$ removed. That's why there is a little hole at the origin in the above sketch.

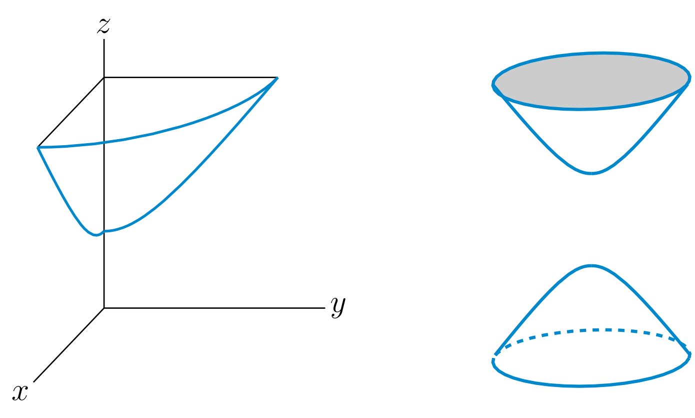
S-5: (a) We can rewrite the equation as

$$x^2 + y^2 = (z-1)^2 - 1$$

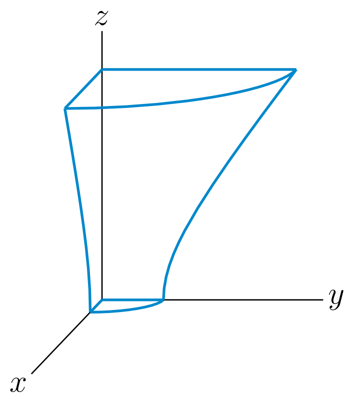
The right hand side is negative for $|z-1| < 1$, i.e. for $0 < z < 2$. So no point on the surface has $0 < z < 2$. For any fixed z, outside that range, the curve $x^2 + y^2 = (z-1)^2 - 1$ is the circle of radius $\sqrt{(z-1)^2 - 1}$ centred on the z–axis. That radius is 0 when $z = 0, 2$ and increases as z moves away from $z = 0, 2$. For very large $|z|$, the radius increases roughly linearly with $|z|$. Here is a sketch of some level curves.

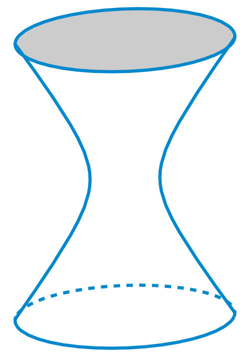


(b) The surface consists of two stacks of circles. One stack starts with radius 0 at $z = 2$. The radius increases as z increases. The other stack starts with radius 0 at $z = 0$. The radius increases as z decreases. This surface is a hyperboloid of two sheets. Here are two sketchs. The sketch on the left is of the part of the surface in the first octant. The sketch on the right of the full surface.

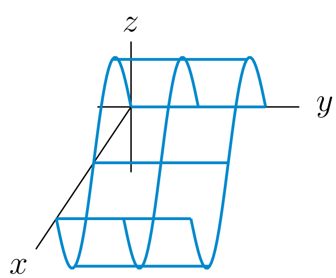


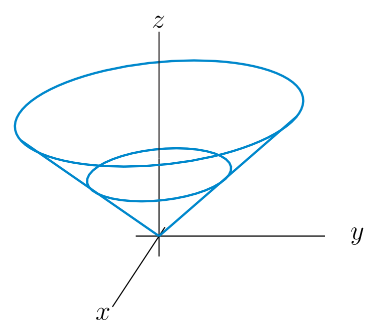
S-6: For each fixed z, $4x^2 + y^2 = 1 + z^2$ is an ellipse. So the surface consists of a stack of ellipses one on top of the other. The semi axes are $\frac{1}{2}\sqrt{1+z^2}$ and $\sqrt{1+z^2}$. These are smallest when $z = 0$ (i.e. for the ellipse in the xy-plane) and increase as $|z|$ increases. The intersection of the surface with the xz-plane (i.e. with the plane $y = 0$) is the hyperbola $4x^2 - z^2 = 1$ and the intersection with the yz-pane (i.e. with the plane $x = 0$) is the hyperbola $y^2 - z^2 = 1$. Here are two sketches of the surface. The sketch on the left only shows the part of the surface in the first octant (with axes).





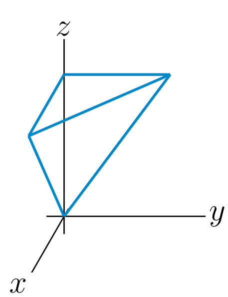
S-7: (a) The graph is $z = \sin x$ with (x, y) running over $0 \leqslant x \leqslant 2\pi$, $0 \leqslant y \leqslant 1$. For each fixed y_0 between 0 and 1, the intersection of this graph with the vertical plane $y = y_0$ is the same sin graph $z = \sin x$ with x running from 0 to 2π. So the whole graph is just a bunch of 2-d sin graphs stacked side-by-side. This gives the graph on the left below.

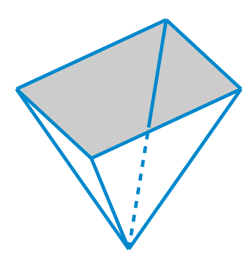




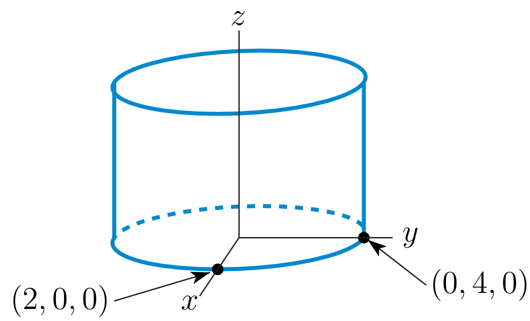
(b) The graph is $z = \sqrt{x^2 + y^2}$. For each fixed $z_0 \geqslant 0$, the intersection of this graph with the horizontal plane $z = z_0$ is the circle $\sqrt{x^2 + y^2} = z_0$. This circle is centred on the z-axis and has radius z_0. So the graph is the upper half of a cone. It is the sketch on the right above.

(c) The graph is $z = |x| + |y|$. For each fixed $z_0 \geqslant 0$, the intersection of this graph with the horizontal plane $z = z_0$ is the square $|x| + |y| = z_0$. The side of the square with $x, y \geqslant 0$ is the straight line $x + y = z_0$. The side of the square with $x \geqslant 0$ and $y \leqslant 0$ is the straight line $x - y = z_0$ and so on. The four corners of the square are $(\pm z_0, 0, z_0)$ and $(0, \pm z_0, z_0)$. So the graph is a stack of squares. It is an upside down four-sided pyramid. The part of the pyramid in the first octant (that is, $x, y, z \geqslant 0$) is the sketch below.

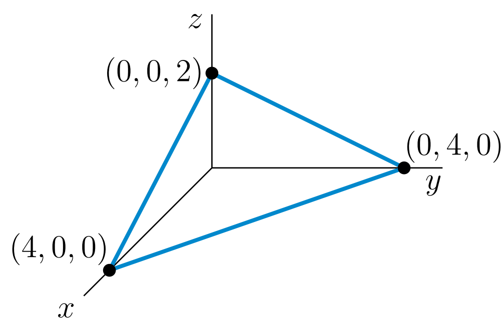




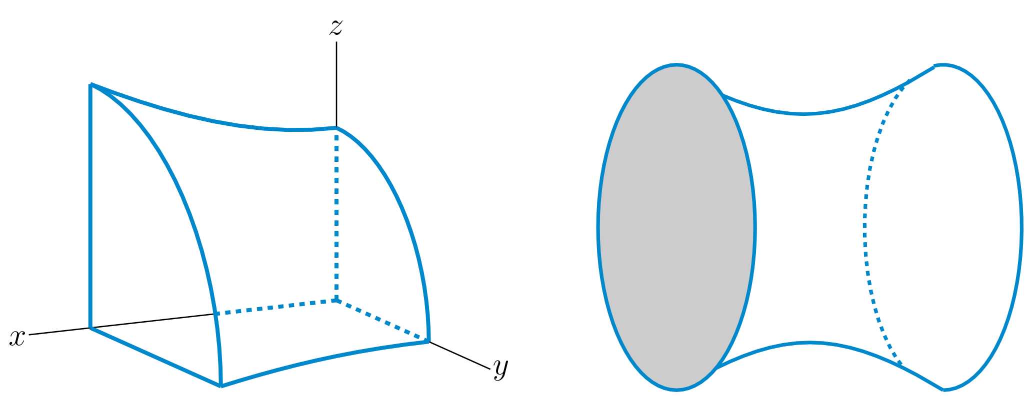
S-8: (a) For each fixed z_0, the $z = z_0$ cross-section (parallel to the xy-plane) of this surface is an ellipse centered on the origin with one semiaxis of length 2 along the x-axis and one semiaxis of length 4 along the y-axis. So this is an elliptic cylinder parallel to the z-axis. Here is a sketch of the part of the surface above the xy–plane.



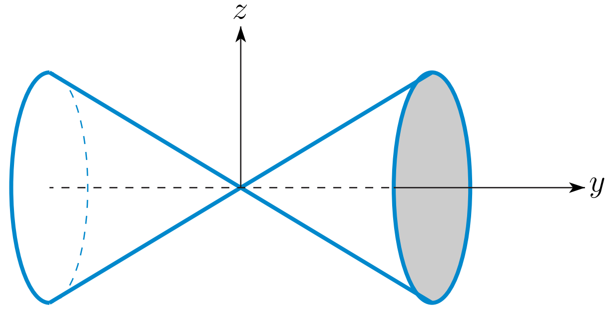
(b) This is a plane through $(4,0,0)$, $(0,4,0)$ and $(0,0,2)$. Here is a sketch of the part of the plane in the first octant.



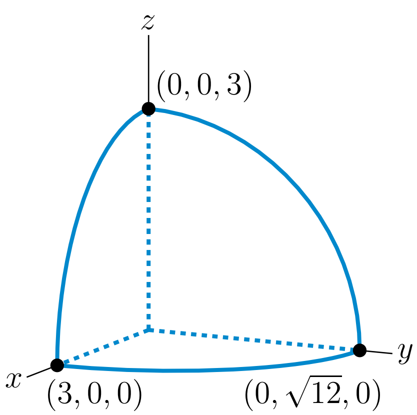
(c) For each fixed x_0, the $x = x_0$ cross-section parallel to the yz-plane is an ellipse with semiaxes $3\sqrt{1+\frac{x_0^2}{16}}$ parallel to the y-axis and $2\sqrt{1+\frac{x_0^2}{16}}$ parallel to the z-axis. As you move out along the x-axis, away from $x = 0$, the ellipses grow at a rate proportional to $\sqrt{1+\frac{x^2}{16}}$, which for large x is approximately $\frac{|x|}{4}$. This is called a hyperboloid of one sheet. Its

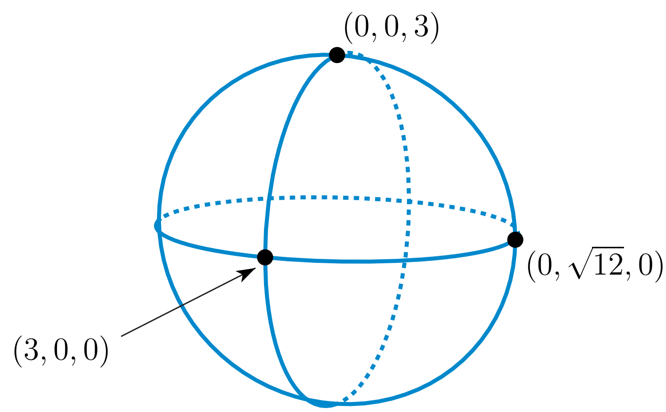


(d) For each fixed y_0, the $y = x_0$ cross-section (parallel to the xz-plane) is a circle of radius $|y|$ centred on the y-axis. When $y_0 = 0$ the radius is 0. As you move further from the xz-plane, in either direction, i.e. as $|y_0|$ increases, the radius grows linearly. The full surface consists of a bunch of these circles stacked sideways. This is a circular cone centred on the y-axis.

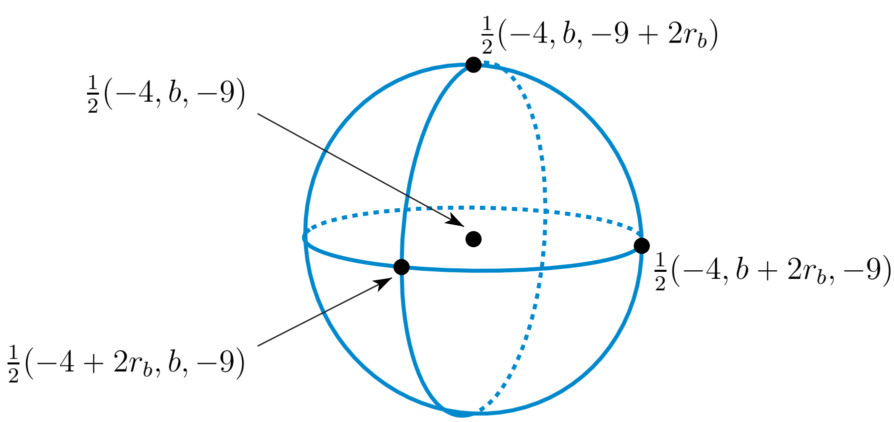


(e) This is an ellipsoid centered on the origin with semiaxes 3, $\sqrt{12} = 2\sqrt{3}$ and 3 along the x, y and z-axes, respectively.

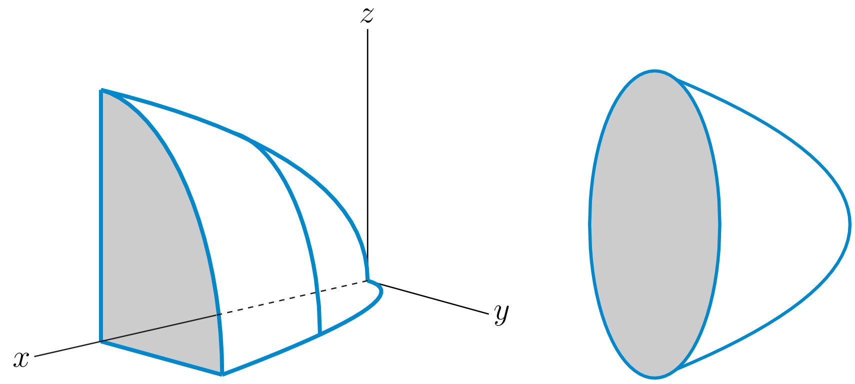




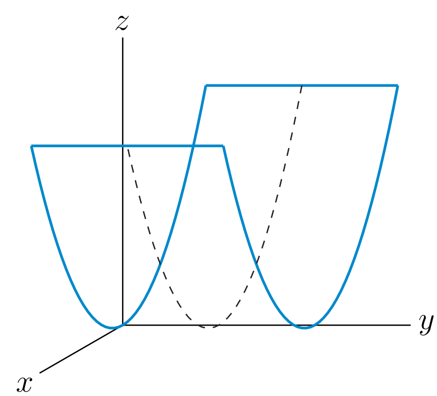
(f) Completing three squares, we have that $x^2 + y^2 + z^2 + 4x - by + 9z - b = 0$ if and only if $(x+2)^2 + \left(y - \frac{b}{2}\right)^2 + \left(z + \frac{9}{2}\right)^2 = b + 4 + \frac{b^2}{4} + \frac{81}{4}$. This is a sphere of radius $r_b = \frac{1}{2}\sqrt{b^2 + 4b + 97}$ centered on $\frac{1}{2}(-4, b, -9)$.



(g) There are no points on the surface with $x < 0$. For each fixed $x_0 > 0$ the cross-section $x = x_0$ parallel to the yz-plane is an ellipse centred on the x–axis with semiaxes $\sqrt{x_0}$ in the y-axis direction and $\frac{3}{2}\sqrt{x_0}$ in the z–axis direction. As you increase x_0, i.e. move out along the x-axis, the ellipses grow at a rate proportional to $\sqrt{x_0}$. This is an elliptic paraboloid with axis the x-axis.



(h) This is called a parabolic cylinder. For any fixed y_0, the $y = y_0$ cross-section (parallel to the xz-plane) is the upward opening parabola $z = x^2$ which has vertex on the y-axis.

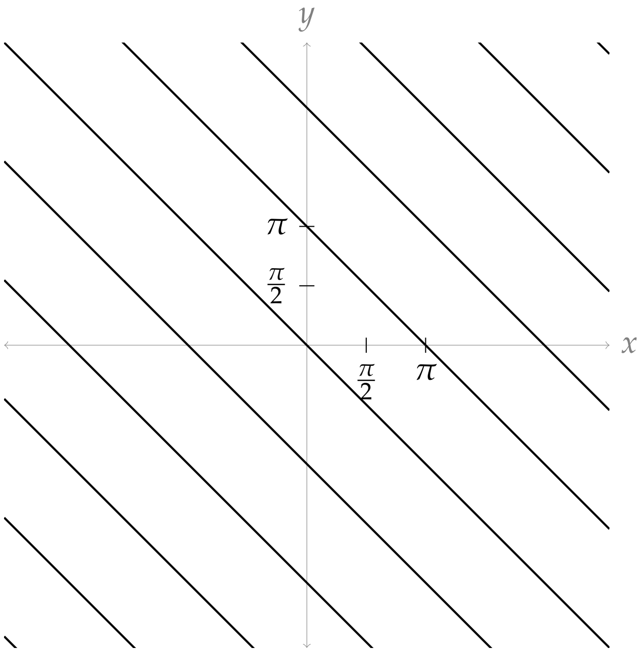


S-9: The level curves of $z = 0$ correspond to all points (x, y) such that $0 = \sin(x + y)$. The angles that make $\sin\theta$ equal to 0 are $\theta = \pi n$ for integer values of n. So, the level curves are lines of the form

$$x + y = \pi n$$

where n is any integer.

So, our level curve has the lines $y = -x$, $y = \pi - x$, $y = 2\pi - x$, etc.

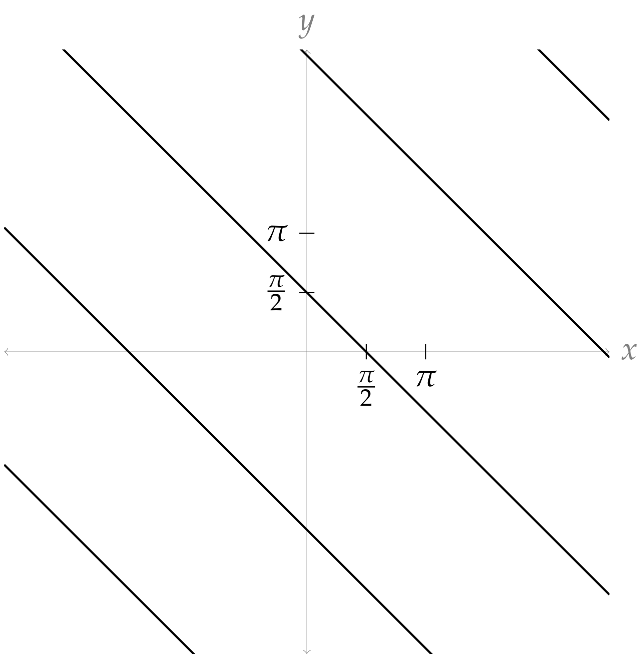


The level curves of $z = 1$ correspond to all points (x, y) such that $1 = \sin(x + y)$. The angles that make $\sin\theta$ equal to 1 are $\theta = \frac{pi}{2} + 2\pi n$ for integer values of n. So, the level curves are lines of the form

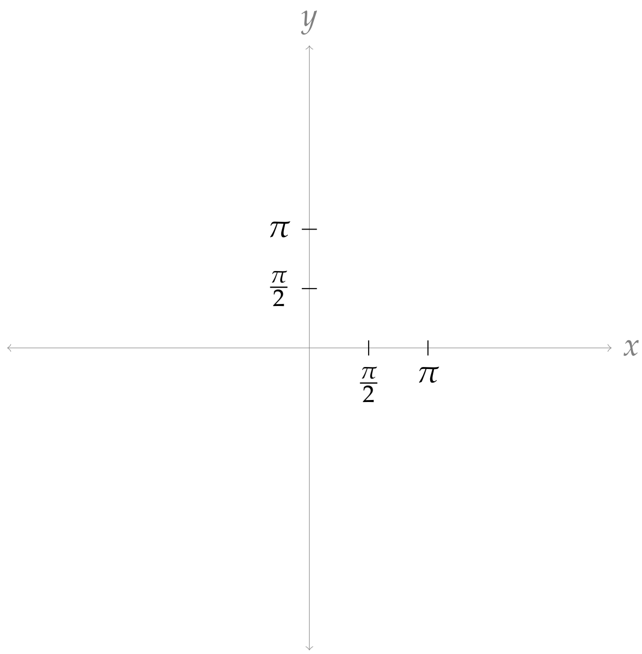
$$x + y = \frac{\pi}{2} + 2\pi n$$

where n is any integer.

So, our level curve has the lines $y = \frac{\pi}{2} - x$, $y = \frac{\pi}{2} + 2\pi - x$, $y = \frac{\pi}{2} + 4\pi - x$, etc.



The equation $2 = \sin(x + y)$ has no solutions, since no angle has sine greater than 1. So the level curve at $z = 2$ has no points:

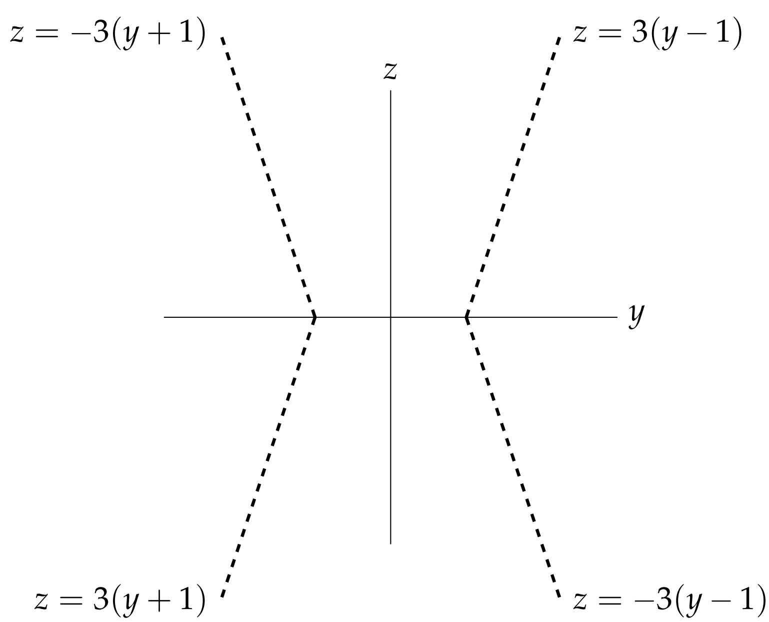


S-10: Since the level curves are circles centred at the origin (in the xy-plane), when z is a constant, the equation will have the form $x^2 + y^2 = c$ for some constant. That is, our equation looks like

$$x^2 + y^2 = g(z),$$

where $g(z)$ is a function depending only on z.

Because our cross-sections are so nicely symmetric, we know the intersection of the figure with the left side of the yz-plane as well: $z = 3(-y-1) = -3(y+1)$ (when $z \geqslant 0$) and $z = -3(-y-1) = 3(y+1)$ (when $z < 0$). Below is the intersection of our surface with the yz plane.



Setting $x = 0$, our equation becomes $y^2 = g(z)$. Looking at the right side of the yz plane, this should lead to: $\begin{Bmatrix} z = 3(y-1) & \text{if } z \geqslant 0,\ y \geqslant 1 \\ z = -3(y-1) & \text{if } z < 0,\ y \geqslant 1 \end{Bmatrix}$. That is:

$$\begin{aligned} |z| &= 3(y-1) \\ \frac{|z|}{3} + 1 &= y \\ \left(\frac{|z|}{3} + 1\right)^2 &= y^2 \qquad (*) \end{aligned}$$

A quick check: when we squared both sides of the equation in $(*)$, we added another solution, $\frac{|z|}{3} + 1 = -y$. Let's make sure we haven't diverged from our diagram.

$$\begin{aligned} & \left(\frac{|z|}{3} + 1\right)^2 = y^2 \\ \Leftrightarrow\quad & \underbrace{\frac{|z|}{3} + 1}_{\text{positive}} = \pm y \\ \Leftrightarrow\quad & \begin{cases} \frac{|z|}{3} + 1 = y & y > 0 \\ \frac{|z|}{3} + 1 = -y & y < 0 \end{cases} \\ \Leftrightarrow\quad & \begin{cases} \frac{|z|}{3} + 1 = y & y \geqslant 1 \\ \frac{|z|}{3} + 1 = -y & y \leqslant -1 \end{cases} \\ \Leftrightarrow\quad & \begin{cases} |z| = 3(y-1) & y \geqslant 1 \\ |z| = -3(y+1) & y \leqslant -1 \end{cases} \\ \Leftrightarrow\quad & \begin{cases} z = \pm \underbrace{3(y-1)}_{\text{positive}} & y \geqslant 1 \\ z = \pm \underbrace{3(y+1)}_{\text{negative}} & y \leqslant -1 \end{cases} \\ \Leftrightarrow\quad & \begin{cases} z = 3(y-1) & y \geqslant 1,\ z \geqslant 0 \\ z = -3(y-1) & y \geqslant 1,\ z \leqslant 0 \\ z = -3(y+1) & y \leqslant -1,\ z \geqslant 0 \\ z = 3(y+1) & y \leqslant -1,\ z \leqslant 0 \end{cases} \end{aligned}$$

This matches our diagram eactly. So, all together, the equation of the surface is

$$x^2 + y^2 = \left(\frac{|z|}{3} + 1\right)^2$$

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S-1:

If $f_y(0,0) < 0$, then $f(0,y)$ decreases as y increases from 0. Thus moving in the positive y direction takes you downhill. This means you aren't at the lowest point in a valley, since you can still move downhill. On the other hand, as $f_y(0,0) < 0$, $f(0,y)$ also decreases as y increases towards 0 from slightly negative values. Thus if you move in the *negative* y-direction from $y = 0$, your height z will *increase*. So you are not at a locally highest point—you're not at a summit.

S-2: The definition of the derivative involves a limit at h goes to 0; we can approximate that limit by choosing a value of h that's close to 0; in our case, 0.1 or -0.1 are the best we can do, using the information on the table.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \approx \frac{f(x+0.1,y) - f(x,y)}{0.1}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \approx \frac{f(x,y+0.1) - f(x,y)}{0.1}$$

(a) To find $f_y(1.5, 2.4)$, we keep x fixed at $x = 1.5$, and vary y. We don't know what happens at $y = 2.5$, but we do know what happens at $y = 2.3$:

$$f_y(1.5, 2.4) \approx \frac{f(1.5, 2.3) - f(1.5, 2.4)}{2.3 - 2.4} = \frac{11.2 - 11.0}{-0.1} = -2$$

(b) To find $f_x(1.7, 1.7)$, we keep y fixed at $y = 1.7$, and vary x. We can choose to use either $x = 1.6$ or $x = 1.8$.

$$f_x(1.7, 1.7) \approx \frac{f(1.8, 1.7) - f(1.7, 1.7)}{1.8 - 1.7} = \frac{16.1 - 15.0}{0.1} = 11$$
$$f_x(1.7, 1.7) \approx \frac{f(1.6, 1.7) - f(1.7, 1.7)}{1.6 - 1.7} = \frac{13.9 - 15.0}{-0.1} = 11$$

(c) To find $f_y(1.7, 1.7)$, we keep x fixed at $x = 1.7$, and vary y. We can choose to use either $y = 1.6$ or $y = 1.8$.

$$f_y(1.7, 1.7) \approx \frac{f(1.7, 1.8) - f(1.7, 1.7)}{1.8 - 1.7} = \frac{14.7 - 15.0}{0.1} = -3$$
$$f_y(1.7, 1.7) \approx \frac{f(1.7, 1.6) - f(1.7, 1.7)}{1.6 - 1.7} = \frac{15.3 - 15.0}{-0.1} = -3$$

(d) To find $f_x(1.1, 2)$, we keep y fixed at $y = 2$, and vary x. We can choose to use either $x = 1.0$ or $x = 1.2$.

$$f_x(1.1, 2) \approx \frac{f(1.2, 2) - f(1.1, 2)}{1.2 - 1.1} = \frac{9.1 - 8.2}{0.1} = 9$$
$$f_x(1.1, 2) \approx \frac{f(1.0, 2) - f(1.1, 2)}{1.0 - 1.1} = \frac{7.3 - 8.2}{-0.1} = 9$$

S-3: (a)

$$\begin{aligned} f_x(x,y,z) &= 3x^2y^4z^5 & f_x(0,-1,-1) &= 0\\ f_y(x,y,z) &= 4x^3y^3z^5 & f_y(0,-1,-1) &= 0\\ f_z(x,y,z) &= 5x^3y^4z^4 & f_z(0,-1,-1) &= 0 \end{aligned}$$

(b)

$$\begin{aligned} w_x(x,y,z) &= \frac{yze^{xyz}}{1+e^{xyz}} & w_x(2,0,-1) &= 0\\ w_y(x,y,z) &= \frac{xze^{xyz}}{1+e^{xyz}} & w_y(2,0,-1) &= -1\\ w_z(x,y,z) &= \frac{xye^{xyz}}{1+e^{xyz}} & w_z(2,0,-1) &= 0 \end{aligned}$$

(c)

$$\begin{aligned} f_x(x,y) &= -\frac{x}{(x^2+y^2)^{3/2}} & f_x(-3,4) &= \frac{3}{125}\\ f_y(x,y) &= -\frac{y}{(x^2+y^2)^{3/2}} & f_y(-3,4) &= -\frac{4}{125} \end{aligned}$$

S-4: By the quotient rule

$$\begin{aligned} \frac{\partial z}{\partial x}(x,y) &= \frac{(1)(x-y)-(x+y)(1)}{(x-y)^2} &&= \frac{-2y}{(x-y)^2}\\ \frac{\partial z}{\partial y}(x,y) &= \frac{(1)(x-y)-(x+y)(-1)}{(x-y)^2} &&= \frac{2x}{(x-y)^2} \end{aligned}$$

Hence

$$x\frac{\partial z}{\partial x}(x,y)+y\frac{\partial z}{\partial y}(x,y) = \frac{-2xy+2yx}{(x-y)^2} = 0$$

S-5: (a) We are told that $z(x,y)$ obeys

$$\begin{aligned} z(x,y)\,y - y + x &= \ln\big(xy\,z(x,y)\big)\\ &= \ln x + \ln y + \ln\big(z(x,y)\big) \end{aligned} \tag{$*$}$$

for all (x,y) (near $(-1,-2)$). Recall the following derivatives:

- The partial derivative of z with respect to x is $\frac{\partial z}{\partial x}$
- The partial derivative of y with respect to x is 0 (since we treat y as a constant)

- The partial derivative of x with respect to x is 1

Differentiating $(*)$ with respect to x gives

$$y\,\frac{\partial z}{\partial x}(x,y)+1=\frac{1}{x}+\frac{\frac{\partial z}{\partial x}(x,y)}{z(x,y)} \implies \frac{\partial z}{\partial x}(x,y)=\frac{\frac{1}{x}-1}{y-\frac{1}{z(x,y)}}$$

or, dropping the arguments (x,y) and multiplying both the numerator and denominator by xz,

$$\frac{\partial z}{\partial x}=\frac{z-xz}{xyz-x}=\frac{z(1-x)}{x(yz-1)}$$

Differentiating $(*)$ with respect to y gives

$$z(x,y)+y\,\frac{\partial z}{\partial y}(x,y)-1=\frac{1}{y}+\frac{\frac{\partial z}{\partial y}(x,y)}{z(x,y)} \implies \frac{\partial z}{\partial y}(x,y)=\frac{\frac{1}{y}+1-z(x,y)}{y-\frac{1}{z(x,y)}}$$

or, dropping the arguments (x,y) and multiplying both the numerator and denominator by yz,

$$\frac{\partial z}{\partial y}=\frac{z+yz-yz^2}{y^2z-y}=\frac{z(1+y-yz)}{y(yz-1)}$$

(b) When $(x,y,z)=(-1,-2,1/2)$,

$$\begin{aligned}\frac{\partial z}{\partial x}(-1,-2)&=\left.\frac{\frac{1}{x}-1}{y-\frac{1}{z}}\right|_{(x,y,z)=(-1,-2,1/2)}=\frac{\frac{1}{-1}-1}{-2-2}=\frac{1}{2}\\ \frac{\partial z}{\partial y}(-1,-2)&=\left.\frac{\frac{1}{y}+1-z}{y-\frac{1}{z}}\right|_{(x,y,z)=(-1,-2,1/2)}=\frac{\frac{1}{-2}+1-\frac{1}{2}}{-2-2}=0\end{aligned}$$

S-6: We are told that the four variables T, U, V, W obey the the single equation $(TU-V)^2\ln(W-UV)=\ln 2$. So they are not all independent variables. Roughly speaking, we can treat any three of them as independent variables and solve the given equation for the fourth as a function of the three chosen independent variables.

We are first asked to find $\frac{\partial U}{\partial T}$. This implicitly tells to treat T, V and W as independent variables and to view U as a function $U(T,V,W)$ that obeys

$$\big(T\,U(T,V,W)-V\big)^2\ln\big(W-U(T,V,W)\,V\big)=\ln 2 \tag{E1}$$

for all (T,U,V,W) sufficiently near $(1,1,2,4)$. Differentiating (E1) with respect to T gives

$$\begin{aligned}2\big(T\,U(T,V,W)-V\big)\left[U(T,V,W)+T\,\frac{\partial U}{\partial T}(T,V,W)\right]\ln\big(W-U(T,V,W)\,V\big)&\\ -\big(T\,U(T,V,W)-V\big)^2\frac{1}{W-U(T,V,W)\,V}\frac{\partial U}{\partial T}(T,V,W)\,V=0&\end{aligned}$$

In particular, for $(T,U,V,W) = (1,1,2,4)$,

$$2\big((1)(1)-2\big)\left[1+(1)\frac{\partial U}{\partial T}(1,2,4)\right]\ln\big(4-(1)(2)\big) \\ -\big((1)(1)-2\big)^2\frac{1}{4-(1)(2)}\frac{\partial U}{\partial T}(1,2,4)\,(2) = 0$$

This simplifies to

$$-2\left[1+\frac{\partial U}{\partial T}(1,2,4)\right]\ln(2) - \frac{\partial U}{\partial T}(1,2,4) = 0 \implies \frac{\partial U}{\partial T}(1,2,4) = -\frac{2\ln(2)}{1+2\ln(2)}$$

We are then asked to find $\frac{\partial T}{\partial V}$. This implicitly tells to treat U, V and W as independent variables and to view T as a function $T(U,V,W)$ that obeys

$$\big(T(U,V,W)\,U - V\big)^2\ln\big(W-U\,V\big) = \ln 2 \tag{E2}$$

for all (T,U,V,W) sufficiently near $(1,1,2,4)$. Differentiating (E2) with respect to V gives

$$2\big(T(U,V,W)\,U-V\big)\ \left[\frac{\partial T}{\partial V}(U,V,W)\,U-1\right]\ln\big(W-U\,V\big) \\ -\big(T(U,V,W)\,U-V\big)^2\frac{U}{W-U\,V} = 0$$

In particular, for $(T,U,V,W) = (1,1,2,4)$,

$$2\big((1)(1)-2\big)\left[(1)\frac{\partial T}{\partial V}(1,2,4)-1\right]\ln\big(4-(1)(2)\big) \\ -\big((1)(1)-2\big)^2\frac{1}{4-(1)(2)} = 0$$

This simplifies to

$$-2\left[\frac{\partial T}{\partial V}(1,2,4)-1\right]\ln(2) - \frac{1}{2} = 0 \implies \frac{\partial T}{\partial V}(1,2,4) = 1 - \frac{1}{4\ln(2)}$$

S-7: The function

$$\begin{aligned} u(\rho,r,\theta) &= \big[\rho r\cos\theta\big]^2 + \big[\rho r\sin\theta\big]\rho r \\ &= \rho^2 r^2\cos^2\theta + \rho^2 r^2\sin\theta \end{aligned}$$

So

$$\frac{\partial u}{\partial r}(\rho,r,\theta) = 2\rho^2 r\cos^2\theta + 2\rho^2 r\sin\theta$$

and

$$\frac{\partial u}{\partial r}(2,3,\pi/2) = 2(2^2)(3)(0)^2 + 2(2^2)(3)(1) = 24$$

S-8: By definition

$$f_x(x_0,y_0) = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x,y_0) - f(x_0,y_0)}{\Delta x} \qquad f_y(x_0,y_0) = \lim_{\Delta y\to 0} \frac{f(x_0,y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

Setting $x_0 = y_0 = 0$,

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x\to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(\Delta x,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{((\Delta x)^2 - 2\times 0^2)/(\Delta x - 0)}{\Delta x} \\ &= \lim_{\Delta x\to 0} 1 = 1 \\ f_y(0,0) &= \lim_{\Delta y\to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{f(0,\Delta y)}{\Delta y} = \lim_{\Delta y\to 0} \frac{(0^2 - 2(\Delta y)^2)/(0 - \Delta y)}{\Delta y} \\ &= \lim_{\Delta y\to 0} 2 = 2 \end{aligned}$$

S-9: As $z(x,y) = f(x^2+y^2)$

$$\begin{aligned} \frac{\partial z}{\partial x}(x,y) &= 2xf'(x^2+y^2) \\ \frac{\partial z}{\partial y}(x,y) &= 2yf'(x^2+y^2) \end{aligned}$$

by the (ordinary single variable) chain rule. So

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y(2x)f'(x^2+y^2) - x(2y)f'(x^2+y^2) = 0$$

and the differential equation is always satisfied, assuming that f is differentiable, so that the chain rule applies.

S-10: By definition

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{\Delta x\to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\frac{(\Delta x+2\times 0)^2}{\Delta x+0} - 0}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{\Delta y\to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} \\ &= \lim_{\Delta y\to 0} \frac{\frac{(0+2\Delta y)^2}{0+\Delta y} - 0}{\Delta y} \\ &= \lim_{\Delta y\to 0} \frac{4\Delta y}{\Delta y} \\ &= 4 \end{aligned}$$

(b) $f(x,y)$ is *not continuous* at $(0,0)$, even though both partial derivatives exist there. To see this, make a change of coordinates from (x,y) to (X,y) with $X = x+y$ (the denominator). Of course, $(x,y) \to (0,0)$ if and only if $(X,y) \to (0,0)$. Now watch what happens when $(X,y) \to (0,0)$ with X a lot smaller than y. For example, $X = ay^2$. Then

$$\frac{(x+2y)^2}{x+y} = \frac{(X+y)^2}{X} = \frac{(ay^2+y)^2}{ay^2} = \frac{(1+ay)^2}{a} \to \frac{1}{a}$$

This depends on a. So approaching $(0,0)$ along different paths gives different limits. (You can see the same effect without changing coordinates by sending $(x,y) \to (0,0)$ with $x = -y + ay^2$.) Even more dramatically, watch what happens when $(X,y) \to (0,0)$ with $X = y^3$. Then

$$\frac{(x+2y)^2}{x+y} = \frac{(X+y)^2}{X} = \frac{(y^3+y)^2}{y^3} = \frac{\left(1+y^2\right)^2}{y} \to \pm\infty$$

S-11: **Solution 1**

Let's start by finding an equation for this surface. Every level curve is a horizontal circle of radius one, so the equation should be of the form

$$(x-f_1)^2 + (y-f_2)^2 = 1$$

where f_1 and f_2 are functions depending only on z. Since the centre of the circle at height z is at position $x = 0$, $y = z$, we see that the equation of our surface is

$$x^2 + (y-z)^2 = 1$$

The height of the surface at the point (x,y) is the $z(x,y)$ found by solving that equation. That is,

$$x^2 + \big(y - z(x,y)\big)^2 = 1 \tag{$*$}$$

We differentiate this equation implicitly to find $z_x(x,y)$ and $z_y(x,y)$ at the desired point $(x,y) = (0,-1)$. First, differentiating $(*)$ with respect to y gives

$$\begin{aligned} 0 + 2\big(y - z(x,y)\big)\big(1 - z_y(x,y)\big) &= 0 \\ 2(-1-0)\big(1 - z_y(0,-1)\big) &= 0 \qquad \text{at } (0,-1,0) \end{aligned}$$

so that the slope looking in the positive y direction is $z_y(0,-1) = 1$. Similarly, differentiating $(*)$ with respect to x gives

$$\begin{aligned} 2x + 2\big(y - z(x,y)\big) \cdot \big(0 - z_x(x,y)\big) &= 0 \\ 2x &= 2\big(y - z(x,y)\big) \cdot z_x(x,y) \\ z_x(x,y) &= \frac{x}{y - z(x,y)} \\ z_x(0,-1) &= 0 \qquad \text{at } (0,-1,0) \end{aligned}$$

The slope looking in the positive x direction is $z_x(0,-1) = 0$.

Solution 2

Standing at $(0,-1,0)$ and looking in the positive y direction, the surface follows the straight line that

- passes through the point $(0, -1, 0)$, and
- is parallel to the central line $z = y, x = 0$ of the cylinder.

Shifting the central line one unit in the y-direction, we get the line $z = y + 1$, $x = 0$. (As a check, notice that $(0, -1, 0)$ is indeed on $z = y + 1$, $x = 0$.) The slope of this line is 1.

Standing at $(0, -1, 0)$ and looking in the positive x direction, the surface follows the circle $x^2 + y^2 = 1$, $z = 0$, which is the intersection of the cylinder with the xy-plane. As we move along that circle our z coordinate stays fixed at 0. So the slope in that direction is 0.

S-12: (a) By definition

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{(\Delta x^2)(0)}{\Delta x^2 + 0^2} - 0}{\Delta x} \\ &= 0\end{aligned}$$

(b) By definition

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{\frac{(0^2)(\Delta y)}{0^2 + \Delta y^2} - 0}{\Delta y} \\ &= 0\end{aligned}$$

(c) By definition

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f(t,t)\Big|_{t=0} &= \lim_{t \to 0} \frac{f(t,t) - f(0,0)}{t} \\ &= \lim_{h \to 0} \frac{\frac{(t^2)(t)}{t^2 + t^2} - 0}{t} \\ &= \lim_{t \to 0} \frac{t/2}{t} \\ &= \frac{1}{2}\end{aligned}$$

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S-1: We have to derive a bunch of equalities.

- Fix any real number x and set $g(y, z) = f_x(x, y, z)$. By (Clairaut's) Theorem 2.2.5 in the CLP-3 text $g_{yz}(y, z) = g_{zy}(y, z)$, so

$$f_{xyz}(x, y, z) = g_{yz}(y, z) = g_{zy}(y, z) = f_{xzy}(x, y, z)$$

- For every fixed real number z, (Clairaut's) Theorem 2.2.5 in the CLP-3 text gives $f_{xy}(x,y,z) = f_{yx}(x,y,z)$. So

$$f_{xyz}(x,y,z) = \frac{\partial}{\partial z} f_{xy}(x,y,z) = \frac{\partial}{\partial z} f_{yx}(x,y,z) = f_{yxz}(x,y,z)$$

  So far, we have

$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z)$$

- Fix any real number y and set $g(x,z) = f_y(x,y,z)$. By (Clairaut's) Theorem 2.2.5 in the CLP-3 text $g_{xz}(x,z) = g_{zx}(x,z)$. So

$$f_{yxz}(x,y,z) = g_{xz}(x,z) = g_{zx}(x,z) = f_{yzx}(x,y,z)$$

  So far, we have

$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{yzx}(x,y,z)$$

- For every fixed real number y, (Clairaut's) Theorem 2.2.5 in the CLP-3 text gives $f_{xz}(x,y,z) = f_{zx}(x,y,z)$. So

$$f_{xzy}(x,y,z) = \frac{\partial}{\partial y} f_{xz}(x,y,z) = \frac{\partial}{\partial y} f_{zx}(x,y,z) = f_{zxy}(x,y,z)$$

  So far, we have

$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z)$$

- Fix any real number z and set $g(x,y) = f_z(x,y,z)$. By (Clairaut's) Theorem 2.2.5 in the CLP-3 text $g_{xy}(x,y) = g_{yx}(x,y)$. So

$$f_{zxy}(x,y,z) = g_{xy}(x,y) = g_{yx}(x,y) = f_{zxy}(x,y,z)$$

  We now have all of

$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z) = f_{zxy}(x,y,z)$$

S-2: No such $f(x,y)$ exists, because if it were to exist, then we would have that $f_{xy}(x,y) = f_{yx}(x,y)$. But

$$f_{xy}(x,y) = \frac{\partial}{\partial y} f_x(x,y) = \frac{\partial}{\partial y} e^y = e^y$$
$$f_{yx}(x,y) = \frac{\partial}{\partial x} f_y(x,y) = \frac{\partial}{\partial x} e^x = e^x$$

are not equal.

S-3: (a) We have

$$f_x(x,y) = 2xy^3 \qquad f_{xx}(x,y) = 2y^3$$
$$f_{xy}(x,y) = 6xy^2 \qquad f_{yxy}(x,y) = f_{xyy}(x,y) = 12xy$$

(b) We have

$$f_x(x,y) = y^2 e^{xy^2} \quad f_{xx}(x,y) = y^4 e^{xy^2} \qquad f_{xxy}(x,y) = 4y^3 e^{xy^2} + 2xy^5 e^{xy^2}$$
$$f_{xy}(x,y) = 2ye^{xy^2} + 2xy^3 e^{xy^2} \quad f_{xyy}(x,y) = \big(2 + 4xy^2 + 6xy^2 + 4x^2y^4\big)e^{xy^2}$$
$$= \big(2 + 10xy^2 + 4x^2y^4\big)e^{xy^2}$$

(c) We have

$$\begin{aligned}
\frac{\partial f}{\partial u}(u,v,w) &= -\frac{1}{(u+2v+3w)^2}\\
\frac{\partial^2 f}{\partial u\,\partial v}(u,v,w) &= \frac{4}{(u+2v+3w)^3}\\
\frac{\partial^3 f}{\partial u\,\partial v\,\partial w}(u,v,w) &= -\frac{36}{(u+2v+3w)^4}
\end{aligned}$$

In particular

$$\frac{\partial^3 f}{\partial u\,\partial v\,\partial w}(3,2,1) = -\frac{36}{(3+2\times 2+3\times 1)^4} = -\frac{36}{10^4} = -\frac{9}{2500}$$

S-4: Let $f(x,y) = \sqrt{x^2+5y^2}$. Then

$$f_x = \frac{x}{\sqrt{x^2+5y^2}} \qquad f_{xx} = \frac{1}{\sqrt{x^2+5y^2}} - \frac{1}{2}\frac{(x)(2x)}{(x^2+5y^2)^{3/2}} \qquad f_{xy} = -\frac{1}{2}\frac{(x)(10y)}{(x^2+5y^2)^{3/2}}$$
$$f_y = \frac{5y}{\sqrt{x^2+5y^2}} \qquad f_{yy} = \frac{5}{\sqrt{x^2+5y^2}} - \frac{1}{2}\frac{(5y)(10y)}{(x^2+5y^2)^{3/2}} \qquad f_{yx} = -\frac{1}{2}\frac{(5y)(2x)}{(x^2+5y^2)^{3/2}}$$

Simplifying, and in particular using that $\frac{1}{\sqrt{x^2+5y^2}} = \frac{x^2+5y^2}{(x^2+5y^2)^{3/2}}$,

$$f_{xx} = \frac{5y^2}{(x^2+5y^2)^{3/2}} \qquad f_{xy} = f_{yx} = -\frac{5xy}{(x^2+5y^2)^{3/2}} \qquad f_{yy} = \frac{5x^2}{(x^2+5y^2)^{3/2}}$$

S-5: (a) As $f(x,y,z) = \arctan\big(e^{\sqrt{xy}}\big)$ is independent of z, we have $f_z(x,y,z) = 0$ and hence

$$f_{xyz}(x,y,z) = f_{zxy}(x,y,z) = 0$$

(b) Write $u(x,y,z) = \arctan\big(e^{\sqrt{xy}}\big)$, $v(x,y,z) = \arctan\big(e^{\sqrt{xz}}\big)$ and $w(x,y,z) = \arctan\big(e^{\sqrt{yz}}\big)$. Then

- As $u(x,y,z) = \arctan\left(e^{\sqrt{xy}}\right)$ is independent of z, we have $u_z(x,y,z) = 0$ and hence $u_{xyz}(x,y,z) = u_{zxy}(x,y,z) = 0$
- As $v(x,y,z) = \arctan\left(e^{\sqrt{xz}}\right)$ is independent of y, we have $v_y(x,y,z) = 0$ and hence $v_{xyz}(x,y,z) = v_{yxz}(x,y,z) = 0$
- As $w(x,y,z) = \arctan\left(e^{\sqrt{yz}}\right)$ is independent of x, we have $w_x(x,y,z) = 0$ and hence $w_{xyz}(x,y,z) = 0$

As $f(x,y,z) = u(x,y,z) + v(x,y,z) + w(x,y,z)$, we have

$$f_{xyz}(x,y,z) = u_{xyz}(x,y,z) + v_{xyz}(x,y,z) + w_{xyz}(x,y,z) = 0$$

(c) In the course of evaluating $f_{xx}(x,0,0)$, both y and z are held fixed at 0. Thus, if we set $g(x) = f(x,0,0)$, then $f_{xx}(x,0,0) = g''(x)$. Now

$$g(x) = f(x,0,0) = \arctan\left(e^{\sqrt{xyz}}\right)\Big|_{y=z=0} = \arctan(1) = \frac{\pi}{4}$$

for all x. So $g'(x) = 0$ and $g''(x) = 0$ for all x. In particular,

$$f_{xx}(1,0,0) = g''(1) = 0$$

S-6: As

$$\begin{aligned}
u_t(x,y,z,t) &= -\frac{3}{2}\frac{1}{t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{1}{4\alpha\, t^{7/2}}(x^2+y^2+z^2)e^{-(x^2+y^2+z^2)/(4\alpha t)} \\
u_x(x,y,z,t) &= -\frac{x}{2\alpha\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} \\
u_{xx}(x,y,z,t) &= -\frac{1}{2\alpha\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{x^2}{4\alpha^2\, t^{7/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} \\
u_{yy}(x,y,z,t) &= -\frac{1}{2\alpha\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{y^2}{4\alpha^2\, t^{7/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} \\
u_{zz}(x,y,z,t) &= -\frac{1}{2\alpha\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{z^2}{4\alpha^2\, t^{7/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)}
\end{aligned}$$

we have

$$\alpha\big(u_{xx} + u_{yy} + u_{zz}\big) = -\frac{3}{2\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{x^2+y^2+z^2}{4\alpha\, t^{7/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} = u_t$$

S-7: It is given in the question that $x, y > 0$. This is important when deciding the sign of the derivatives. Here, if you have trouble deciding the sign, it might be helpful to fix some value for x and y.

(a) $u(x,y) = x^{0.5}y^{0.5}$: First let us check for x.

   (i) $u_x = 0.5x^{-0.5}y^{0.5} > 0$

   (ii) $u_{xx} = -0.25x^{-1.5}y^{0.5} < 0$

(iii) $u_x \to \infty$ as $x \to \infty$

Computations for y is very similar to x since $u(x, y)$ is symmetric in terms of x and y. So we can just switch the roles of x and y to see the conditions are satisfied for y. So, this utility function satisfies the required properties.

(b) $u(x, y) = \dfrac{x^{0.5}}{y^{0.5}}$: First let us check for y.

(i) $u_y = -0.5x^{0.5}y^{-1.5} < 0$

(ii) $u_{yy} = (-0.5)(-1.5)x^{0.5}y^{-2.5} > 0$

(iii) $u_y \to -\infty$ as $y \to \infty$

We can see right away that this utility function does not satisfy the all required properties! No need to check for them for x. (It does satisfy them when we check for x).

(c) $u(x, y) = \ln(x) + \ln(y)$: First let us first find u_y.

$$u_y = \ln(x) + \frac{1}{y}$$

This is not always positive nor always negative. For example, if $x = e^{-2}$ and $y = 1$, then $u_y = -2 + 1 = -1$. So, this utility function does not satisfy all the required properties! No need to check the rest.

(d) $u(x) = \dfrac{x^{1-a}}{1-a}$. Note that u only depends on x here, so we only check the properties for x. We have

$$u_x = \frac{1-a}{1-a}x^{1-a-1} = x^{-a}, \qquad u_{xx} = -ax^{-a-1}$$

Now $x^{-a} > 0$ as $x > 0$, no matter the sign of a so condition (i) is always satisfied. For the sign of u_{xx}, note that x^{-a-1} is always positive as $x > 0$. We need to work with three cases:

(1) First if $a = 0$ then $u(x) = x$ and $u_{xx} = 0$, so in this case $u(x)$ will not satisfy all of our conditions and we do not need to check for the rest of the conditions.

(2) If $a > 0$ then $u_{xx} < 0$.

(3) If $a < 0$ then $u_{xx} > 0$ which, similarly to case (1), $u(x)$ will not satisfy all of our conditions and we do not need to check for the rest of the conditions.

So the only chance that $u(x)$ can satisfy all of our properties is when $a > 0$. In this case, however, $u_x(x) = x^{-a}$ tends to 0 as x tends to infinity. This means that for any value of a, u will not satisfy all of our conditions.

S-8: The definition of the derivative involves a limit at h goes to 0; we can approximate that limit by choosing a value of h that's close to 0; in our case, 0.1 or -0.1 are the best we can do, using the information on the table.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \approx \frac{f(x+0.1,y) - f(x,y)}{0.1}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \approx \frac{f(x,y+0.1) - f(x,y)}{0.1}$$

The same holds for the second derivative:

$$\begin{aligned} f_{xy}(x,y) = \left(f_x(x,y)\right)_y &= \lim_{h\to 0} \frac{f_x(x,y+h) - f_x(x,y)}{h} \\ &\approx \frac{f_x(x,y+0.1) - f_x(x,y)}{0.1} \\ &= \frac{\left[\lim\limits_{h\to 0} \frac{f(x+h,y+0.1)-f(x,y+0.1)}{h}\right] - \left[\lim\limits_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}\right]}{0.1} \\ &\approx \frac{\left[\frac{f(x+0.1,y+0.1)-f(x,y+0.1)}{0.1}\right] - \left[\frac{f(x+0.1,y)-f(x,y)}{0.1}\right]}{0.1} \end{aligned}$$

These are the ideas we'll use in the approximations below.

The second partial derivative $f_{xy}(x,y)$ of f is the partial derivative of $f_x(x,y)$ with respect to y. That is:

$$f_{xy}(1.8,2.0) = \lim_{h\to 0} \frac{f_x(1.8,2.0+h) - f_x(1.8,2.0)}{h}$$

For our approximation, we can choose $h = 0.1$ or $h = -0.1$. There's no compelling reason to choose one over the other. Let's use $h = 0.1$.

$$\begin{aligned} &\approx \frac{f_x(1.8,2.1) - f_x(1.8,2.0)}{0.1} \\ &= \frac{\left[\lim\limits_{h\to 0} \frac{f(1.8+h,2.1)-f(1.8,2.1)}{h}\right] - \left[\lim\limits_{h\to 0} \frac{f(1.8+h,2.0)-f(1.8,2.0)}{h}\right]}{0.1} \end{aligned}$$

Once again, there's no compelling reason to choose $h = 0.1$ over $h = -0.1$. We could even choose different signs for the two limits. We'll just choose $h = 0.1$ again, because after all, we do have to choose something.

$$\begin{aligned} &\approx \frac{\left[\frac{f(1.9,2.1)-f(1.8,2.1)}{0.1}\right] - \left[\frac{f(1.9,2.0)-f(1.8,2.0)}{0.1}\right]}{0.1} \\ &= 100\Big[\big(f(1.9,2.1) - f(1.8,2.1)\big) - \big(f_x(1.9,2.0) - f_x(1.8,2.0)\big)\Big] \\ &= 100\Big[\big(16.0 - 14.9\big) - \big(16.3 - 15.2\big)\Big] \\ &= 0 \end{aligned}$$

Remark: different choices of h all end up with the same approximation.

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S-1: a) (i) ∇f is zero or does not exist at critical points. The point T is a local maximum and the point U is a saddle point. The remaining points P, R, S, are not critical points.

(a) (ii) Only U is a saddle point.

(a) (iii) We have $f_y(x,y) > 0$ if f increases as you move vertically upward through (x,y). Looking at the diagram, we see

$$f_y(P) < 0 \qquad f_y(Q) < 0 \qquad f_y(R) = 0 \qquad f_y(S) > 0 \qquad f_y(T) = 0 \qquad f_y(U) = 0$$

So only S works.

(b) (i) The function $z = F(x,2)$ is increasing at $x = 1$, because the $y = 2.0$ graph in the diagram has positive slope at $x = 1$. So $F_x(1,2) > 0$.

(b) (ii) The function $z = F(x,2)$ is also increasing (though slowly) at $x = 2$, because the $y = 2.0$ graph in the diagram has positive slope at $x = 2$. So $F_x(2,2) > 0$. So F does *not* have a critical point at $(2,2)$.

(b) (iii) From the diagram the looks like $F_x(1,1.9) > F_x(1,2.0) > F_x(1,2.1)$. That is, it looks like the slope of the $y = 1.9$ graph at $x = 1$ is larger than the slope of the $y = 2.0$ graph at $x = 1$, which in turn is larger than the slope of the $y = 2.1$ graph at $x = 1$. So it looks like $F_x(1,y)$ decreases as y increases through $y = 2$, and consequently $F_{xy}(1,2) < 0$.

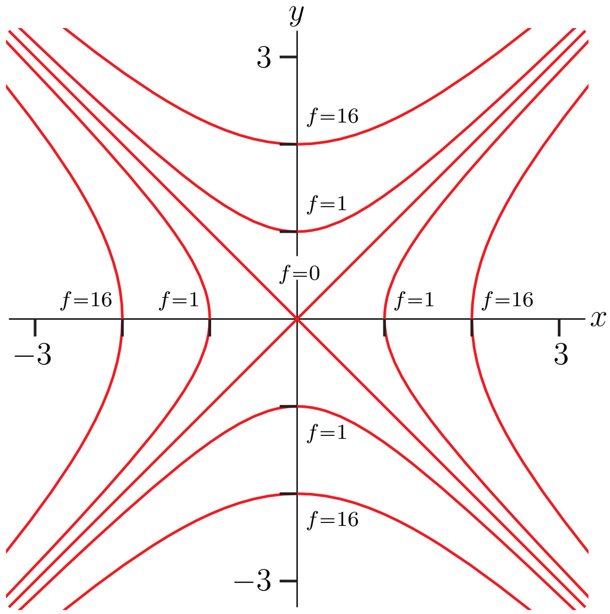
S-2: (a)

- The level curve $z = 0$ is $y^2 - x^2 = 0$, which is the pair of $45°$ lines $y = \pm x$.
- When $C > 0$, the level curve $z = C^4$ is $\left(y^2 - x^2\right)^2 = C^4$, which is the pair of hyperbolae $y^2 - x^2 = C^2$, $y^2 - x^2 = -C^2$ or

$$y = \pm\sqrt{x^2 + C^2} \qquad x = \pm\sqrt{y^2 + C^2}$$

  The hyperbola $y^2 - x^2 = C^2$ crosses the y–axis (i.e. the line $x = 0$) at $(0, \pm C)$. The hyperbola $y^2 - x^2 = -C^2$ crosses the x–axis (i.e. the line $y = 0$) at $(\pm C, 0)$.

Here is a sketch showing the level curves $z = 0$, $z = 1$ (i.e. $C = 1$), and $z = 16$ (i.e. $C = 2$).



(b) As $f_x(x,y) = -4x(y^2 - x^2)$ and $f_y(x,y) = 4y(y^2 - x^2)$, we have $f_x(0,0) = f_y(0,0) = 0$ so that $(0,0)$ is a critical point. Note that

- $f(0,0) = 0$,
- $f(x,y) \geqslant 0$ for all x and y.

So $(0,0)$ is a local (and also absolute) minimum.

(c) Note that

$$\begin{aligned} f_{xx}(x,y) &= -4y^2 + 12x^2 & \qquad f_{xx}(x,y) &= 0 \\ f_{yy}(x,y) &= 12y^2 - 4x^2 & f_{yy}(x,y) &= 0 \\ f_{xy}(x,y) &= -8xy & f_{xx}(x,y) &= 0 \end{aligned}$$

As $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0$, the Second Derivative Test (Theorem 2.3.15 in the text) tells us absolutely nothing.

S-3: Write $f(x,y) = x^2 + cxy + y^2$. Then

$$\begin{aligned} f_x(x,y) &= 2x + cy & \qquad f_x(0,0) &= 0 \\ f_y(x,y) &= cx + 2y & f_y(0,0) &= 0 \\ f_{xx}(x,y) &= 2 \\ f_{xy}(x,y) &= c \\ f_{yy}(x,y) &= 2 \end{aligned}$$

As $f_x(0,0) = f_y(0,0) = 0$, we have that $(0,0)$ is always a critical point for f. According to the Second Derivative Test, $(0,0)$ is also a saddle point for f if

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 < 0 \iff 4 - c^2 < 0 \iff |c| > 2$$

As a remark, the Second Derivative Test provides no information when the expression $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0$, i.e. when $c = \pm 2$. But when $c = \pm 2$,

$$f(x,y) = x^2 \pm 2xy + y^2 = (x \pm y)^2$$

and f has a local minimum, not a saddle point, at $(0,0)$.

S-4: To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 - y^3 - 2xy + 6 \\ f_x &= 3x^2 - 2y \qquad & f_{xx} &= 6x \qquad & f_{xy} &= -2 \\ f_y &= -3y^2 - 2x \qquad & f_{yy} &= -6y \qquad & f_{yx} &= -2 \end{aligned}$$

The gradient is defined everywhere so the critical points are the solutions of

$$f_x = 3x^2 - 2y = 0 \qquad f_y = -3y^2 - 2x = 0$$

Substituting $y = \frac{3}{2}x^2$, from the first equation, into the second equation gives

$$\begin{aligned} -3\left(\frac{3}{2}x^2\right)^2 - 2x = 0 &\iff -2x\left(\frac{3^3}{2^3}x^3 + 1\right) = 0 \\ &\iff x = 0, -\frac{2}{3} \end{aligned}$$

So there are two critical points: $(0,0)$, $\left(-\frac{2}{3}, \frac{2}{3}\right)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-2)^2 < 0$		saddle point
$\left(-\frac{2}{3}, \frac{2}{3}\right)$	$(-4) \times (-4) - (-2)^2 > 0$	-4	local max

S-5: To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + x^2y + xy^2 - 9x \\ f_x &= 3x^2 + 2xy + y^2 - 9 \qquad & f_{xx} &= 6x + 2y \qquad & f_{xy} &= 2x + 2y \\ f_y &= x^2 + 2xy \qquad & f_{yy} &= 2x \qquad & f_{yx} &= 2x + 2y \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

f_x and f_y are polynomials (in two variables) and so they are defined everywhere. Therefore the critical points are the solutions of

$$f_x = 3x^2 + 2xy + y^2 - 9 = 0 \tag{E1}$$
$$f_y = x(x + 2y) \qquad = 0 \tag{E2}$$

Equation (E2) is satisfied if at least one of $x = 0$, $x = -2y$.

- If $x = 0$, equation (E1) reduces to $y^2 - 9 = 0$, which is satisfied if $y = \pm 3$.
- If $x = -2y$, equation (E1) reduces to

$$0 = 3(-2y)^2 + 2(-2y)y + y^2 - 9 = 9y^2 - 9$$

which is satisfied if $y = \pm 1$.

So there are four critical points: $(0,3)$, $(0,-3)$, $(-2,1)$ and $(2,-1)$. The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,3)$	$(6) \times (0) - (6)^2 < 0$		saddle point
$(0,-3)$	$(-6) \times (0) - (-6)^2 < 0$		saddle point
$(-2,1)$	$(-10) \times (-4) - (-2)^2 > 0$	-10	local max
$(2,-1)$	$(10) \times (4) - (2)^2 > 0$	10	local min

S-6: To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^2 + y^2 + x^2y + 4 \\ f_x &= 2x + 2xy \qquad f_{xx} = 2 + 2y \qquad f_{xy} = 2x \\ f_y &= 2y + x^2 \qquad f_{yy} = 2 \end{aligned}$$

The first partial derivatives are defined everywhere so the critical points are the solutions of

$$\begin{aligned} & f_x = 0 && f_y = 0 \\ \iff\quad & 2x(1+y) = 0 && 2y + x^2 = 0 \\ \iff\quad & x = 0 \text{ or } y = -1 && 2y + x^2 = 0 \end{aligned}$$

When $x = 0$, y must be 0. When $y = -1$, x^2 must be 2. So, there are three critical points: $(0,0)$, $\big(\pm\sqrt{2}, -1\big)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$2 \times 2 - 0^2 > 0$	$2 > 0$	local min
$(\sqrt{2},-1)$	$0 \times 2 - (2\sqrt{2})^2 < 0$		saddle point
$(-\sqrt{2},-1)$	$0 \times 2 - (-2\sqrt{2})^2 < 0$		saddle point

S-7: To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + x^2 - 2xy + y^2 - x \\ f_x &= 3x^2 + 2x - 2y - 1 \qquad f_{xx} = 6x + 2 \qquad f_{xy} = -2 \\ f_y &= -2x + 2y \qquad f_{yy} = 2 \qquad f_{yx} = -2 \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The gradient exists everywhere so the critical points are the solutions of

$$\begin{aligned} f_x &= 3x^2 + 2x - 2y - 1 = 0 && \text{(E1)} \\ f_y &= -2x + 2y \qquad\quad\;\; = 0 && \text{(E2)} \end{aligned}$$

Substituting $y = x$, from (E2), into (E1) gives

$$3x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{3}} = 0$$

So there are two critical points: $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$(2\sqrt{3}+2) \times (2) - (-2)^2 > 0$	$2\sqrt{3}+2 > 0$	local min
$-\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$(-2\sqrt{3}+2) \times (2) - (-2)^2 < 0$		saddle point

S-8: To find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + xy^2 - 3x^2 - 4y^2 + 4 \\ f_x &= 3x^2 + y^2 - 6x \qquad f_{xx} = 6x - 6 \qquad f_{xy} = 2y \\ f_y &= 2xy - 8y \qquad\qquad f_{yy} = 2x - 8 \qquad f_{yx} = 2y \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The first partial derivatives exist everywhere so the critical points are the solutions of

$$f_x = 3x^2 + y^2 - 6x = 0 \qquad f_y = 2(x-4)y = 0$$

The second equation is satisfied if at least one of $x = 4$, $y = 0$ are satisfied.

- If $x = 4$, the first equation reduces to $y^2 = -24$, which has no real solutions.
- If $y = 0$, the first equation reduces to $3x(x-2) = 0$, which is satisfied if either $x = 0$ or $x = 2$.

So there are two critical points: $(0,0)$, $(2,0)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6) \times (-8) - (0)^2 > 0$	-6	local max
$(2,0)$	$6 \times (-4) - (0)^2 < 0$		saddle point

S-9: (a) To find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + 3xy + 3y^2 - 6x - 3y - 6 \\ f_x &= 3x^2 + 3y - 6 \qquad f_{xx} = 6x \qquad f_{xy} = 3 \\ f_y &= 3x + 6y - 3 \qquad f_{yy} = 6 \qquad f_{yx} = 3 \end{aligned}$$

The first partial derivatives exists everywhere (as they are polynomials with two variables) and so the gradient exists everywhere. So the critical points are the solutions of

$$f_x = 3x^2 + 3y - 6 = 0 \qquad f_y = 3x + 6y - 3 = 0$$

Subtracting the second equation from 2 times the first equation gives

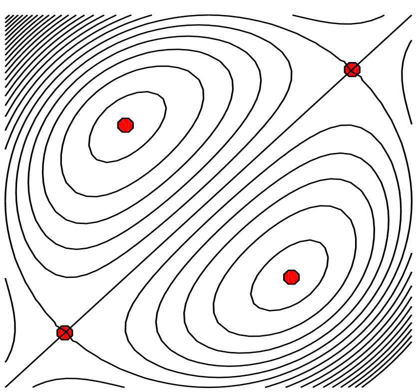
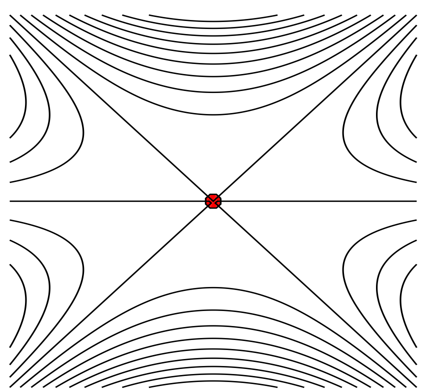
$$6x^2 - 3x - 9 = 0 \iff 3(2x-3)(x+1) = 0 \iff x = \frac{3}{2}, -1$$

Since $y = \frac{1-x}{2}$ (from the second equation), the critical points are $(\frac{3}{2}, -\frac{1}{4})$, $(-1, 1)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(\frac{3}{2}, -\frac{1}{4})$	$(9) \times (6) - (3)^2 > 0$	9	local min
$(-1, 1)$	$(-6) \times (6) - (3)^2 < 0$		saddle point

(b) Notice that the lines $x = y$, $x = -y$ and $y = 0$ are all level curves of the function $f(x, y) = y(x+y)(x-y) + 1$ (i.e. of (iii)) with $f = 1$. So the first picture goes with (iii). And the second picture goes with (i).

Here are the pictures with critical points marked on them. There are saddle points where level curves cross and there are local max's or min's at "bull's eyes".

(i)  (ii) 

S-10: To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + 3xy + 3y^2 - 6x - 3y - 6 \\ f_x &= 3x^2 + 3y - 6 \qquad f_{xx} = 6x \qquad f_{xy} = 3 \\ f_y &= 3x + 6y - 3 \qquad f_{yy} = 6 \qquad f_{yx} = 3 \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The gradient is defined everywhere and so the critical points are the solutions of

$$f_x = 3x^2 + 3y - 6 = 0 \tag{E1}$$
$$f_y = 3x \ + 6y - 3 \ = 0 \tag{E2}$$

Subtracting equation (E2) from twice equation (E1) gives

$$6x^2 - 3x - 9 = 0 \iff (2x-3)(3x+3) = 0$$

So we must have either $x = \frac{3}{2}$ or $x = -1$.

- If $x = \frac{3}{2}$, (E2) reduces to $\frac{9}{2} + 6y - 3 = 0$ so $y = -\frac{1}{4}$.
- If $x = -1$, (E2) reduces to $-3 + 6y - 3 = 0$ so $y = 1$.

So there are two critical points: $\left(\frac{3}{2}, -\frac{1}{4}\right)$ and $(-1, 1)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$\left(\frac{3}{2}, -\frac{1}{4}\right)$	$(9) \times (6) - (3)^2 > 0$	9	local min
$(-1, 1)$	$(-6) \times (6) - (3)^2 < 0$		saddle point

S-11: Thinking a little way ahead, to find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= 3x^2y + y^3 - 3x^2 - 3y^2 + 4 \\ f_x &= 6xy - 6x \qquad f_{xx} = 6y - 6 \qquad f_{xy} = 6x \\ f_y &= 3x^2 + 3y^2 - 6y \qquad f_{yy} = 6y - 6 \qquad f_{yx} = 6x \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The first partial derivatives are defined everywhere and so the critical points are the solutions of

$$f_x = 6x(y-1) = 0 \qquad f_y = 3x^2 + 3y^2 - 6y = 0$$

The first equation is satisfied if at least one of $x = 0$, $y = 1$ are satisfied.

- If $x = 0$, the second equation reduces to $3y^2 - 6y = 0$, which is satisfied if either $y = 0$ or $y = 2$.
- If $y = 1$, the second equation reduces to $3x^2 - 3 = 0$ which is satisfied if $x = \pm 1$.

So there are four critical points: $(0,0)$, $(0,2)$, $(1,1)$, $(-1,1)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6) \times (-6) - (0)^2 > 0$	-6	local max
$(0,2)$	$6 \times 6 - (0)^2 > 0$	6	local min
$(1,1)$	$0 \times 0 - (6)^2 < 0$		saddle point
$(-1,1)$	$0 \times 0 - (-6)^2 < 0$		saddle point

S-12: We have

$$\begin{aligned} f(x,y) &= x^4 + y^4 - 4xy + 2 & f_x(x,y) &= 4x^3 - 4y & f_{xx}(x,y) &= 12x^2 \\ & & f_y(x,y) &= 4y^3 - 4x & f_{yy}(x,y) &= 12y^2 \\ & & & & f_{xy}(x,y) &= -4 \end{aligned}$$

The partial first derivatives are defined everywhere. So the critical point are the solutions of

$$\begin{aligned} f_x(x,y) = f_y(x,y) = 0 &\iff y = x^3 \text{ and } x = y^3 \\ &\iff x = x^9 \text{ and } y = x^3 \\ &\iff x(x^8 - 1) = 0,\ y = x^3 \\ &\iff (x,y) = (0,0) \text{ or } (1,1) \text{ or } (-1,-1) \end{aligned}$$

Here is a table giving the classification of each of the three critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-4)^2 < 0$		saddle point
$(1,1)$	$12 \times 12 - (-4)^2 > 0$	12	local min
$(-1,-1)$	$12 \times 12 - (-4)^2 > 0$	12	local min

S-13: We have

$$\begin{aligned} f(x,y) &= x^4 + y^4 - 4xy & f_x(x,y) &= 4x^3 - 4y & f_{xx}(x,y) &= 12x^2 \\ & & f_y(x,y) &= 4y^3 - 4x & f_{yy}(x,y) &= 12y^2 \\ & & & & f_{xy}(x,y) &= -4 \end{aligned}$$

The first partial derivatives are defined everywhere. So the critical points are the solution of

$$
\begin{aligned}
f_x(x,y) = f_y(x,y) = 0 &\iff y = x^3 \text{ and } x = y^3 \iff x = x^9 \text{ and } y = x^3 \\
&\iff x(x^8 - 1) = 0,\ y = x^3 \\
&\iff (x,y) = (0,0) \text{ or } (1,1) \text{ or } (-1,-1)
\end{aligned}
$$

Here is a table giving the classification of each of the three critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-4)^2 < 0$		saddle point
$(1,1)$	$12 \times 12 - (-4)^2 > 0$	12	local min
$(-1,-1)$	$12 \times 12 - (-4)^2 > 0$	12	local min

S-14: We have

$$
\begin{aligned}
f(x,y) &= x^3 + xy^2 - x & f_x(x,y) &= 3x^2 + y^2 - 1 & f_{xx}(x,y) &= 6x \\
& & f_y(x,y) &= 2xy & f_{yy}(x,y) &= 2x \\
& & & & f_{xy}(x,y) &= 2y
\end{aligned}
$$

The first partial derivatives are defined everywhere. So the critical points are the solution of

$$
\begin{aligned}
f_x(x,y) = f_y(x,y) = 0 &\iff xy = 0 \text{ and } 3x^2 + y^2 = 1 \\
&\iff \{x = 0 \text{ or } y = 0\} \text{ and } 3x^2 + y^2 = 1 \\
&\iff (x,y) = (0,1) \text{ or } (0,-1) \text{ or } \left(\frac{1}{\sqrt{3}}, 0\right) \text{ or } \left(-\frac{1}{\sqrt{3}}, 0\right)
\end{aligned}
$$

Here is a table giving the classification of each of the four critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,1)$	$0 \times 0 - 2^2 < 0$		saddle point
$(0,-1)$	$0 \times 0 - (-2)^2 < 0$		saddle point
$(\frac{1}{\sqrt{3}}, 0)$	$2\sqrt{3} \times \frac{2}{\sqrt{3}} - 0^2 > 0$	$2\sqrt{3}$	local min
$(-\frac{1}{\sqrt{3}}, 0)$	$-2\sqrt{3} \times (-\frac{2}{\sqrt{3}}) - 0^2 > 0$	$-2\sqrt{3}$	local max

S-15: We have

$$
\begin{aligned}
f(x,y) &= x^3 - 3xy^2 - 3x^2 - 3y^2 & f_x(x,y) &= 3x^2 - 3y^2 - 6x & f_{xx}(x,y) &= 6x - 6 \\
& & f_y(x,y) &= -6xy - 6y & f_{yy}(x,y) &= -6x - 6 \\
& & & & f_{xy}(x,y) &= -6y
\end{aligned}
$$

The first partial derivatives are defined everywhere. So the critical points are the solution of

$$
\begin{aligned}
f_x(x,y) = f_y(x,y) = 0 &\iff 3(x^2 - y^2 - 2x) = 0 \text{ and } -6y(x+1) = 0 \\
&\iff \{x = -1 \text{ or } y = 0\} \text{ and } x^2 - y^2 - 2x = 0 \\
&\iff (x,y) = (-1, \sqrt{3}) \text{ or } (-1, -\sqrt{3}) \text{ or } (0,0) \text{ or } (2,0)
\end{aligned}
$$

Here is a table giving the classification of each of the four critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6)\times(-6) - 0^2 > 0$	-6	local max
$(2,0)$	$6\times(-18) - 0^2 < 0$		saddle point
$(-1,\sqrt{3})$	$(-12)\times 0 - (-6\sqrt{3})^2 < 0$		saddle point
$(-1,-\sqrt{3})$	$(-12)\times 0 - (6\sqrt{3})^2 < 0$		saddle point

S-16: To find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= 3kx^2y + y^3 - 3x^2 - 3y^2 + 4 \\ f_x &= 6kxy - 6x & f_{xx} &= 6ky - 6 & f_{xy} &= 6kx \\ f_y &= 3kx^2 + 3y^2 - 6y & f_{yy} &= 6y - 6 & f_{yx} &= 6kx \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The first partial derivatives are defined everywhere. So the critical points are the solution of

$$f_x = 6x(ky-1) = 0 \qquad f_y = 3kx^2 + 3y^2 - 6y = 0$$

The first equation is satisfied if at least one of $x = 0$, $y = 1/k$ are satisfied. (Recall that $k > 0$.)

- If $x = 0$, the second equation reduces to $3y(y-2) = 0$, which is satisfied if either $y = 0$ or $y = 2$.
- If $y = 1/k$, the second equation reduces to $3kx^2 + \frac{3}{k^2} - \frac{6}{k} = 3kx^2 + \frac{3}{k^2}(1-2k) = 0$.

Case $k < \frac{1}{2}$: If $k < \frac{1}{2}$, then $\frac{3}{k^2}(1-2k) > 0$ and the equation $3kx^2 + \frac{3}{k^2}(1-2k) = 0$ has no real solutions. In this case there are two critical points: $(0,0)$, $(0,2)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6)\times(-6) - (0)^2 > 0$	-6	local max
$(0,2)$	$(12k-6)\times 6 - (0)^2 < 0$		saddle point

Case $k = \frac{1}{2}$: If $k = \frac{1}{2}$, then $\frac{3}{k^2}(1-2k) = 0$ and the equation $3kx^2 + \frac{3}{k^2}(1-2k) = 0$ reduces to $3kx^2 = 0$ which has as its only solution $x = 0$. We have already seen this third critical point, $x = 0$, $y = 1/k = 2$. So there are again two critical points: $(0,0)$, $(0,2)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6)\times(-6) - (0)^2 > 0$	-6	local max
$(0,2)$	$(12k-6)\times 6 - (0)^2 = 0$		unknown

Case $k > \frac{1}{2}$: If $k > \frac{1}{2}$, then $\frac{3}{k^2}(1-2k) < 0$ and the equation $3kx^2 + \frac{3}{k^2}(1-2k) = 0$ reduces to $3kx^2 = \frac{3}{k^2}(2k-1)$ which has two solutions, namely $x = \pm\sqrt{\frac{1}{k^3}(2k-1)}$. So there are four critical points: $(0,0)$, $(0,2)$, $\left(\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$ and $\left(-\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6)\times(-6)-(0)^2 > 0$	-6	local max
$(0,2)$	$(12k-6)\times 6-(0)^2 > 0$	$12k-6>0$	local min
$\left(\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$	$(6-6)\times(\frac{6}{k}-6)-(>0)^2 < 0$		saddle point
$\left(-\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$	$(6-6)\times(\frac{6}{k}-6)-(<0)^2 < 0$		saddle point

S-17: We wish to choose m and b so as to minimize the (square of the) rms error $E(m,b) = \sum\limits_{i=1}^{n}(mx_i + b - y_i)^2$.

$$\begin{aligned}
0 = \frac{\partial E}{\partial m} &= \sum_{i=1}^{n} 2(mx_i+b-y_i)x_i = m\Big[\sum_{i=1}^{n} 2x_i^2\Big] + b\Big[\sum_{i=1}^{n} 2x_i\Big] - \Big[\sum_{i=1}^{n} 2x_iy_i\Big]\\
0 = \frac{\partial E}{\partial b} &= \sum_{i=1}^{n} 2(mx_i+b-y_i) \quad = m\Big[\sum_{i=1}^{n} 2x_i\Big] + b\Big[\sum_{i=1}^{n} 2\Big] - \Big[\sum_{i=1}^{n} 2y_i\Big]
\end{aligned}$$

Here, the first partial derivatives $\frac{\partial E}{\partial m}$ and $= \frac{\partial E}{\partial b}$ are defined everywhere and the critical points are the solution of

$$\begin{aligned}
0 = \frac{\partial E}{\partial m} &= \sum_{i=1}^{n} 2(mx_i+b-y_i)x_i = m\Big[\sum_{i=1}^{n} 2x_i^2\Big] + b\Big[\sum_{i=1}^{n} 2x_i\Big] - \Big[\sum_{i=1}^{n} 2x_iy_i\Big]\\
0 = \frac{\partial E}{\partial b} &= \sum_{i=1}^{n} 2(mx_i+b-y_i) \quad = m\Big[\sum_{i=1}^{n} 2x_i\Big] + b\Big[\sum_{i=1}^{n} 2\Big] - \Big[\sum_{i=1}^{n} 2y_i\Big]
\end{aligned}$$

There are a lot of symbols in those two equations. But remember that only two of them, namely m and b, are unknowns. All of the x_i's and y_i's are given data. We can make the equations look a lot less imposing if we define $S_x = \sum_{i=1}^n x_i$, $S_y = \sum_{i=1}^n y_i$, $S_{x^2} = \sum_{i=1}^n x_i^2$ and $S_{xy} = \sum_{i=1}^n x_iy_i$. In terms of this notation, the two equations are (after dividing by two)

$$S_{x^2}\, m + S_x\, b = S_{xy} \tag{1}$$
$$S_x\, m + n\, b = S_y \tag{2}$$

This is a system of two linear equations in two unknowns. One way[5] to solve them, is to use one of the two equations to solve for one of the two unknowns in terms of the other unknown. For example, equation (2) gives that

$$b = \frac{1}{n}(S_y - S_x\, m)$$

5 This procedure is probably not the most efficient one. But it has the advantage that it always works, it does not require any ingenuity on the part of the solver, and it generalizes easily to larger linear systems of equations.

If we now substitute this into equation (1) we get

$$S_{x^2}\, m + \frac{S_x}{n}\big(S_y - S_x\, m\big) = S_{xy} \implies \left(S_{x^2} - \frac{S_x^2}{n}\right) m = S_{xy} - \frac{S_x S_y}{n}$$

which is a single equation in the single unkown m. We can easily solve it for m. It tells us that

$$m = \frac{nS_{xy} - S_x S_y}{nS_{x^2} - S_x^2}$$

Then substituting this back into $b = \frac{1}{n}\big(S_y - S_x\, m\big)$ gives us

$$b = \frac{S_y}{n} - \frac{S_x}{n}\left(\frac{nS_{xy} - S_x S_y}{nS_{x^2} - S_x^2}\right) = \frac{S_y S_{x^2} - S_x S_{xy}}{nS_{x^2} - S_x^2}$$

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S-1: False. A common mistake is to think that the intercepts of a circle are somehow "endpoints," in the same way that the interval $[-1, 1]$ has endpoints -1 and 1. But circles don't have endpoints!

When[6] we're finding the extrema of a function over a closed curve, we use the equation of the curve (or its parametrization) to get a function of one variable. Then we look for critical points and endpoints of that function. These may *or may not* occur at $x = \pm 1$ or $y = \pm 1$.

Now, you might notice that school problems often end up having their extrema at the extreme values of x and/or y in the boundary. This is a result of writing problems with relatively easy algebra, rather than the result of some universal law.

S-2: The height $\sqrt{x^2 + y^2}$ at (x, y) is the distance from (x, y) to $(0, 0)$. So the minimum height is zero at $(0, 0, 0)$. The surface is a cone. The cone has a point at $(0, 0, 0)$ and the derivatives z_x and z_y do not exist there. The maximum height is achieved when (x, y) is as far as possible from $(0, 0)$. The highest points are at $(\pm 1, \pm 1, \sqrt{2})$. There z_x and z_y exist but are not zero. These points would not be the highest points if it were not for the restriction $|x|, |y| \leqslant 1$.

S-3: The specified function and its first order derivatives are

$$f(x, y) = xy - x^3y^2 \qquad f_x(x, y) = y - 3x^2y^2 \qquad f_y(x, y) = x - 2x^3y$$

- First, we find the critical points. The first partial derivatives are defined everywhere and so the critical points are the solution of

$$\begin{aligned} f_x = 0 \quad &\iff \quad y(1 - 3x^2y) = 0 \quad &\iff \quad y = 0 \text{ or } 3x^2y = 1 \\ f_y = 0 \quad &\iff \quad x(1 - 2x^2y) = 0 \quad &\iff \quad x = 0 \text{ or } 2x^2y = 1 \end{aligned}$$

6 At least in this section, this is how we do it.. but we'll learn other ways that also don't involve optimizing x and y separately

– If $y = 0$, we cannot have $2x^2y = 1$, so we must have $x = 0$.

– If $3x^2y = 1$, we cannot have $x = 0$, so we must have $2x^2y = 1$. Dividing gives $1 = \frac{3x^2y}{2x^2y} = \frac{3}{2}$ which is impossible.

So the only critical point in the square is $(0, 0)$. There $f = 0$.

- Next, we look at the part of the boundary with $x = 0$. There $f = 0$.
- Next, we look at the part of the boundary with $y = 0$. There $f = 0$.
- Next, we look at the part of the boundary with $x = 1$. There $f = y - y^2$. As $\frac{\mathrm{d}}{\mathrm{d}y}(y - y^2) = 1 - 2y$, the max and min of $y - y^2$ for $0 \leqslant y \leqslant 1$ must occur either at $y = 0$, where $f = 0$, or at $y = \frac{1}{2}$, where $f = \frac{1}{4}$, or at $y = 1$, where $f = 0$.
- Next, we look at the part of the boundary with $y = 1$. There $f = x - x^3$. As $\frac{\mathrm{d}}{\mathrm{d}x}(x - x^3) = 1 - 3x^2$, the max and min of $x - x^3$ for $0 \leqslant x \leqslant 1$ must occur either at $x = 0$, where $f = 0$, or at $x = \frac{1}{\sqrt{3}}$, where $f = \frac{2}{3\sqrt{3}}$, or at $x = 1$, where $f = 0$.

All together, we have the following candidates for max and min.

point	$(0,0)$	$x = 0$	$y = 0$	$(1,0)$	$(1,\frac{1}{2})$	$(1,1)$	$(0,1)$	$(\frac{1}{\sqrt{3}},1)$	$(1,1)$
value of f	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{2}{3\sqrt{3}}$	0
	min	min	min	min		min	min	max	min

The largest and smallest values of f in this table are

$$\text{min} = 0 \qquad \text{max} = \frac{2}{3\sqrt{3}} \approx 0.385$$

S-4: (a) To find the critical points we will need the gradient of h and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} h &= y(4 - x^2 - y^2) \\ h_x &= -2xy \qquad & h_{xx} &= -2y \qquad & h_{xy} &= -2x \\ h_y &= 4 - x^2 - 3y^2 \qquad & h_{yy} &= -6y \qquad & h_{yx} &= -2x \end{aligned}$$

(Of course, h_{xy} and h_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The first partial derivatives are defined everywhere and so the critical points are the solutions of

$$h_x = -2xy = 0 \qquad h_y = 4 - x^2 - 3y^2 = 0$$

The first equation is satisfied if at least one of $x = 0$, $y = 0$ are satisfied.

- If $x = 0$, the second equation reduces to $4 - 3y^2 = 0$, which is satisfied if $y = \pm\frac{2}{\sqrt{3}}$.
- If $y = 0$, the second equation reduces to $4 - x^2 = 0$ which is satisfied if $x = \pm 2$.

So there are four critical points: $\left(0, \frac{2}{\sqrt{3}}\right)$, $\left(0, -\frac{2}{\sqrt{3}}\right)$, $(2,0)$, $(-2,0)$.

The classification is

critical point	$h_{xx}h_{yy} - h_{xy}^2$	h_{xx}	type
$\left(0, \frac{2}{\sqrt{3}}\right)$	$\left(\frac{-4}{\sqrt{3}}\right) \times \left(-\frac{12}{\sqrt{3}}\right) - (0)^2 > 0$	$\frac{-4}{\sqrt{3}}$	local max
$\left(0, -\frac{2}{\sqrt{3}}\right)$	$\left(\frac{4}{\sqrt{3}}\right) \times \left(\frac{12}{\sqrt{3}}\right) - (0)^2 > 0$	$\frac{4}{\sqrt{3}}$	local min
$(2,0)$	$0 \times 0 - (-4)^2 < 0$		saddle point
$(-2,0)$	$0 \times 0 - (4)^2 < 0$		saddle point

(b) The absolute max and min can occur either in the interior of the disk or on the boundary of the disk. The boundary of the disk is the circle $x^2 + y^2 = 1$.

- Any absolute max or min in the interior of the disk must also be a local max or min and, must also be a critical point of h. We found all of the critical points of h in part (a). Since $2 > 1$ and $\frac{2}{\sqrt{3}} > 1$ none of the critical points are in the disk.
- At each point of $x^2 + y^2 = 1$ we have $h(x,y) = 3y$ with $-1 \leqslant y \leqslant 1$. Clearly the maximum value is 3 (at $(0,1)$) and the minimum value is -3 (at $(0,-1)$).

So all together, the maximum and minimum values of $h(x,y)$ in $x^2 + y^2 \leqslant 1$ are 3 (at $(0,1)$) and -3 (at $(0,-1)$), respectively.

S-5: The maximum and minimum must either occur at a critical point or on the boundary of R.

- The critical points are the points where the gradient is zero or it does not exist. Here $f_x(x,y) = 2 - 2x$ and $f_y(x,y) = -8y$ and so they are defined everywhere. Therefore, the critical points are the solutions of

$$0 = f_x(x,y) = 2 - 2x$$
$$0 = f_y(x,y) = -8y$$

  So the only critical point is $(1,0)$.
- On the side $x = -1$, $-1 \leqslant y \leqslant 1$ of the boundary of R

$$f(-1,y) = 2 - 4y^2$$

  This function decreases as $|y|$ increases. So its maximum value on $-1 \leqslant y \leqslant 1$ is achieved at $y = 0$ and its minimum value is achieved at $y = \pm 1$.
- On the side $x = 3$, $-1 \leqslant y \leqslant 1$ of the boundary of R

$$f(3,y) = 2 - 4y^2$$

  This function decreases as $|y|$ increases. So its maximum value on $-1 \leqslant y \leqslant 1$ is achieved at $y = 0$ and its minimum value is achieved at $y = \pm 1$.

- On both sides $y = \pm 1$, $-1 \leqslant x \leqslant 3$ of the boundary of R

$$f(x, \pm 1) = 1 + 2x - x^2 = 2 - (x-1)^2$$

  This function decreases as $|x-1|$ increases. So its maximum value on $-1 \leqslant x \leqslant 3$ is achieved at $x = 1$ and its minimum value is achieved at $x = 3$ and $x = -1$ (both of whom are a distance 2 from $x = 1$).

So we have the following candidates for the locations of the min and max

point	$(1,0)$	$(-1,0)$	$(1,\pm1)$	$(-1,\pm1)$	$(3,0)$	$(3,\pm1)$
value of f	6	2	2	-2	2	-2
	max			min		min

So the minimum is -2 and the maximum is 6.

S-6: Since $\nabla h = \langle -4\,, -2\rangle$ exists and is never zero, h has no critical points and the minimum of h on the disk $x^2 + y^2 \leqslant 1$ must be taken on the boundary, $x^2 + y^2 = 1$, of the disk.

Solution 1 This is a good candidate for parametrization (if you read that optional section). A non-parametrization solution is given below this one.

To find the minimum on the boundary, we parametrize $x^2 + y^2 \leqslant 1$ by $x = \cos\theta$, $y = \sin\theta$ and find the minimum of

$$H(\theta) = -4\cos\theta - 2\sin\theta + 6$$

Since

$$0 = H'(\theta) = 4\sin\theta - 2\cos\theta \implies x = \cos\theta = 2\sin\theta = 2y$$

So

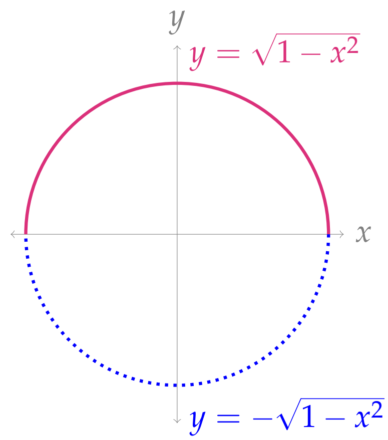
$$1 = x^2 + y^2 = 4y^2 + y^2 = 5y^2 \implies y = \pm\frac{1}{\sqrt{5}},\ x = \pm\frac{2}{\sqrt{5}}$$

At these two points

$$h = 6 - 4x - 2y = 6 - 10y = 6 \mp \frac{10}{\sqrt{5}} = 6 \mp 2\sqrt{5}$$

The minimum is $6 - 2\sqrt{5}$.

Solution 2 To find the minimum on the boundary, we need to use the equation $x^2 + y^2 \leqslant 1$ to turn $h(x, y)$ into a function of one variable. We can break the boundary up into two pieces: $y = \sqrt{1-x^2}$, $-1 \leqslant x \leqslant 1$, and $y = -\sqrt{1-x^2}$, $-1 \leqslant x \leqslant 1$.



- Define $g_1(x)$ as the value of h along the boundary curve $y = \sqrt{1-x^2}$, $-1 \leqslant x \leqslant 1$.

$$\begin{aligned} g_1(x) = h\left(x, \sqrt{1-x^2}\right) &= -4x - 2\left(\underbrace{\sqrt{1-x^2}}_{y}\right) + 6 \\ &= -4x - 2\sqrt{1-x^2} + 6 \end{aligned}$$

To find the minimum of $g_1(x)$, we first find its critical points.

$$\begin{aligned} g_1'(x) &= -4 - 2\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = -4 + \frac{2x}{\sqrt{1-x^2}} \\ 0 &= -4 + \frac{2x}{\sqrt{1-x^2}} \\ 4 &= \frac{2x}{\sqrt{1-x^2}} \\ 2\sqrt{1-x^2} &= x \qquad (*) \end{aligned}$$

Squaring both sides,

$$\begin{aligned} 4(1-x^2) &= x^2 \\ 4 &= 5x^2 \\ \frac{4}{5} &= x^2 \\ x &= \pm\frac{2}{\sqrt{5}} \end{aligned}$$

From line (*), we see x must be positive, so the only one of these roots that actually solves our equation is the positive one

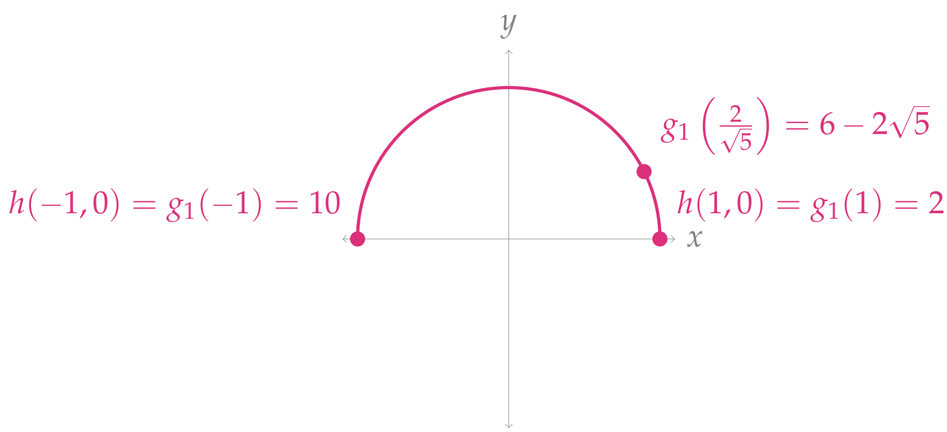
$$x = \frac{2}{\sqrt{5}}$$

So, the minimum of $g_1(x)$ will occur at its CP $x = \frac{2}{\sqrt{5}}$ or at an endpoint $x = 1$ or

$x = -1$.

$$\begin{aligned}
g_1(-1) &= -4(-1) - 2\sqrt{1-1} + 6 = 10 \\
g_1(1) &= -4(1) - 2\sqrt{1-1} + 6 = 2 \\
g_1\left(\frac{2}{\sqrt{5}}\right) &= -4\left(\frac{2}{\sqrt{5}}\right) - 2\sqrt{1 - \left(\frac{2}{\sqrt{5}}\right)^2} + 6 \\
&= -\frac{8}{\sqrt{5}} - 2\sqrt{\frac{1}{5}} + 6 \\
&= -\frac{10}{\sqrt{5}} + 6 \\
&= -2\sqrt{5} + 6 \approx 1.53
\end{aligned}$$

So the minimum of $g_1(x)$ is $g_1 = \left(\frac{2}{\sqrt{5}}\right) = 6 - 2\sqrt{5}$.



- Define $g_2(x)$ as the value of h along the boundary curve $y = -\sqrt{1-x^2}$, $-1 \leqslant x \leqslant 1$.

$$\begin{aligned}
g_2(x) &= h\left(x, -\sqrt{1-x^2}\right) = -4x - 2\left(-\sqrt{1-x^2}\right) + 6 \\
&= -4x + 2\sqrt{1-x^2} + 6 \\
g_2'(x) &= -4 + 2\left(\frac{-2x}{2\sqrt{1-x^2}}\right) \\
&= -4 - \frac{2x}{\sqrt{1-x^2}} \\
4 &= -\frac{2x}{\sqrt{1-x^2}} \\
-2\sqrt{1-x^2} &= x \qquad \text{(*)} \\
4(1-x^2) &= x^2 \\
4 &= 5x^2 \\
x &= \pm\frac{2}{\sqrt{5}}
\end{aligned}$$

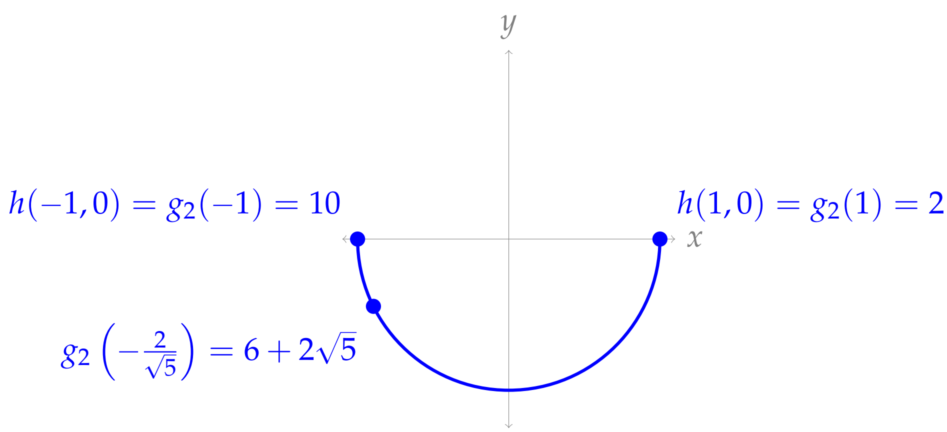
From line (*), we see that x must be negative, so the only solution that works it the negative one

$$x = -\frac{2}{\sqrt{5}}$$

We see that the minimum of $g_2(x)$ will occur at its sole critical point $x = -\frac{2}{\sqrt{5}}$, or at its endpoints $x = \pm 1$.

$$\begin{aligned}
g_2(-1) &= -4(-1) + 2\sqrt{1-(-1)^2} + 6 = 4 + 6 = 10 \\
g_2(1) &= -4(1) + 2\sqrt{1-(1)^2} + 6 = -4 + 6 = 2 \\
g_2\left(-\frac{2}{\sqrt{5}}\right) &= -4\left(-\frac{2}{\sqrt{5}}\right) + 2\sqrt{1-\left(-\frac{2}{\sqrt{5}}\right)^2} + 6 \\
&= \frac{8}{\sqrt{5}} + 2\sqrt{\frac{1}{5}} + 6 = \frac{10}{\sqrt{5}} + 6 \\
&= 2\sqrt{5} + 6 \approx 10.47
\end{aligned}$$

So, the minimum of $g_2(x)$ is $g_2(1) = 2$.



All together, the minimum value h achieves over the boundary $x^2 + y^2 = 1$ is $6 - 2\sqrt{5}$. Since we already decided the global minimum would occur on the boundary, that tells us our global minimum is $6 - 2\sqrt{5}$.

S-7: (a) Thinking a little way ahead, to find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.3.15 in the text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned}
f &= xy(x+y-3) \\
f_x &= 2xy + y^2 - 3y \qquad f_{xx} = 2y \qquad f_{xy} = 2x + 2y - 3 \\
f_y &= x^2 + 2xy - 3x \qquad f_{yy} = 2x \qquad f_{yx} = 2x + 2y - 3
\end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The gradient is defined everywhere and so the critical points are the solutions of

$$f_x = y(2x + y - 3) = 0 \qquad f_y = x(x + 2y - 3) = 0$$

The first equation is satisfied if at least one of $y = 0$, $y = 3 - 2x$ are satisfied.

- If $y = 0$, the second equation reduces to $x(x - 3) = 0$, which is satisfied if either $x = 0$ or $x = 3$.
- If $y = 3 - 2x$, the second equation reduces to $x(x + 6 - 4x - 3) = x(3 - 3x) = 0$ which is satisfied if $x = 0$ or $x = 1$.

So there are four critical points: $(0,0)$, $(3,0)$, $(0,3)$, $(1,1)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-3)^2 < 0$		saddle point
$(3,0)$	$0 \times 6 - (3)^2 < 0$		saddle point
$(0,3)$	$6 \times 0 - (3)^2 < 0$		saddle point
$(1,1)$	$2 \times 2 - (1)^2 > 0$	2	local min

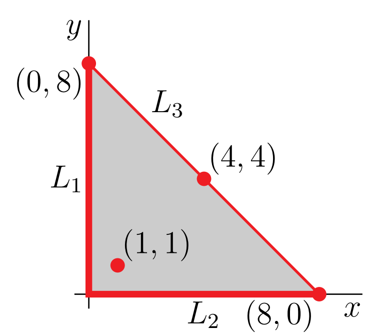
(b) The absolute max and min can occur either in the interior of the triangle or on the boundary of the triangle. The boundary of the triangle consists of the three line segments.

$$L_1 = \{\, (x,y) \mid x = 0,\ 0 \leqslant y \leqslant 8 \,\}$$
$$L_2 = \{\, (x,y) \mid y = 0,\ 0 \leqslant x \leqslant 8 \,\}$$
$$L_3 = \{\, (x,y) \mid x + y = 8,\ 0 \leqslant x \leqslant 8 \,\}$$

- Any absolute max or min in the interior of the triangle must also be a local max or min and, must also be a critical point of f. We found all of the critical points of f in part (a). Only one of them, namely $(1,1)$ is in the interior of the triangle. (The other three critical points are all on the boundary of the triangle.) We have $f(1,1) = -1$.
- At each point of L_1 we have $x = 0$ and so $f(x,y) = 0$.
- At each point of L_2 we have $y = 0$ and so $f(x,y) = 0$.
- At each point of L_3 we have $f(x,y) = x(8 - x)(5) = 40x - 5x^2 = 5[8x - x^2]$ with $0 \leqslant x \leqslant 8$. As $\frac{\mathrm{d}}{\mathrm{d}x}(40x - 5x^2) = 40 - 10x$, the max and min of $40x - 5x^2$ on $0 \leqslant x \leqslant 8$ must be one of $5\big[8x - x^2\big]_{x=0} = 0$ or $5\big[8x - x^2\big]_{x=8} = 0$ or $5\big[8x - x^2\big]_{x=4} = 80$.

So all together, we have the following candidates for max and min, with the max and min indicated.

point(s)	$(1,1)$	L_1	L_2	$(0,8)$	$(8,0)$	$(4,4)$
value of f	-1	0	0	0	0	80
	min					max



S-8: (a) Since

$$\begin{array}{lll} f = 2x^3 - 6xy + y^2 + 4y & & \\ f_x = 6x^2 - 6y & f_{xx} = 12x & f_{xy} = -6 \\ f_y = -6x + 2y + 4 & f_{yy} = 2 & \end{array}$$

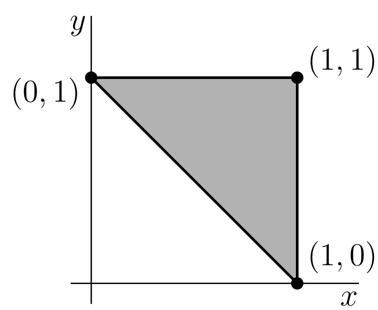
the gradient is defined everywhere and the critical points are the solutions of

$$\begin{array}{rll} & f_x = 0 & f_y = 0 \\ \iff & y = x^2 & y - 3x + 2 = 0 \\ \iff & y = x^2 & x^2 - 3x + 2 = 0 \\ \iff & y = x^2 & x = 1 \text{ or } 2 \end{array}$$

So, there are two critical points: $(1, 1)$, $(2, 4)$.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(1, 1)$	$12 \times 2 - (-6)^2 < 0$		saddle point
$(2, 4)$	$24 \times 2 - (-6)^2 > 0$	24	local min

(b) There are no critical points in the interior of the allowed region, so both the maximum and the minimum occur only on the boundary. The boundary consists of the line segments (i) $x = 1$, $0 \leqslant y \leqslant 1$, (ii) $y = 1$, $0 \leqslant x \leqslant 1$ and (iii) $y = 1 - x$, $0 \leqslant x \leqslant 1$.



- First, we look at the part of the boundary with $x = 1$. There $f = y^2 - 2y + 2$. As $\frac{d}{dy}(y^2 - 2y + 2) = 2y - 2$ vanishes only at $y = 1$, the max and min of $y^2 - 2y + 2$ for $0 \leqslant y \leqslant 1$ must occur either at $y = 0$, where $f = 2$, or at $y = 1$, where $f = 1$.
- Next, we look at the part of the boundary with $y = 1$. There $f = 2x^3 - 6x + 5$. As $\frac{d}{dx}(2x^3 - 6x + 5) = 6x^2 - 6$, the max and min of $2x^3 - 6x + 5$ for $0 \leqslant x \leqslant 1$ must occur either at $x = 0$, where $f = 5$, or at $x = 1$, where $f = 1$.
- Next, we look at the part of the boundary with $y = 1 - x$. There $f = 2x^3 - 6x(1 - x) + (1 - x)^2 + 4(1 - x) = 2x^3 + 7x^2 - 12x + 5$. As $\frac{d}{dx}(2x^3 + 7x^2 - 12x + 5) = 6x^2 + 14x - 12 = 2\big(3x^2 + 7x - 6\big) = 2(3x - 2)(x + 3)$, the max and min of $2x^3 + 7x^2 - 12x + 5$ for $0 \leqslant x \leqslant 1$ must occur either at $x = 0$, where $f = 5$, or at $x = 1$, where $f = 2$, or at $x = \frac{2}{3}$, where $f = 2(\frac{8}{27}) - 6(\frac{2}{3})(\frac{1}{3}) + \frac{1}{9} + \frac{4}{3} = \frac{16 - 36 + 3 + 36}{27} = \frac{19}{27}$.

So all together, we have the following candidates for max and min, with the max and min indicated.

point	$(1,0)$	$(1,1)$	$(0,1)$	$(\frac{2}{3},\frac{1}{3})$
value of f	2	1	5	$\frac{19}{27}$
			max	min

S-9: (a) We have

$$\begin{aligned} f(x,y) &= xy(x+2y-6) & f_x(x,y) &= 2xy+2y^2-6y & f_{xx}(x,y) &= 2y \\ & & f_y(x,y) &= x^2+4xy-6x & f_{yy}(x,y) &= 4x \\ & & & & f_{xy}(x,y) &= 2x+4y-6 \end{aligned}$$

The first partial derivatives are defined everywhere. So the critical points are the solution of

$$\begin{aligned} f_x(x,y) = f_y(x,y) = 0 &\iff 2y(x+y-3) = 0 \text{ and } x(x+4y-6) = 0 \\ &\iff \{y=0 \text{ or } x+y=3\} \text{ and } \{x=0 \text{ or } x+4y=6\} \\ &\iff \{x=y=0\} \text{ or } \{y=0,\ x+4y=6\} \\ &\qquad \text{or } \{x+y=3,\ x=0\} \text{ or } \{x+y=3,\ x+4y=6\} \\ &\iff (x,y) = (0,0) \text{ or } (6,0) \text{ or } (0,3) \text{ or } (2,1) \end{aligned}$$

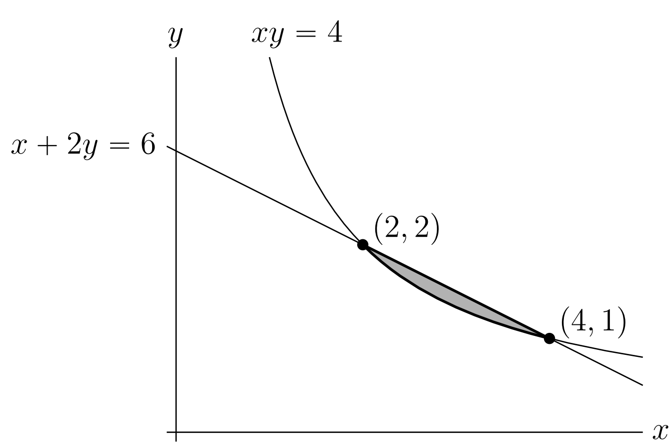
Here is a table giving the classification of each of the four critical points.

critical point	$f_{xx}f_{yy}-f_{xy}^2$	f_{xx}	type
$(0,0)$	$0\times 0-(-6)^2<0$		saddle point
$(6,0)$	$0\times 24-6^2<0$		saddle point
$(0,3)$	$6\times 0-6^2<0$		saddle point
$(2,1)$	$2\times 8-2^2>0$	2	local min

(b) Observe that $xy=4$ and $x+2y=6$ intersect when $x=6-2y$ and

$$\begin{aligned} (6-2y)y=4 &\iff 2y^2-6y+4=0 \iff 2(y-1)(y-2)=0 \\ &\iff (x,y) = (4,1) \text{ or } (2,2) \end{aligned}$$

The shaded region in the sketch below is D.



None of the critical points are in D. So the max and min must occur at either $(2,2)$ or $(4,1)$ or on $xy=4$, $2<x<4$ (in which case $F(x)=f\left(x,\frac{4}{x}\right)=4\left(x+\frac{8}{x}-6\right)$ obeys $F'(x)=4-\frac{32}{x^2}=0 \iff x=\pm 2\sqrt{2}$) or on $x+2y=6$, $2<x<4$ (in which case $f(x,y)$ is identically zero). So the min and max must occur at one of

(x,y)	$f(x,y)$
$(2,2)$	$2\times 2(2+2\times 2-6)=0$
$(4,1)$	$4\times 1(4+2\times 1-6)=0$
$(2\sqrt{2},\sqrt{2})$	$4(2\sqrt{2}+2\sqrt{2}-6)<0$

The maximum value is 0 and the minimum value is $4(4\sqrt{2}-6)\approx -1.37$.

S-10: The coldest point must be either on the boundary of the plate or in the interior of the plate.

- On the semi–circular part of the boundary $0\leqslant y\leqslant 2$ and $x^2+y^2=4$ so that $T=\ln\left(1+x^2+y^2\right)-y=\ln 5-y$. The smallest value of $\ln 5-y$ is taken when y is as large as possible, i.e. when $y=2$, and is $\ln 5-2\approx -0.391$.
- On the flat part of the boundary, $y=0$ and $-2\leqslant x\leqslant 2$ so that $T=\ln\left(1+x^2+y^2\right)-y=\ln\left(1+x^2\right)$. The smallest value of $\ln\left(1+x^2\right)$ is taken when x is as small as possible, i.e. when $x=0$, and is 0.
- If the coldest point is in the interior of the plate, it must be at a critical point of $T(x,y)$. Since

  $$T_x(x,y)=\frac{2x}{1+x^2+y^2}\qquad T_y(x,y)=\frac{2y}{1+x^2+y^2}-1$$

  a critical point must have $x=0$ and $\frac{2y}{1+x^2+y^2}-1=0$, which is the case if and only if $x=0$ and $2y-1-y^2=0$. So the only critical point is $x=0,\ y=1$, where $T=\ln 2-1\approx -0.307$.

Since $-0.391<-0.307<0$, the coldest temperature is -0.391 and the coldest point is $(0,2)$.

S-11: (a) We have

$$\begin{aligned} g(x,y)&=x^2-10y-y^2 &\quad g_x(x,y)&=2x &\quad g_{xx}(x,y)&=2\\ && g_y(x,y)&=-10-2y & g_{yy}(x,y)&=-2\\ &&&& g_{xy}(x,y)&=0 \end{aligned}$$

The first partial derivatives are defined everywhere. So the critical points are the solution of

$$g_x(x,y)=g_y(x,y)=0 \iff 2x=0 \text{ and } -10-2y=0 \iff (x,y)=(0,-5)$$

Since $g_{xx}(0,-5)g_{yy}(0,-5)-g_{xy}(0,-5)^2=2\times(-2)-0^2<0$, the critical point is a saddle point.

(b) The extrema must be either on the boundary of the region or in the interior of the region.

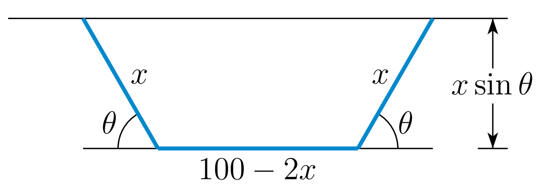
- On the semi-elliptical part of the boundary $-2 \leqslant y \leqslant 0$ and $x^2 + 4y^2 = 16$ so that $g = x^2 - 10y - y^2 = 16 - 10y - 5y^2 = 21 - 5(y+1)^2$. This has a minimum value of 16 (at $y = 0, -2$) and a maximum value of 21 (at $y = -1$). You could also come to this conclusion by checking the critical point of $16 - 10y - 5y^2$ (i.e. solving $\frac{\mathrm{d}}{\mathrm{d}y}(16 - 10y - 5y^2) = 0$) and checking the end points of the allowed interval (namely $y = 0$ and $y = -2$).
- On the flat part of the boundary $y = 0$ and $-4 \leqslant x \leqslant 4$ so that $g = x^2$. The smallest value is taken when $x = 0$ and is 0 and the largest value is taken when $x = \pm 4$ and is 16.
- If an extremum is in the interior of the plate, it must be at a critical point of $g(x, y)$. The only critical point is not in the prescribed region.

Here is a table giving all candidates for extrema:

(x, y)	$g(x, y)$
$(0, -2)$	16
$(\pm 4, 0)$	16
$(\pm\sqrt{12}, -1)$	21
$(0, 0)$	0

From the table the smallest value of g is 0 at $(0, 0)$ and the largest value is 21 at $(\pm 2\sqrt{3}, -1)$.

S-12: Suppose that the bends are made a distance x from the ends of the fence and that the bends are through an angle θ. Here is a sketch of the enclosure.



It consists of a rectangle, with side lengths $100 - 2x$ and $x \sin\theta$, together with two triangles, each of height $x \sin\theta$ and base length $x \cos\theta$. So the enclosure has area

$$\begin{aligned} A(x, \theta) &= (100 - 2x)x \sin\theta + 2 \cdot \tfrac{1}{2} \cdot x \sin\theta \cdot x \cos\theta \\ &= (100x - 2x^2) \sin\theta + \tfrac{1}{2} x^2 \sin(2\theta) \end{aligned}$$

The maximize the area, we need find the critical points.

$$\begin{aligned} A_x &= (100 - 4x) \sin\theta + x \sin(2\theta) \\ A_\theta &= (100x - 2x^2) \cos\theta + x^2 \cos(2\theta) \end{aligned}$$

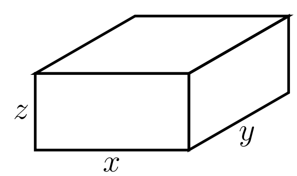
Note that A_x and A_θ are define everywhere in their domain and so to find the critical points we only needed to find the points where the gradient vanishes.

$$\begin{aligned} 0 = A_x &= (100-4x)\sin\theta + x\sin(2\theta) &\implies&& (100-4x) + 2x\cos\theta = 0 \\ 0 = A_\theta &= (100x-2x^2)\cos\theta + x^2\cos(2\theta) &\implies&& (100-2x)\cos\theta + x\cos(2\theta) = 0 \end{aligned}$$

Here we have used that the fence of maximum area cannot have $\sin\theta = 0$ or $x = 0$, because in either of these two cases, the area enclosed will be zero. The first equation forces $\cos\theta = -\frac{100-4x}{2x}$ and hence $\cos(2\theta) = 2\cos^2\theta - 1 = \frac{(100-4x)^2}{2x^2} - 1$. Substituting these into the second equation gives

$$\begin{aligned} & -(100-2x)\frac{100-4x}{2x} + x\Big[\frac{(100-4x)^2}{2x^2} - 1\Big] = 0 \\ \implies & -(100-2x)(100-4x) + (100-4x)^2 - 2x^2 = 0 \\ \implies & 6x^2 - 200x = 0 \\ \implies & x = \frac{100}{3} \quad \cos\theta = -\frac{-100/3}{200/3} = \frac{1}{2} \quad \theta = 60^\circ \\ & A = \Big(100\frac{100}{3} - 2\frac{100^2}{3^2}\Big)\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{100^2}{3^2}\frac{\sqrt{3}}{2} = \frac{2500}{\sqrt{3}} \end{aligned}$$

S-13: Suppose that the box has side lengths x, y and z. Here is a sketch.



Because the box has to have volume V we need that $V = xyz$. We wish to minimize the area $A = xy + 2yz + 2xz$ of the four sides and bottom. Substituting in $z = \frac{V}{xy}$,

$$\begin{aligned} A &= xy + 2\frac{V}{x} + 2\frac{V}{y} \\ A_x &= y - 2\frac{V}{x^2} \\ A_y &= x - 2\frac{V}{y^2} \end{aligned}$$

To minimize, we want $A_x = A_y = 0$, which is the case when $yx^2 = 2V,\ xy^2 = 2V$. This forces $yx^2 = xy^2$. Since $V = xyz$ is nonzero, neither x nor y may be zero. So $x = y = (2V)^{1/3}$, $z = 2^{-2/3}V^{1/3}$.

S-14: (a) The maximum and minimum can occur either in the interior of the disk or on the boundary of the disk. The boundary of the disk is the circle $x^2 + y^2 = 4$.

- Any absolute max or min in the interior of the disk must also be a local max or min and must also be a critical point of h. Since $T_x = -8x$ and $T_y = -2y$, the only critical point is $(x, y) = (0, 0)$, where $T = 20$. Since $4x^2 + y^2 \geqslant 0$, we have $T(x, y) = 20 - 4x^2 - y^2 \leqslant 20$. So the maximum value of T (even in $\mathbb{R}^2$) is 20.
- At each point of $x^2 + y^2 = 4$ we have $T(x, y) = 20 - 4x^2 - y^2 = 20 - 4x^2 - (4 - x^2) = 16 - 3x^2$ with $-2 \leqslant x \leqslant 2$. So T is a minimum when x^2 is a maximum. Thus the minimum value of T on the disk is $16 - 3(\pm 2)^2 = 4$.

So all together, the maximum and minimum values of $T(x, y)$ in $x^2 + y^2 \leqslant 4$ are 20 (at $(0, 0)$) and 4 (at $(\pm 2, 0)$), respectively.

(b) We are being asked to find the $(x, y) = (x, 2 - x^2)$ which maximizes

$$T\big(x, 2 - x^2\big) = 20 - 4x^2 - \big(2 - x^2\big)^2 = 16 - x^4$$

The maximum of $16 - x^4$ is obviously 16 at $x = 0$. So the ant should go to $\big(0, 2 - 0^2\big) = (0, 2)$.

S-15: The region of interest is

$$D = \big\{\ (x, y, z)\ \big|\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0,\ 2x + y + z = 5\ \big\}$$

First observe that, on the boundary of this region, at least one of x, y and z is zero. So $f(x, y, z) = x^2y^2z$ is zero on the boundary. As f takes values which are strictly bigger than zero at all points of D that are not on the boundary, the minimum value of f is 0 on

$$\partial D = \big\{\ (x, y, z)\ \big|\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0,\ 2x + y + z = 5,\ \text{at least one of } x, y, z \text{ zero}\ \big\}$$

The maximum value of f will be taken at a critical point. On D

$$f = x^2y^2(5 - 2x - y) = 5x^2y^2 - 2x^3y^2 - x^2y^3$$

So the critical points are the solutions of

$$\begin{aligned} 0 &= f_x(x, y) = 10xy^2 - 6x^2y^2 - 2xy^3 \\ 0 &= f_y(x, y) = 10x^2y - 4x^3y - 3x^2y^2 \end{aligned}$$

(note that the gradient is defined everywhere) or, dividing by the first equation by xy^2 and the second equation by x^2y, (recall that $x, y \neq 0$)

$$\begin{aligned} 10 - 6x - 2y = 0 &\qquad \text{or} \qquad 3x + y = 5 \\ 10 - 4x - 3y = 0 &\qquad \text{or} \qquad 4x + 3y = 10 \end{aligned}$$

Substituting $y = 5 - 3x$, from the first equation, into the second equation gives

$$4x + 3(5 - 3x) = 10 \implies -5x + 15 = 10 \implies x = 1,\ y = 5 - 3(1) = 2$$

So the maximum value of f is $(1)^2(2)^2(5 - 2 - 2) = 4$ at $(1, 2, 1)$.

S-16: (a) For $x, y > 0$, f_x and f_y are well-defined and so the critical points are the solutions of

$$f_x = 2 - \frac{1}{x^2 y} = 0 \iff y = \frac{1}{2x^2}$$
$$f_y = 4 - \frac{1}{xy^2} = 0$$

Substituting $y = \frac{1}{2x^2}$, from the first equation, into the second gives $4 - 4x^3 = 0$ which forces $x = 1, y = \frac{1}{2}$. At $x = 1, y = \frac{1}{2}$,

$$f\big(1, \tfrac{1}{2}\big) = 2 + 2 + 2 = 6$$

(b) The second derivatives are

$$f_{xx}(x,y) = \frac{2}{x^3 y} \qquad f_{xy}(x,y) = \frac{1}{x^2 y^2} \qquad f_{yy}(x,y) = \frac{2}{xy^3}$$

In particular

$$f_{xx}\big(1, \tfrac{1}{2}\big) = 4 \qquad f_{xy}\big(1, \tfrac{1}{2}\big) = 4 \qquad f_{yy}\big(1, \tfrac{1}{2}\big) = 16$$

Since $f_{xx}\big(1, \frac{1}{2}\big) f_{yy}\big(1, \frac{1}{2}\big) - f_{xy}\big(1, \frac{1}{2}\big)^2 = 4 \times 16 - 4^2 = 48 > 0$ and $f_{xx}\big(1, \frac{1}{2}\big) = 4 > 0$, the point $\big(1, \frac{1}{2}\big)$ is a local minimum.

(c) As x or y tends to infinity (with the other at least zero), $2x + 4y$ tends to $+\infty$. As (x, y) tends to any point on the first quadrant part of the x- and y–axes, $\frac{1}{xy}$ tends to $+\infty$. Hence as x or y tends to the boundary of the first quadrant (counting infinity as part of the boundary), $f(x, y)$ tends to $+\infty$. As a result $\big(1, \frac{1}{2}\big)$ is a global (and not just local) minimum for f in the first quadrant. Hence $f(x, y) \geqslant f\big(1, \frac{1}{2}\big) = 6$ for all $x, y > 0$.

S-17: First, let's visualize what's going on. Our surface looks like a bowl, sitting on the origin, opening upwards. It is radially symmetric about the z-axis, with circular level curves. That means every point on a level curve is equidistant from the z-axis. Since the point $(0, 0, a)$ is on the z-axis, if there is a point (x, y, z) that has minimum distance to the point, then its entire level curve has the same minimum distance. So we expect our answer to look like a circle (or possibly a single point – a "circle" of radius 0). If a is a negative number, it seems natural that the closest point would be $(0, 0, 0)$.

The distance from $(0, 0, a)$ to an arbitrary point (x, y, z) is $\sqrt{x^2 + y^2 + (z - a)^2}$. If the point (x, y, z) is on our surface, then $z = x^2 + y^2$. Rather than deal with square roots, we'll minimize the distance squared:

$$f(x, y) = x^2 + y^2 + \Big(x^2 + y^2 - a\Big)^2$$

From our observations above, there will be no global maximum; the global minimum will be a local minimum; the global minimum will depend on a in a less-than-simple way; and there are likely to be multiple points that are all minimum distance to $(0, 0, a)$.

We start by finding critical points.

$$\begin{aligned}
f_x(x,y) &= 2x + 2x\cdot 2\left(x^2+y^2-a\right)\\
&= 2x\left(1+2\left(x^2+y^2-a\right)\right)\\
&= 4x\left(x^2+y^2+\frac{1}{2}-a\right)\\
f_y(x,y) &= 2y + 2y\cdot 2\left(x^2+y^2-a\right)\\
&= 2y\left(1+2\left(x^2+y^2-a\right)\right)\\
&= 4y\left(x^2+y^2+\frac{1}{2}-a\right)
\end{aligned}$$

- For any value of a, $(x,y)=(0,0)$ is a critical point.
- If $a<\frac{1}{2}$, then the only critical point is $(x,y)=(0,0)$.
- If $a\geqslant\frac{1}{2}$, then all points on the level curve $x^2+y^2=a-\frac{1}{2}$ are critical points.

So if $a<\frac{1}{2}$, we're done: the single closest point on the surface is $(0,0,0)$.

Suppose $a\geqslant\frac{1}{2}$. Now we need to decide whether $(0,0,a)$ is closer to the origin or to a poitn on the level curve $x^2+y^2=a-\frac{1}{2}$.

- $f(0,0)=0+0+(0-a)^2=a^2$
- If $x^2+y^2=a-\frac{1}{2}$, then:

$$\begin{aligned}
f(x,y) &= x^2+y^2+\left(x^2+y^2-a\right)^2\\
&= \left(a-\frac{1}{2}\right)+\left(a-\frac{1}{2}-a\right)^2\\
&= \left(a-\frac{1}{2}\right)+\frac{1}{4}\\
&= a-\frac{1}{4}
\end{aligned}$$

- All together, the origin is closer than the level curve when:

$$\begin{aligned}
a^2 &< a-\frac{1}{4}\\
a^2-a+\frac{1}{4} &< 0\\
\left(a-\frac{1}{2}\right)^2 &< 0
\end{aligned}$$

  which never happens. So the origin is never closer than the level curve, again provided $a\geqslant\frac{1}{2}$.

So, all together: if $a < \frac{1}{2}$, then the closest point is the origin. If $a \geqslant \frac{1}{2}$, then the closest points are the level curve where $z = a - \frac{1}{2}$.

S-18:

(a) Let us first find the profit equations for each of the paper sizes separately and then we sum them up to get the total profit function.

$$\Pi_4(x) = f(x)(6) - x(1) = 15x^{0.8} - x \qquad \text{(profit for A4)}$$
$$\Pi_3(y) = g(y)(8) - y(3) = 80y^{0.6} - 3y \qquad \text{(profit for A3)}$$

and therefore, the total profit equation is given by

$$\begin{aligned} \Pi(x,y) &= \Pi_{4(x)} + \Pi_3(y) \\ &= (15x^{0.8} - x) + (80y^{0.6} - 3y) \end{aligned}$$

Note that the production functions of the two paper types aren't really linked. It's as if one firm is doing all the A4, and a different firm is doing all the A3. So to maximize $\Pi(x,y)$, we can just find the maximum value of Π_4 and the maximum value of Π_3 separately.

(b) Note that $x_4, x_3, x_2 \geqslant 0$ as we cannot produce negative amount of papers. (Maybe that would mean turning papers into trees?) Note also:

$$\Pi_4(0) = 0 \qquad \lim_{x\to\infty} \Pi_4(x) = -\infty$$
$$\Pi_3(0) = 0 \qquad \lim_{y\to\infty} \Pi_3(y) = -\infty$$

Now let's consider critical points of each function.

$$\begin{aligned} \frac{d\Pi_4}{dx} &= 15(0.8)x^{-0.2} - 1 = 12x^{-0.2} - 1 = 0 \implies x = 12^5 \\ \Pi_4\left(12^5\right) &= 15\left(12^5\right)^{4/5} - 12^5 = 15\left(12^4\right) - 12^5 = 3 \cdot 12^4 \\ \frac{d\Pi_3}{dy} &= 80(0.6)y^{-0.4} - 3 = 48y^{-0.4} - 3 = 0 \implies y = 2^{10} \\ \Pi_3\left(2^{10}\right) &= 80\left(2^{10}\right)^{6/10} - 3 \cdot 2^{10} = 5 \cdot 2^{10} - 3 \cdot 2^{10} = 2^{11} \end{aligned}$$

(Also $x = 0$ and $y = 0$ are critical points, since the derivatives are undefined there, but we've already considered them when we thought about endpoints.)

Since $\Pi_4\left(12^5\right) > \Pi_4(0)$ and $\Pi_3\left(2^{10}\right) > \Pi_3(0)$, we see our maximum will occur when $x = 12^5$ and $y = 2^{10}$. Then the number of reams produced will be:

$$f\left(12^5\right) = \frac{5}{2}\left(12^5\right)^{4/5} = 51840$$
$$g\left(2^{10}\right) = 10\left(2^{10}\right)^{6/10} = 640$$

(c) As we saw before, the two reams are optimized separately. So the optimal production of A3 isn't affected by how much A4 is produced. That is, the branch should stick with $y = 1,024$ leading to 640 reams of A3.

S-19:

(a) To find Ayan's profit equation, which we denote by Π_A, we just plug in the information we are given in the general profit equation (revenue minus cost).

$$\begin{aligned}
\Pi_A(q_A) &= \underbrace{q_A\left[121 - 2(q_A + q_P)\right]}_{\text{revenue}} - \underbrace{q_A(1)}_{\text{cost}} \\
&= 121q_A - 2q_A^2 - 2q_Aq_P - q_A \\
&= -2q_A^2 + 120q_A - 2q_Aq_P
\end{aligned}$$

This is a parabola pointing down, so its maximum will be at its only critical point.

$$\begin{aligned}
\frac{\mathrm{d}\Pi_a}{\mathrm{d}q_A} &= -4q_A + 120 - 2q_P = 0 \\
4q_A &= 120 - 2q_P \\
q_1 &= 30 - \frac{1}{2}q_P
\end{aligned}$$

So Ayan would maximize their profit by selling $30 - \frac{1}{2}q_P$ servings of lemonade.

(b) This is very similar to the last part. We find Pipe's profit function.

$$\begin{aligned}
\Pi_P(q_P) &= q_P\left[121 - 2(q_A + q_P)\right] - q_P(1) \\
&= 121q_P - 2q_P^2 - 2q_Pq_A - q_P \\
&= -2q_P^2 + 120q_P - 2q_Pq_A
\end{aligned}$$

Note that this is Π_A if we switch the places of q_A and q_P. So Pipe would maximize their profit by selling $30 - \frac{1}{2}q_A$ pitchers of lemonade.

(c) Ayan's and Pipe's cost and price for every pitcher of lemonade produced are the same. Their businesses are identical. So we predict that they will sell the same amount of lemonade to maximize their respective profits.

(d) To find how much each seller will sell when they are working separately, find out which values of q_A and q_P end up with both individual profit functions being maximized. Therefore we solve the system of equations we get from (a) and (b).

$$\begin{cases} q_A &= 30 - \frac{1}{2}q_P \\ q_P &= 30 - \frac{1}{2}q_A \end{cases}$$

$$\begin{aligned}
&\implies q_P = 30 - \frac{1}{2}\underbrace{\left(30 - \frac{1}{2}q_P\right)}_{q_A} = 15 + \frac{1}{4}q_P \\
&\implies q_P = 20 \\
&\implies q_A = 30 - \frac{1}{2}\underbrace{(20)}_{q_P} = 20
\end{aligned}$$

So, as predicted, both sellers sell the same number of pitchers.

(e) We need to plug in $q_P = q_A = 20$ in Π_A and Π_P:

$$\Pi_A(20)\big|_{q_P=20} = -2(20)^2 + 120(20) - 2(20)(20) = 800$$

And similarly, $\Pi_P(20)\big|_{q_A=20} = 800$. So, they would each make 800 dollars in profit.

(f) The joint profit function is $\Pi(q_A, q_P) = \Pi_A(q_A) + \Pi_P(q_P)$. Note that here, Ayan and Pipe are helping each other to make the most profit, instead of competing. Using the same intuition as before, we can conclude that $q_A = q_P$ in this case too. (So they share the workload fairly!)

So to make things easier let us assume $q_A = q_P$ and denote this quantity by q. Then $\Pi_A(q) = \Pi_P(q) = -4q^2 + 120q$. This means

$$\begin{aligned}\Pi_{\text{joint}}(q) &= \Pi_A(q) + \Pi_P(q) = 2\Pi_A(q)\\ &= 2(-4q^2 + 120q)\\ &= -8q^2 + 240q\end{aligned}$$

This is a parabola pointing down, so its global min is at its sole critical point, $q = 15$.

So $q = q_A = q_P = 15$ maximizes the joint profit. Let us compute the corresponding joint profit

$$\Pi_{\text{joint}}(15) = -8(15)^2 + 240(15) = 1,800$$

So their optimal joint profit will be $1,800$ dollars. But, they need to share this profit among the two of them. So if they collaborate, they will each earn 900 dollars. This is more than their individual optimal profit in the scenario where they are competing found in part (e) (we found this to be \$800). So it is better for them to collaborate!

(g) When the two sellers collaborate, they sell fewer lemonades (30 pitchers total instead of 40 total) and the lemonade costs more (\$60 instead of \$40). So it's better for consumers when the sellers compete.

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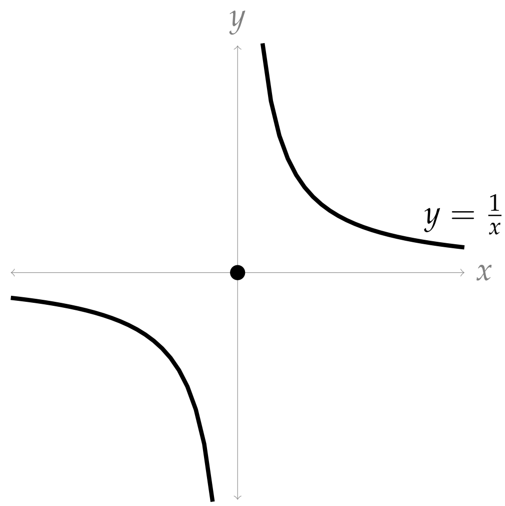
S-1: (a) $f(x, y) = x^2 + y^2$ is the square of the distance from the point (x, y) to the origin. There are points on the curve $xy = 1$ that have either x or y arbitrarily large and so whose distance from the origin is arbitrarily large. So f has no maximum on the curve.



On the other hand f will have a minimum, achieved at the points of $xy = 1$ that are closest to the origin.

(b) On the curve $xy = 1$ we have $y = \frac{1}{x}$ and hence $f = x^2 + \frac{1}{x^2}$. As

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + \frac{1}{x^2}\right) = 2x - \frac{2}{x^3} = \frac{2}{x^3}(x^4 - 1)$$

and as no point of the curve has $x = 0$, the minimum is achieved when $x = \pm 1$. So the minima are at $\pm(1, 1)$, where f takes the value 2.

Remark: this is less a question specifically about Lagrange multipliers and more a question about the existence of extrema on unbounded curves, as in section 2.5.1 in the text.

S-2: The easiest (cheapest?) way out is to think of a function $z = k(x)$ with local but not absolute extrema, then consider the constraint $y = 0$. This puts our function in the xz-plane, effectively making it look just like the function of one-variable $y = f(x)$.

For example, we can set $f(x, y) = x^3 - x$, with constraint function $g(x, y) = y = 0$.

Using techniques from last semester, the function $z = x^3 - x$ has local max at $x = -\frac{1}{\sqrt{3}}$ and local min at $x = \frac{1}{\sqrt{3}}$; but it has no absolute extrema because $\lim\limits_{x\to\infty}(x^3 - x) = \infty$ and $\lim\limits_{x\to-\infty}(x^3 - x) = -\infty$.

Similarly, $f(x, y)$ has a local constrained max resp. min at $\left(-\frac{1}{\sqrt{3}}, 0\right)$ resp. $\left(-\frac{1}{\sqrt{3}}, 0\right)$; but has no absolute extrema.

S-3: So we are to minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = x^2 y - 1 = 0$.

The constraint is not a closed curve, so we need to be a little more careful than average. We can interpret our objective function as the distance from the origin squared. So we're trying to find the point on the curve $y = \frac{1}{x^2}$ that is closest to the origin. The distance from points on that curve to the origin can be arbitrarily large, so the system has no absolute maximum. It does have an absolute minimum, which will also be a local minimum, so it will be a solution to the system of Lagrange equations.

According to the method of Lagrange multipliers, we need to find all solutions to

$$\begin{aligned} f_x &= \lambda g_x & 2x &= \lambda(2xy) && \text{(E1)} \\ f_y &= \lambda g_y & 2y &= \lambda x^2 && \text{(E2)} \\ g(x,y) &= 0 & x^2y &= 1 && \text{(E3)} \end{aligned}$$

- If $g_x \neq 0$ and $g_y \neq 0$, then $\lambda = \frac{2x}{2xy} = \frac{1}{y}$ by (E1) and $\lambda = \frac{2y}{x^2}$ by (E2).

$$\begin{aligned} \frac{1}{y} &= \frac{2y}{x^2} \\ x^2 &= 2y^2 \\ x &= \pm\sqrt{2}y \end{aligned}$$

  Using (E3):

$$\begin{aligned} 1 = x^2y &= \left(\pm\sqrt{2}y\right)^2 y \\ &= 2y^3 \\ y &= \frac{1}{\sqrt[3]{2}} \\ x &= \pm\sqrt{2}\cdot\frac{1}{\sqrt[3]{2}} = \pm 2^{\frac{1}{2}-\frac{1}{3}} = \pm 2^{\frac{1}{6}} \end{aligned}$$

  This gives us two solutions: $\left(\pm 2^{1/6}, 2^{-1/3}\right)$.

- If $g_x = 0$, then $0 = 2xy$. By (E1), $x = 0$; then by(E2), $y = 0$. Then (E3) fails, so there are no solutions of this type.

- If $g_y = 0$, then $0 = x^2$, so $0 = x$. By (E2), $y = 0$. Then (E3) fails, so there are no solutions of this type.

So the two points to check are $\left(2^{\frac{1}{6}}, 2^{-\frac{1}{3}}\right)$ and $\left(-2^{\frac{1}{6}}, 2^{-\frac{1}{3}}\right)$. For both of these critical points,

$$x^2 + y^2 = 2^{\frac{1}{3}} + 2^{-\frac{2}{3}} = 2^{\frac{1}{3}} + \frac{1}{2}2^{\frac{1}{3}} = \frac{3}{2}\sqrt[3]{2} = \frac{3}{\sqrt[3]{4}}$$

S-4: For this problem the objective function is $f(x,y) = xy$ and the constraint function is $g(x,y) = x^2 + 2y^2 - 1$. To apply the method of Lagrange multipliers we need ∇f and ∇g. So we start by computing the first order derivatives of these functions.

$$f_x = y \qquad f_y = x \qquad g_x = 2x \qquad g_y = 4y$$

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$y = \lambda(2x) \tag{E1}$$
$$x = \lambda(4y) \tag{E2}$$
$$x^2 + 2y^2 - 1 = 0 \tag{E3}$$

- If $g_x \neq 0$ and $g_y \neq 0$, then $\lambda = \frac{y}{2x}$ (E1) and $\lambda = \frac{x}{4y}$.

  $$\frac{y}{2x} = \frac{x}{4y}$$
  $$2y^2 = x^2$$

  From (E3):

  $$2y^2 + 2y^2 - 1 = 0$$
  $$4y^2 = 1$$
  $$y = \pm\frac{1}{2}$$
  $$x = \pm\sqrt{2}y = \pm\frac{1}{\sqrt{2}}$$

  So four solutions to the system are $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$.

- If $g_x = 0$ then $x = 0$; by (E1), $y = 0$; then (E3) fails.
- If $g_y = 0$ then $y = 0$; by (E2), $x = 0$; then (E3) fails.

The method of Lagrange multipliers, Theorem 2.5.2 in the text, gives that the only possible locations of the maximum and minimum of the function f are $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$.

point	$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$
$f(x,y)$	$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
	max	min	min	max

So the maximum and minimum values of f are $\frac{1}{2\sqrt{2}}$ and $-\frac{1}{2\sqrt{2}}$, respectively.

S-5: This is a constrained optimization problem with the objective function being $f(x,y) = x^2 + y^2$ and the constraint function being $g(x,y) = x^4 + y^4 - 1$. By Theorem 2.5.2 in the text, any minimum or maximum (x,y) must obey the Lagrange multiplier equations

$$f_x = g_x \qquad 2x = 4\lambda x^3 \tag{E1}$$
$$f_y = g_y \qquad 2y = 4\lambda y^3 \tag{E2}$$
$$g(x,y) = 1 \qquad x^4 + y^4 = 1 \tag{E3}$$

- If $g_x \neq 0$ and $g_y \neq 0$, then $\lambda = \frac{2x}{4x^3} = \frac{1}{2x^2}$ (E2) and $\lambda = \frac{2y}{4y^3} = \frac{1}{2y^2}$ (E2). So $x^2 = \frac{1}{2\lambda} = y^2$. Then (E3) reduces to

$$2x^4 = 1$$

so that $x^2 = y^2 = \frac{1}{\sqrt{2}}$ and $x = \pm 2^{-1/4}$, $y = \pm 2^{-1/4}$. At all four of these points, we have $f = \sqrt{2}$.

- If $g_x = 0$, then $x = 0$. (E1) holds for any λ, so by choosing λ correctly we can make (E2) hold as well. (E3) reduces to $y^4 = 1$ or $y = \pm 1$. At both $\big(0, \pm 1\big)$ we have $f\big(0, \pm 1\big) = 1$.
- If $g_y = 0$, then $y = 0$. (E2) holds for any λ, so by choosing λ correctly (E1) holds as well. (E3) reduces to $x^4 = 1$ or $x = \pm 1$. At both $\big(\pm 1, 0\big)$ we have $f\big(\pm 1, 0\big) = 1$.

So the minimum value of f on $x^4 + y^4 = 1$ is 1 and the maximum value of f on $x^4 + y^4 = 1$ is $\sqrt{2}$.

S-6:

$$\begin{aligned} f_x &= \lambda g_x & 4x^3 &= \lambda \cdot 2x && \text{(E1)} \\ f_y &= \lambda g_y & 4y^3 + 4y^5 &= \lambda \cdot 2y && \text{(E2)} \\ g(x,y) &= 1 & x^2 + y^2 &= 1 && \text{(E3)} \end{aligned}$$

- If $g_x \neq 0$ and $g_y \neq 0$, then $\lambda = \frac{4x^3}{2x} = 2x^2$ (E1) and $\lambda = \frac{4y^3 + 4y^5}{2y} = (2y^2 + 2y^4)$ (E2). So, $x^2 = \frac{\lambda}{2} = y^2 + y^4$. From (E3):

$$\begin{aligned} (y^2 + y^4) + y^2 &= 1 \\ y^4 + 2y^2 - 1 &= 0 \\ y^2 &= \frac{-2 \pm \sqrt{4 - 4(-1)}}{2} \\ &= -1 \pm \sqrt{2} \\ y^2 &= \sqrt{2} - 1 \end{aligned}$$

In this case, $x^2 = 1 - y^2 = 2 - \sqrt{2}$. So, we should check $(\pm\sqrt{2 - \sqrt{2}}\,,\ \pm\sqrt{\sqrt{2} - 1})$.

- If $g_x = 0$, then x=0. Then (E1) is true for any λ, which means we can make $(E2)$ be true by choosing λ accordingly. By (E3), $x = 0 \implies y = \pm 1$, so we should check $(0, \pm 1)$
- If $g_y = 0$, then $y = 0$. Then (E2) is true for any λ, which means we can make $(E1)$ be true by choosing λ accordingly. By (E3), $y = 0 \implies x = \pm 1$, so we should check $(\pm 1, 0)$

Comparing:

- $f(0, \pm 1) = 0 + 1 + \frac{2}{3} = \frac{5}{3}$

- $f(\pm 1, 0) = 1 + 0 + 0 = 1$
- When $x^2 = 2 - \sqrt{2}$ and $y^2 = \sqrt{2} - 1$, then

  $$\begin{aligned} f(x,y) &= (2-\sqrt{2})^2 + (\sqrt{2}-1)^2 + \frac{2}{3}(\sqrt{2}-1)^3 \\ &= \frac{13 - 8\sqrt{2}}{3} \end{aligned}$$

  Since $\sqrt{2} > \frac{5}{4}$, we see

  $$\frac{13-8\sqrt{2}}{3} < \frac{13 - 8(5/4)}{3} = \frac{13-10}{3} = 1$$

  So, our absolute min over the constraint is $\frac{13-8\sqrt{2}}{3}$, and our absolute max over the constraint is $\frac{5}{3}$.

S-7: (It's possible to solve this without Lagrange, but we were asked to use Lagrange to practice the technique.)

We want to minimize $\sqrt{x^2+y^2}$, the distance from the origin to a point (x,y). Note the minimum of that function will occur at the same (x,y)-values as the minimum of its square, x^2+y^2. Since that's easier to minimize, we use it as our objective function: $f(x,y) = x^2 + y^2$.

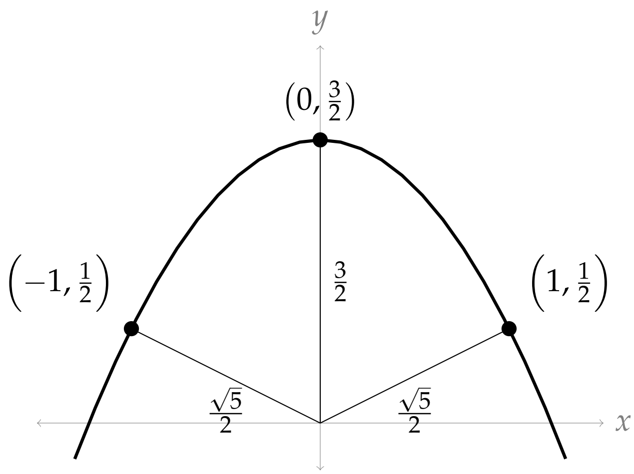
We only care about coordinates that are actually on the parabola, so our constraint function is $g(x,y) = y + x^2 = \frac{3}{2}$.

Our constraint function is not a closed curve. We can keep travelling along the parabola to end up arbitrarily far from the origin. So there's no global maximum distance, but there is a global minimum distance. The global minimum will also be a local minimum, so it will be a solution to the Lagrange equations.

$$\begin{aligned} f_x &= \lambda g_x & 2x &= \lambda 2x & \text{(E1)} \\ f_y &= \lambda g_y & 2y &= \lambda & \text{(E2)} \\ g(x,y) &= \frac{3}{2} & y + x^2 &= \frac{3}{2} & \text{(E3)} \end{aligned}$$

- If $g_x \neq 0$ and $g_y \neq 0$, then $\lambda = 1$ from (E1) and $\lambda = 2y$ from (E2), so $1 = 2y$, i.e. $y = \frac{1}{2}$. From (E3), then $x = \pm 1$.
- If $g_x = 0$, then $x = 0$, so (E1) is true for any λ. Then we can make (E2) true by choosing the appropriate λ; from (E3), $y = \frac{3}{2}$. So another point solving the system is $(0, \frac{3}{2})$.
- There are no points corresponding to $g_y = 0$.

$f\left(0, \frac{3}{2}\right) = \frac{9}{4}$ and $f\left(\pm 1, \frac{1}{2}\right) = \frac{5}{4}$. So, the closest points to the origin on the parabola are the points $(-1, 1/2)$ and $(1, 1/2)$.



S-8:

To find extrema over a region, we check CPs and the boundary.
$f(x, y) = xy$, so $f_x = y$ and $f_y = x$. Then the only CP is $(0, 0)$.

To check the boundary, we need to know the extreme values of $f(x, y) = xy$ over the ellipse $x^2 - 2xy + 5y^2 = 1$. It seems tough to do this with plugging in, so we use Lagrange.

$$f_x = \lambda g_x \qquad y = \lambda(2x - 2y) \qquad \text{(E1)}$$
$$f_y = \lambda g_y \qquad x = \lambda(-2x + 10y) \qquad \text{(E2)}$$
$$g(x, y) = 1 \qquad x^2 - 2xy + 5y^2 = 1 \qquad \text{(E3)}$$

- If $g_x \neq 0$ and $g_y \neq 0$, then $\lambda = \frac{y}{2(x-y)}$ and $\lambda = \frac{x}{2(5y-x)}$:

$$\frac{y}{2(x - y)} = \frac{x}{2(5y - x)}$$
$$5y^2 - xy = x^2 - xy$$
$$x = \pm\sqrt{5}y$$

From (E3), if $x = \sqrt{5}y$:

$$\begin{aligned} 1 &= 5y^2 - 2(\sqrt{5}y)y + 5y^2 \\ &= (10 - 2\sqrt{5})y^2 \\ \frac{1}{10 - 2\sqrt{5}} &= y^2 \\ y &= \pm\frac{1}{\sqrt{10 - 2\sqrt{5}}} \end{aligned}$$

From (E3), if $x = -\sqrt{5}y$:

$$\begin{aligned} 1 &= (10 + 2\sqrt{5})y^2 \\ y &= \pm\frac{1}{\sqrt{10 + 2\sqrt{5}}} \end{aligned}$$

This gives us four points to check: $\left(\sqrt{\frac{5}{10-2\sqrt{5}}}, \pm\frac{1}{\sqrt{10-2\sqrt{5}}}\right)$ and $\left(\sqrt{\frac{5}{10+2\sqrt{5}}}, \pm\frac{1}{\sqrt{10+2\sqrt{5}}}\right)$.

- If $g_x = 0$, then (E1) $y = 0$, so $0 = g_x = 2x - 2y = 2x$, hence $x = 0$. But then (E3) fails.
- If $g_y = 0$, then (E2) $x = 0$, so $0 = g_y = -2x + 10y = 10y$, hence $y = 0$. But then (E3) fails.

All together, we've identified 5 possible locations of extrema.

- $f(0,0) = 0$
- $f\left(\sqrt{\frac{5}{10-2\sqrt{5}}}, \frac{1}{\sqrt{10-2\sqrt{5}}}\right) = \frac{\sqrt{5}}{10-2\sqrt{5}}$
- $f\left(\sqrt{\frac{5}{10-2\sqrt{5}}}, -\frac{1}{\sqrt{10-2\sqrt{5}}}\right) = -\frac{\sqrt{5}}{10-2\sqrt{5}}$
- $f\left(\sqrt{\frac{5}{10+2\sqrt{5}}}, \frac{1}{\sqrt{10+2\sqrt{5}}}\right) = \frac{\sqrt{5}}{10+2\sqrt{5}}$
- $f\left(\sqrt{\frac{5}{10+2\sqrt{5}}}, -\frac{1}{\sqrt{10+2\sqrt{5}}}\right) = -\frac{\sqrt{5}}{10+2\sqrt{5}}$

The largest and smallest of these are $\frac{\sqrt{5}}{10-2\sqrt{5}}$ and $\frac{-\sqrt{5}}{10-2\sqrt{5}}$, respectively.

S-9: By way of preparation, we have

$$\frac{\partial T}{\partial x}(x,y) = 2x\,e^y \qquad \frac{\partial T}{\partial y}(x,y) = e^y\big(x^2 + y^2 + 2y\big)$$

(a) (i) For this problem the objective function is $T(x,y) = e^y\big(x^2 + y^2\big)$ and the constraint function is $g(x,y) = x^2 + y^2 - 100$. According to the method of Lagrange multipliers, Theorem 2.5.2 in the text, we need to find all solutions to

$$\begin{aligned} T_x &= \lambda g_x & 2x\,e^y &= \lambda(2x) && \text{(E1)}\\ T_y &= \lambda g_y & e^y\big(x^2+y^2+2y\big) &= \lambda(2y) && \text{(E2)}\\ g(x,y) &= 100 & x^2+y^2 &= 100 && \text{(E3)} \end{aligned}$$

(a) (ii)

- If $g_x \neq 0$ and $g_y \neq 0$, then (E1) $\lambda = e^y$ and (E2) $\lambda = \frac{e^y(x^2+y^2+2y)}{2y}$.

$$\begin{aligned} e^y &= \frac{e^y(x^2+y^2+2y)}{2y}\\ 2y &= x^2 + y^2 + 2y\\ 0 &= x^2 + y^2 \end{aligned}$$

but this conflicts with (E3). So $g_x \neq 0$ and $g_y \neq 0$ doesn't lead to any solutions.

- If $g_x = 0$, then $x = 0$ and (E1) is true; then we can choose the appropriate l to make (E2) true. From (E3), $y = \pm 10$. So $(0, \pm 10)$ gives a solution.
- If $g_y = 0$, then $y = 0$. By (E2), $x = 0$, which conflicts with (E3).

So the only possible locations of the maximum and minimum of the function T are $(0, 10)$ and $(0, -10)$. To complete this part of the problem, we only have to compute T at those points.

point	$(0,10)$	$(0,-10)$
value of T	$100e^{10}$	$100e^{-10}$
	max	min

Hence the maximum value of $T(x,y) = e^y\left(x^2+y^2\right)$ on $x^2+y^2 = 100$ is $100e^{10}$ at $(0,10)$ and the minimum value is $100e^{-10}$ at $(0,-10)$.

We remark that, on $x^2+y^2=100$, the objective function $T(x,y) = e^y\left(x^2+y^2\right) = 100e^y$. So of course the maximum value of T is achieved when y is a maximum, i.e. when $y = 10$, and the minimum value of T is achieved when y is a minimum, i.e. when $y = -10$.

(b) (i) By definition, the point (x,y) is a critical point of $T(x,y)$ if and only if the gradient at that point does not exist or it is zero. The first partial derivatives

$$\begin{aligned} T_x &= 2x\,e^y \\ T_y &= e^y\left(x^2+y^2+2y\right) \end{aligned}$$

are well defined everywhere and so the critical points are exactly the point where

$$T_x = 2x\,e^y = 0 \tag{E1}$$

$$T_y = e^y\left(x^2+y^2+2y\right) = 0 \tag{E2}$$

(b) (ii) Equation (E1) forces $x = 0$. When $x = 0$, equation (E2) reduces to

$$e^y\left(y^2+2y\right) = 0 \iff y(y+2) = 0 \iff y = 0 \text{ or } y = -2$$

So there are two critical points, namely $(0,0)$ and $(0,-2)$.

(c) Note that $T(x,y) = e^y\left(x^2+y^2\right) \geqslant 0$ on all of $\mathbb{R}^2$. As $T(x,y) = 0$ only at $(0,0)$, it is obvious that $(0,0)$ is the coolest point.

In case you didn't notice that, here is a more conventional solution.

The coolest point on the solid disc $x^2+y^2 \leqslant 100$ must either be on the boundary, $x^2+y^2 = 100$, of the disc or be in the interior, $x^2+y^2 < 100$, of the disc.

In part (a) (ii) we found that the coolest point on the boundary is $(0,-10)$, where $T = 100e^{-10}$.

If the coolest point is in the interior, it must be a critical point and so must be either $(0,0)$, where $T = 0$, or $(0,-2)$, where $T = 4e^{-2}$.

So the coolest point is $(0,0)$.

S-10: Since $x \geqslant 0$ and $y \geqslant 0$, our constraint function has endpoints $(x, y) = (0, 400)$ and $(x, y) = (25, 0)$. Absolute extrema will occur at these endpoints or at points that solve the system of Lagrange equations.

$$\begin{aligned} f_x &= \lambda g_x & 3x^{-\frac{2}{3}}y^{\frac{2}{3}} &= 3200\lambda && \text{(E1)} \\ f_y &= \lambda g_y & 6x^{\frac{1}{3}}y^{-\frac{1}{3}} &= 200\lambda && \text{(E2)} \\ g(x,y) &= 80,000 & 3200x + 200y &= 80,000 && \text{(E3)} \end{aligned}$$

Since g_x and g_y are always nonzero, we only have one of our usual three cases.

$$\begin{aligned} 3x^{-\frac{2}{3}}y^{\frac{2}{3}} \cdot \frac{1}{3200} &= 6x^{\frac{1}{3}}y^{-\frac{1}{3}} \cdot \frac{1}{200} \\ x^{-\frac{2}{3}}y^{\frac{2}{3}} &= 32x^{\frac{1}{3}}y^{-\frac{1}{3}} \\ y^{\frac{1}{3}}y^{\frac{2}{3}} &= 32x^{\frac{1}{3}}x^{\frac{2}{3}} \\ y &= 32x \\ 3200x + 200(32x) &= 80,000 \\ x &= \frac{25}{3} \\ y &= \frac{25 \cdot 32}{3} = \frac{800}{3} \end{aligned}$$

Now we compare our three points of interest.

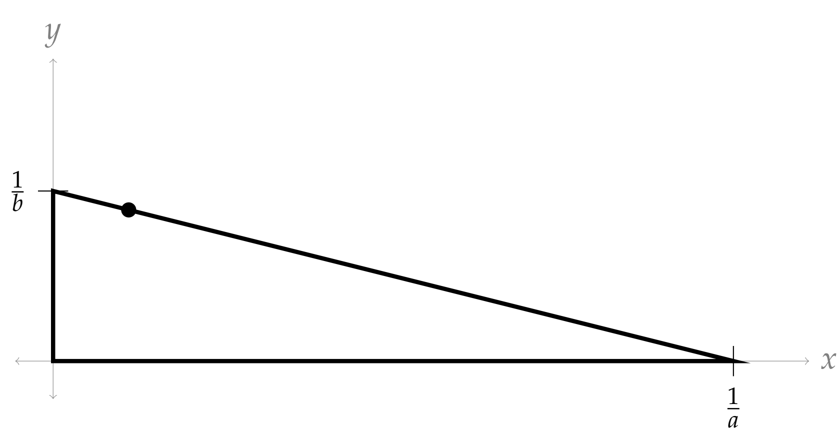
point	$(0, 400)$	$(25, 0)$	$\left(\frac{25}{3}, \frac{800}{3}\right)$
$f(x,y)$	0	0	$75 \cdot 2^{10/3}$
	min	min	max

S-11: The constraint tells us

$$g(a,b) = a + 2b = 1$$

The triangle formed is a right triangle with area $\frac{1}{2}bh$. Its base and height are the two intercepts of the line. That is, its base is $\frac{1}{a}$, and its height is $\frac{1}{b}$. So, the area (which we want to minimize) is

$$f(x,y) = \frac{1}{2} \cdot \frac{1}{a} \cdot \frac{1}{b}$$



By choosing lines with slopes close to 0, or large negative slopes, we can make triangles with arbitrarily large area. So the absolute minimum will occur somewhere in between at a local minimum value. So we can find the absolute minimum using the method of Lagrange multipliers.

$$f_a = \lambda g_a \qquad -\frac{1}{2a^2b} = \lambda(1) \tag{E1}$$

$$f_b = \lambda g_b \qquad -\frac{1}{2ab^2} = \lambda(2) \tag{E1}$$

Since g_a and g_b can't be 0, we have only one of our usual three cases.

$$\begin{aligned} -\frac{1}{2a^2b} &= -\frac{1}{2}\cdot\frac{1}{2ab^2} \\ \frac{1}{a} &= \frac{1}{2b} \\ a &= 2b \end{aligned}$$

Using our constraint,

$$\begin{aligned} 2b + 2b &= 1 \\ b &= \frac{1}{4} \\ a &= \frac{1}{2} \end{aligned}$$

So the minimum area is achieved by the line $\frac{1}{2}x + \frac{1}{4}y = 1$. That area is $\frac{1}{2}\cdot 4\cdot 2 = 4$.

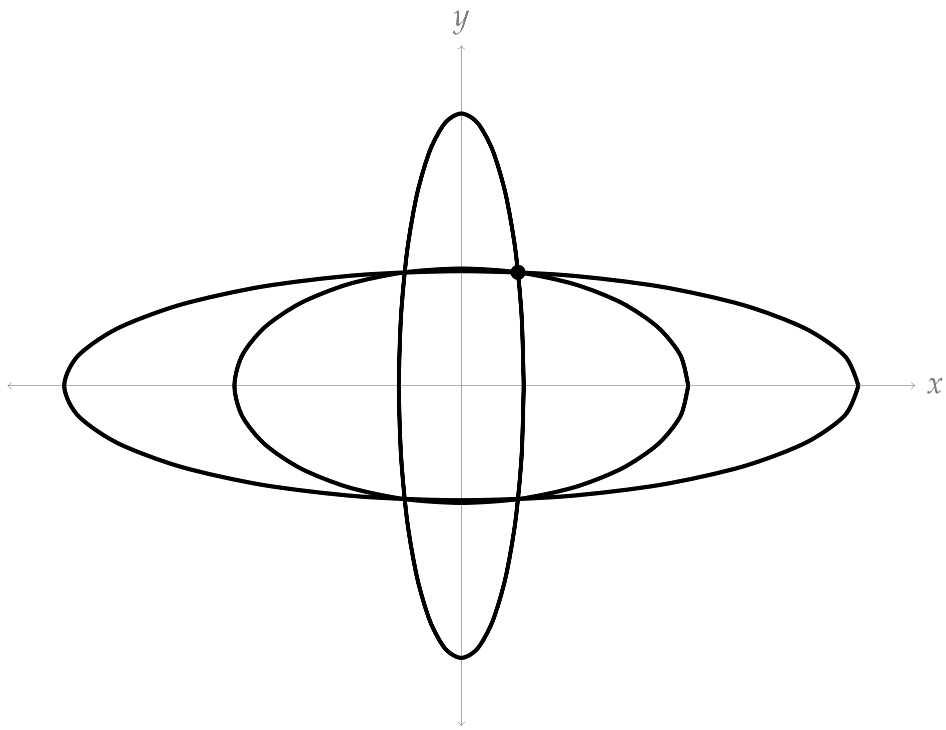
S-12: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point $(1, 2)$ if and only if $\frac{1}{a^2} + \frac{4}{b^2} = 1$. We are to minimize

$$f(a, b) = \pi ab$$

subject to the constraint that

$$g(a, b) = \frac{1}{a^2} + \frac{4}{b^2} - 1 = 0.$$

We can imagine ellipses centred at the origin passing through $(1, 2)$ of arbitrarily large size.



For large values of a (and corresponding values of b approaching 2), we have a large area. Similarly, for large values of b (and corresponding values of a approaching 2), we have a large area. So there's no absolute maximum, but there is a "sweet spot" where a and b are both not too large and we have a global minimum. It will also be a local minimum.

According to the method of Lagrange multipliers, we need to find all solutions to the system:

$$f_a = \lambda g_a \qquad \pi b = -\frac{2\lambda}{a^3} \tag{E1}$$

$$f_b = \lambda g_b \qquad \pi a = -\frac{8\lambda}{b^3} \tag{E2}$$

$$g(a,b) = 0 \quad \frac{1}{a^2} + \frac{4}{b^2} = 1 \tag{E3}$$

- If $g_a \neq 0$ and $g_b \neq 0$, then (E1) $\lambda = -\frac{\pi a^3 b}{2}$ and (E2) $\lambda = -\frac{\pi a b^3}{8}$.

$$\begin{aligned} -\frac{\pi a^3 b}{2} &= -\frac{\pi a b^3}{8} \\ 4a^3 b &= ab^3 \\ 4a^3 b - ab^3 &= 0 \\ ab(4a^2 - b^2) &= 0 \end{aligned}$$

  This last equation has solutions $a = 0$, $b = 0$, and $4a^2 = b^2$. The first two aren't in our model domain, since a and b are positive. In the third case:

$$\begin{aligned} 1 &= \frac{1}{a^2} + \frac{4}{4a^2} \\ &= \frac{1}{a^2} + \frac{1}{a^2} = \frac{2}{a^2} \end{aligned}$$

Remember $a > 0$ and $b > 0$.

$$\begin{aligned} a &= \sqrt{2} \\ b^2 &= 4a^2 = 4 \cdot 2 \\ b &= 2\sqrt{2} \end{aligned}$$

- If $g_a = 0$ or $g_b = 0$, then the constraint fails.

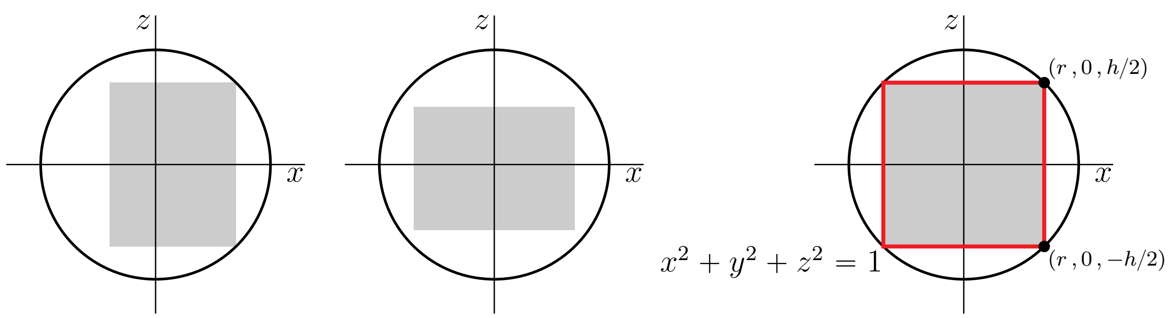
So, the only possible location of a local extremum is $a = \sqrt{2}$, $b = 2\sqrt{2}$. This is the location of our absolute minimum.

S-13: Let r and h denote the radius and height, respectively, of the cylinder. We can always choose our coordinate system so that the axis of the cylinder is parallel to the z–axis.

- If the axis of the cylinder does not lie exactly on the z–axis, we can enlarge the cylinder sideways. (See the figure on the left below. It shows the $y = 0$ cross–section of the cylinder.) So we can assume that the axis of the cylinder lies on the z–axis
- If the top and/or the bottom of the cylinder does not touch the sphere $x^2 + y^2 + z^2 = 1$, we can enlarge the cylinder vertically. (See the central figure below.)
- So we may assume that the cylinder is

$$\{\, (x, y, z) \mid x^2 + y^2 \leqslant r^2,\ -h/2 \leqslant z \leqslant h/2 \,\}$$

with $r^2 + (h/2)^2 = 1$. See the figure on the right below.



So we are to maximize the volume, $f(r, h) = \pi r^2 h$, of the cylinder subject to the constraint $g(r, h) = r^2 + \frac{h^2}{4} - 1 = 0$. According to the method of Lagrange multipliers, we need to find all solutions to

$$f_r = g_r \qquad 2\pi rh = 2\lambda r \tag{E1}$$

$$f_h = g_h \qquad \pi r^2 = \lambda \frac{h}{2} \tag{E2}$$

$$g(r, h) = 1 \qquad r^2 + \frac{h^2}{4} = 1 \tag{E3}$$

- If $g_r \neq 0$ and $g_h \neq 0$, then (E1) gives us $\lambda = \frac{2\pi rh}{2r} = \pi h$ and (E2) gives us $\lambda = \frac{\pi r^2}{h/2} = \frac{2\pi r^2}{h}$.

$$\begin{aligned}\pi h &= \frac{2\pi r^2}{h}\\ \frac{h^2}{2} &= r^2\end{aligned}$$

Now from (E3):

$$\begin{aligned}1 &= r^2 + \frac{h^2}{4} = \frac{h^2}{2} + \frac{h^2}{4}\\ &= \frac{3}{4}h^2\\ h^2 &= \frac{4}{3}\end{aligned}$$

Since h and r are nonnegative,

$$\begin{aligned}h &= \frac{2}{\sqrt{3}}\\ r &= \sqrt{\frac{h^2}{2}} = \frac{h}{\sqrt{2}} = \sqrt{\frac{2}{3}}\end{aligned}$$

So one point to check is $r = \sqrt{\frac{2}{3}}$, $h = \frac{2}{\sqrt{3}}$.

- If $g_r = 0$, then $r = 0$. Then (E1) is true for any λ and any r. From (E2), $h = 0$. But then (E3) fails.
- If $g_h = 0$, then $h = 0$. From (E2), $r = 0$. But then (E3) fails.

So the only solution to all three equations with $r > 0$ and $h > 0$ is $r = \sqrt{\frac{2}{3}}$, $h = \frac{2}{\sqrt{3}}$. Since we restricted our domain to non-negative values of r and h, the points with $r = 0$ or with $h = 0$ are "endpoints" of the region we're considering. At these points, our volume is 0, so they give us the global minimum value over our model domain.

So, $r = \sqrt{\frac{2}{3}}$, $h = \frac{2}{\sqrt{3}}$ give the cylinder with maximum volume.

S-14:

The function we want to minimize is surface area, so this is our objective function:

$$f(x, y) = 2(2x \cdot x) + 2(2x \cdot y) + 2(x \cdot y) = 4x^2 + 6xy$$

Our constraint is that the volume must be 72 cubic centimetres.

$$g(x, y) = x \cdot 2x \cdot y = 2x^2y = 72$$

This is not a closed curve. If we think of y as a function of x, then our constraint gives us $y = \frac{36}{x^2}$, $x > 0$, $y > 0$. So this curve has domain $0 < x$. Note that as x approaches 0, then y

approaches infinity, and vice-versa. (That is: to have a very very short box with fixed volume, the box must be very wide.) Then our objective function goes to infinity as well. So this system *has no global maximum*, but it does have a global minimum. That global minimum will also be a local minimum, so it will be a solution to the system of Lagrange equations.

$$f_x = \lambda g_x \qquad 8x + 6y = \lambda(4xy) \tag{E1}$$
$$f_y = \lambda g_y \qquad 6x = \lambda(2x^2) \tag{E2}$$
$$g(x, y) = 72 \qquad 2x^2y = 72 \tag{E3}$$

- If $g_x \neq 0$ and $g_y \neq 0$, then (E1) $\lambda = \frac{8x+6y}{4xy} = \frac{4x+3y}{2xy}$ and (E2) $\lambda = \frac{6x}{2x^2} = \frac{3}{x}$:

$$\begin{aligned} \frac{4x+3y}{2xy} &= \frac{3}{x} \\ \implies 4x^2 + 3xy &= 6xy \\ \implies 4x^2 - 3xy &= 0 \\ \implies x(4x - 3y) &= 0 \\ \implies x = 0 \text{ or } (4x - 3y) &= 0 \end{aligned}$$

  From (E3), we see $x \neq 0$, so the only point to consider is when $4x = 3y$. Plugging this into our constraint function,

$$\begin{aligned} 72 = 2x^2y = 2x^2\left(\frac{4}{3}x\right) &= 3x^3 \\ \implies 27 &= x^3 \\ \implies 3 &= x \\ \implies y &= \frac{4}{3} \cdot 3 = 4 \end{aligned}$$

  So the point to consider is $(3, 4)$.

- If $g_x = 0$, then $x = 0$ or $y = 0$, both of which make (E3) false.
- If $g_y = 0$, then $x = 0$, which makes (E3) false.

So the only point to consider is $(3, 4)$.

We aren't considering a region with a closed curve bounding it, so we'll need some thought to decide whether this is, in fact, a minimum. Note that our model domain is that x and y must both be positive numbers. We see that as x or y goes to 0, while the other one stays constant, our surface area function goes to infinity. Similarly as x or y goes to infinity, while the other one stays constant, our surface area function goes to infinity. So the function must have a minimum somewhere well away from its "boundaries" near and far from the x and y axes.

So, the dimensions of the box with smallest surface area are:
$x = 3, \quad 2x = 6, \quad y = 4$

S-15: Note that if (x,y) obeys $g(x,y) = xy - 1 = 0$, then x is necessarily nonzero. So we may assume that $x \neq 0$. Then

$$\begin{aligned}
&\text{There is a } \lambda \text{ such that } (x,y,\lambda) \text{ obeys (E1)} \\
&\quad \Longleftrightarrow \text{ there is a } \lambda \text{ such that } f_x(x,y) = \lambda g_x(x,y), \quad f_y(x,y) = \lambda g_y(x,y), \quad g(x,y) = 0 \\
&\quad \Longleftrightarrow \text{ there is a } \lambda \text{ such that } f_x(x,y) = \lambda y, \quad f_y(x,y) = \lambda x, \quad xy = 1 \\
&\quad \Longleftrightarrow \text{ there is a } \lambda \text{ such that } \frac{1}{y} f_x(x,y) = \frac{1}{x} f_y(x,y) = \lambda, \quad xy = 1 \\
&\quad \Longleftrightarrow \frac{1}{y} f_x(x,y) = \frac{1}{x} f_y(x,y), \quad xy = 1 \\
&\quad \Longleftrightarrow x f_x\Big(x, \frac{1}{x}\Big) = \frac{1}{x} f_y\Big(x, \frac{1}{x}\Big), \quad y = \frac{1}{x} \\
&\quad \Longleftrightarrow F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f\Big(x, \frac{1}{x}\Big) = f_x\Big(x, \frac{1}{x}\Big) - \frac{1}{x^2} f_y\Big(x, \frac{1}{x}\Big) = 0, \quad y = \frac{1}{x}
\end{aligned}$$

Solutions to Exercises 2.6 — Jump to TABLE OF CONTENTS

S-1: In Marshallian demand, the budget is a constant, while utility is maximized. This is the strategy of Consumer A. In Hicksian demand, cost is minimized, subject to a minimum acceptable utility. This the strategy of Consumer B.

S-2: Following Definition 2.6.7, the price effect of p_x resp. p_y is the partial derivative of x^m with respect to those variables.

$$x^m(p_x, p_y, I) = \begin{cases} \frac{I}{2(p_x - p_y)} & \text{if } p_x \geqslant 2p_y \\ \frac{I}{p_x} & \text{if } p_x < p_y \end{cases} = \begin{cases} \frac{I}{2}(p_x - p_y)^{-1} & \text{if } p_x \geqslant 2p_y \\ I(p_x)^{-1} & \text{if } p_x < p_y \end{cases}$$

$$\frac{\partial x^m(p_x, p_y, I)}{\partial p_x} = \begin{cases} \frac{-I}{2}(p_x - p_y)^{-2} & \text{if } p_x \geqslant 2p_y \\ -I(p_x)^{-2} & \text{if } p_x < p_y \end{cases} = \begin{cases} \frac{-I}{2(p_x - p_y)^2} & \text{if } p_x \geqslant 2p_y \\ -\frac{I}{p_x^2} & \text{if } p_x < p_y \end{cases}$$

$$\frac{\partial x^m(p_x, p_y, I)}{\partial p_y} = \begin{cases} \frac{-I}{2}(p_x - p_y)^{-2} \cdot (-1) & \text{if } p_x \geqslant 2p_y \\ 0 & \text{if } p_x < p_y \end{cases} = \begin{cases} \frac{I}{2(p_x - p_y)^2} & \text{if } p_x \geqslant 2p_y \\ 0 & \text{if } p_x < p_y \end{cases}$$

S-3: To answer the question, according to Definition 2.6.5 in the text, we need to decide whether $\frac{\partial x^m}{\partial I}$ is positive or negative.

$$\frac{\partial x^m}{\partial I} = \frac{\partial}{\partial I}\left[\frac{p_y^2 - Ip_x}{4p_y^2 - 2p_x^2}\right] = \frac{\partial}{\partial I}\left[\frac{p_y^2}{4p_y^2 - 2p_x^2} - \frac{p_x}{4p_y^2 - 2p_x^2} I\right]$$
$$= -\frac{p_x}{4p_y^2 - 2p_x^2}$$

The differentiation is easier than it looks, because if we treat I as the only variable, then x^m is just a line. Now we need to determine the sign of the derivative. If $p_x^2 \leqslant 2p_y^2$, then $4p_y^2 - 2p_x^2 \geqslant 0$, so the derivative is negative. Therefore, X is an inferior good.

S-4: The constraint here is Luiza's budget which can be described by

$$3f + c = 10$$

To solve the problem using the method of Lagrange multipliers, we identify the constraint function as $g(f, c) = 3f + c - 10$ and we note that $u(f, c)$ and $g(f, c)$ have continuous first partial derivatives in their domain assuming f is non-zero (the derivative $\sqrt{f}$ is not defined at $f = 0$). Assume for now that f is not zero. Moreover, we note that $\nabla g \neq 0$.

The Lagrange multiplier rule tells us that to optimize the utility function u given Luiza's budget, we need to find the points (f, c) such that

$$\nabla u(f, c) = \lambda \nabla g(f, c)$$

for some real number λ (Lagrange multiplier) and

$$g(f, c) = 0$$

This gives us three equations (with three unknowns):

$$u_f = \frac{1}{2\sqrt{f}} = 3\lambda = \lambda g_f \tag{E1}$$
$$u_c = \frac{1}{10} = \lambda = \lambda g_c \tag{E2}$$
$$3f + c = 10 \tag{E3}$$

Equation (E2) tells us that $\lambda = \frac{1}{10}$. Putting this into (E1) give us

$$\frac{1}{2\sqrt{f}} = \frac{3}{10} \implies \sqrt{f} = \frac{5}{3} \implies f = \frac{25}{9}$$

To find c we put the value we found for f in equation (E3):

$$3(\frac{25}{9}) + c = 10 \implies c = 10 - \frac{25}{3} = \frac{5}{3}$$

Now note that if $f = 0$, then the utility function becomes $u(c) = \frac{c}{10}$ for $c \geqslant 0$. This function has a global min at $c = 0$ and it does not achieve a maximum. Therefore we can safely assume that in order to maximize our utility function u, f is non-zero.

So we do not have any other candidate points that optimizes u given the constraint $g = 0$, so we do not need to compare them and check which maximizes u. The answer to part (a) is $\frac{5}{3}$ and to part (b) is $\frac{25}{9}$.

Finally, for part (c), we multiply the quantities we found by their respective cost:

1. For food (\$3) $\dfrac{25}{9} = \$\dfrac{25}{3} \approx \$8.33$;
2. for coffee (\$1) $\dfrac{5}{3} = \$\dfrac{5}{3} \approx \$1.67$.

S-5: As suggested by the hint, we use the method of Lagrange Multipliers (Theorem 2.5.2 in the text). The budget constraint is expressed by

$$80m + 20s = 1000$$

We can take the constraint function $g(m, s)$ to be

$$g(m, s) = 80m + 20s - 1000$$

We wish to find the points (m, s) that maximizes the utility function u while $g(m, s) = 0$. Note that $s \geqslant 0$ (we cannot buy negative amount of shares) and $m > 0$ (being the argument of log function). The functions u and g have continuous first partial derivatives in their domain and $\nabla g(c, s, w)$ is never zero. Then the Lagrange Multipliers Theorem tells us that we need to find the point(s) (m, s) such that

$$\nabla u(m, s) = \lambda \nabla g(m, s)$$

for some real number λ while $g(m, s) = 0$. This means that we have to solve the following system of equations

$$u_m = \frac{1}{m} = \lambda(80) = g_m \tag{E1}$$
$$u_s = \frac{1}{16} = \lambda(20) = g_s \tag{E2}$$
$$80m + 20s = 1000 \tag{E3}$$

The equation (E2) implies that

$$\lambda = \frac{1}{16 \cdot 20} = \frac{1}{320}$$

putting these (E1) tells us that

$$\frac{1}{m} = \left(\frac{1}{320}\right) 80 = \frac{1}{4} \implies m = 4$$

Finally, this information along with (E3) gives

$$80(4) + 20s = 1000 \implies 20s = 1000 - 320 \implies s = 34$$

We should check that this is not secretly a minimum. Since $\lim_{m \to 0} \ln m = -\infty$, certainly $m = 0$ is not ideal. On the other hand, let's consider $s = 0$. Then $m = 12.5$, and

$u(12.5, 0) = \ln(12.5) \approx 2.52$. On the other hand, $u(4, 34) = \ln 4 + 3.4 > 2.52$. The Lagrange point does better.

So Franco should buy four shares of Inter de Milan and thirty two shares of La Spezia with his money.

S-6: We solve this problem using the method of Lagrange multipliers (Theorem 2.5.2 in the text). Note that $c, s > 0$ (being the argument of a log function). The budget constraint can be expressed by

$$5c + 10s = 100$$

So we may define the constraint function by

$$g(c, s) = 5c + 10s - 100$$

$$\begin{cases} u_c & = \lambda g_c \\ u_s & = \lambda g_s \\ g(c, s) & = 0 \end{cases} \implies \begin{cases} \frac{3}{c} & = \lambda \cdot 5 \\ \frac{4}{s} & = \lambda \cdot 10 \\ g(c, s) & = 0 \end{cases}$$

From the first two equations,

$$\lambda = \frac{3}{5c} = \frac{4}{10s}$$
$$s = \frac{2}{3}c$$

Now to satisfy the constraint,

$$0 = 5c + 10s - 100 = 5c + 10\left(\frac{2}{3}c\right) - 100$$
$$c = \frac{60}{7}$$
$$s = \frac{2}{3}c = \frac{2}{3} \cdot \frac{60}{7} = \frac{40}{7}$$

Since the utility function goes to negative infinity as $c \to 0$ or $s \to 0$, we can be confident that the point we've found is a maximum, not a minimum.

S-7: We use the method of Lagrange multipliers. If either k or n is zero then our utility function u would become zero. This does not maximize (actually minimizes) the utility function. So, we may assume none of our variables are zero. The budget constraint is given by

$$4k + 12n = 84$$

which implies that we can take the constraint function to be

$$g(k, n) = 4k + 12n - 84$$

$$\begin{cases} u_k & = \lambda g_k \\ u_n & = \lambda g_n \\ 0 & = g(k,n) \end{cases} \implies \begin{cases} \dfrac{n^{0.2}}{2k^{0.5}} & = \lambda \cdot 4 \\ \dfrac{k^{0.5}}{5n^{0.8}} & = \lambda \cdot 12 \\ 0 & = 4k + 12n - 120 \end{cases}$$

From the first two equations, we see

$$\begin{aligned} \lambda = \frac{n^{0.2}}{8k^{0.5}} &= \frac{k^{0.5}}{60n^{0.8}} \\ 60n &= 8k \\ k &= \frac{15}{2}n \end{aligned}$$

To satisfy the constraint,

$$\begin{aligned} 0 = 4k + 12n - 120 = 4\left(\frac{15}{2}n\right) &+ 12n - 84 \\ n &= 2 \\ k &= \frac{15}{2}n = \frac{15}{2} \cdot 2 = 15 \end{aligned}$$

So the optimal solution is to buy 15 expansion packs for Keitu and 2 for Nefred.

S-8:

(a) Like the previous examples we solve this question using Theorem 2.5.2 in the CLP-3 text. The budget constraint is given by

$$4.5p + 2s = 20$$

and therefore we let the constraint function $g(p,s)$ be

$$g(p,s) = 4.5p + 2s - 20$$

Let us assume for now, s and p are non-zero (we will deal with the case that at least one of them is zero later). The Lagrange Multipliers Theorem tells us that we need to find the point(s) $(p,s) \in \mathbb{R}^2$ such that

$$\nabla u(p,s) = \lambda \nabla g(p,s)$$

for some real number λ. This gives us a system of three equations and three unknowns:

$$u_p = 0.4p^{-0.6}(s+p)^{0.6} + 0.6p^{0.4}(s+p)^{-0.4} = \lambda(4.5) = \lambda g_p \tag{E1}$$

$$u_s = 0.6(s+p)^{-0.4}p^{0.4} = \lambda(2) = \lambda g_s \tag{E2}$$

$$4.5p + 2s = 20 \tag{E3}$$

If we divide (E1) by (E2) we get

$$\begin{aligned}
\frac{4}{6}p^{-1}(s+p)+1 &= \frac{9}{4}\\
\frac{2}{3}p^{-1}(s+p) &= \frac{5}{4}\\
\frac{s}{p}+1 &= \frac{5}{4}\cdot\frac{3}{2} = \frac{15}{8}\\
\frac{s}{p} &= \frac{7}{8}\\
s &= \frac{7}{8}p
\end{aligned}$$

Note that, when we divided (E1) by (E2), we have assumed $\lambda \neq= 0$. If $\lambda = 0$, then (E2) tells us that $p = 0$ which we have assumed not to be the case. Moving on, we put $s = \frac{7}{8}p$ into (E3):

$$4.5p + \frac{7}{4}p = 20 \implies \frac{25}{4}p = 20 \implies p = \frac{80}{25}$$

So $p = 3.2$ units or $p = 320$ gm and

$$s = \frac{7}{8}p = \frac{7}{8}(\frac{80}{25}) = \frac{70}{25} = 2.8$$

units or $s = 280$ ml.

Note that

(1) If $s = 0$, then the utility function becomes $u = p$. This function achieves its global minimum at $p = 0$ and has no local or global maximum. Since we are interested in maximizing the utility function u, we can dismiss this as a solution.

(2) Similarly, if $p = 0$ then $u = 0$ which minimizes the utility function (for all values of s). So we can dismiss this case too.

(b) (i) To check this we only need to multiply the given quantities with their respective price per unit. Here, $p = 4.2$ and $s = 1.6$ which means, at the normal price, the total would be

$$4.5(4.2) + 2(1.6) = \$22.1$$

which is more than Coral's budget and therefore she would not have been able to afford this without the combo price.

(ii) To see if the combo is a better deal for Coral we compare the utility levels of her original optimal consumption with that of the combo. Let us first find her original optimal consumption utility level where $p = 3.2$ and $s = 2.8$:

$$(3.2)^{0.4}(2.8+3.2)^{0.6} \approx 4.666$$

This is while the utility level of the combo where $p = 4.2$ and $s = 1.6$ is given by

$$(4.2)^{0.4}(1.6+4.2)^{0.6} \approx 5.097$$

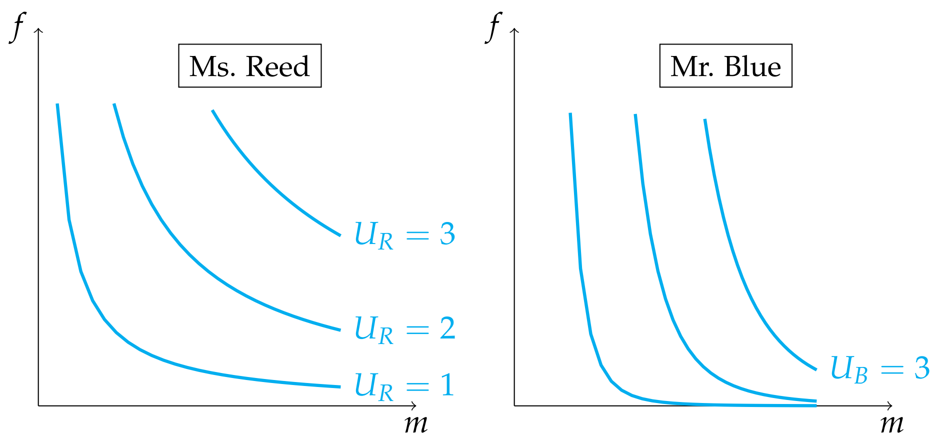
This implies that the combo is a better deal for Coral!

S-9:

(a) The budget constraint is given by

$$30m + 30f = I$$

(b) To get the level curves for Mr. Blue's utility function and Ms. Reed's utility function, we assign some values to U_B and U_R. In the graphs below, we have fixed U_B and U_R to be equal to 1, 2, and 3.



To help us draw these curves, we may use some algebra. For example, when $U_B = 1$ we have

$$U_B = 1 = m^{0.9} f^{0.1} \implies m^9 f^1 = 1^{10} = 1 \implies f = \frac{1}{m^9}$$

When $U_R = 2$ then

$$U_R = 2 = \sqrt{mf} \implies mf = 2^2 \implies f = \frac{4}{m}$$

The computations for other values of U_B and U_R are very similar.

Geometrically, for a fixed value of U_B (a level curve), we can see that the graph is more sensitive to increments in values of m compared to increments in values of f. This is while for any value of U_R (which measures preference here), the level curve is symmetric about the line $f = m$. This tells us that Mr. Reed prefers male officers to female officers while Ms. Reed does not prefer one to the other.

Even though in this question we are asked to argue geometrically, we can look at the algebraic expressions to confirm our answer. For values $f, m > 1$, $m^{0.9}$ contributes more than $f^{0.1}$ to the utility function U_B. (To see this better, you may give some numerical examples.)

(c) Before we apply Lagrange multipliers method, note that we can assume $f, m > 0$. If m or f is zero, then the utility functions U_B and U_R become zero. Since we wish to find the maximum utility, we can assume f and m are non-zero.

(i) Let us first find the optimal combination of m and f for Mr. Blue. We take the constraint function to be

$$g(m, f) = 30m + 30f - I$$

Then Lagrange multipliers method tells us that in order to find the points (m, f) that optimize U_B, we need to solve the following system of equations

$$(U_B)_m = (0.9)m^{-0.1}f^{0.1} = \lambda(30) = \lambda g_m \tag{E1}$$

$$(U_B)_f = (0.1)m^{0.9}f^{-0.9} = \lambda(30) = \lambda g_f \tag{E2}$$

$$30m + 30f = I \tag{E3}$$

For some real number λ. Since m and f are non-negative, the left hand side of equation (E1) is non-zero ($(U_B)_m$ is non-zero), then the right hand side is also non-zero. So in particular, λ is non-zero. So we cab divide equation (E1) by (E2) to get

$$\frac{\text{E1}}{\text{E2}} \implies 9m^{-1}f = 1 \implies m = 9f$$

Now we put this into (E3)

$$30(9f) + 30f = 300f = I \implies f = \frac{I}{300}$$

and

$$m = 9f \implies m = \frac{9I}{300}$$

This maximizes Mr. Blue's utility function U_B. We may denote this m and f that optimize U_B by m_B and f_B (to avoid confusion):

$$f_B = \frac{I}{300}$$

$$m_B = \frac{9I}{300}$$

Note that, we can see from our solution that for every new female police officer that Mr. Blue hires, Mr. Blue hires nine new male police officers.

(ii) Now let us do the same for Ms. Reed. We can guess that whatever the solution would be, we must have that $m = f$, as the U_R which determines Ms. Reed's preference is symmetric in m and f. Having this in mind, we apply Lagrange multipliers method. The the constraint function is the same as before

$$g(m, f) = 30m + 30f - I$$

We need to find the points (m, f) that satisfy the following system of equations

$$(U_R)_m = (0.5)m^{-0.5}f^{0.5} = \lambda(30) = \lambda g_m \tag{E1}$$

$$(U_R)_f = (0.5)m^{0.5}f^{-0.5} = \lambda(30) = \lambda g_f \tag{E2}$$

$$30m + 30f = I \tag{E3}$$

for some real number λ. Again, using equation (E1) (or (E2)) we can infer that λ is non-zero. This means that divide (E1) by (E2) to get

$$\frac{\text{E1}}{\text{E2}} \implies (1)m^{-1}f = 1 \implies m = f$$

as we predicted! Now we put this into (E3)

$$30(f) + 30f = 60f = I \implies f = \frac{I}{60} \text{ and } m = \frac{I}{60}$$

We denote these values by m_R and f_R

$$f_R = \frac{I}{60}$$
$$m_R = \frac{I}{60}$$

So Mr. Blue will hire a higher proportion of male police officers than Ms. Reed as

$$m_B = \frac{9I}{300} > \frac{I}{60} = m_R$$

for any value of I.

(d) (i) We start by Mr. Blue. Like the previous part, we may assume m and f are nonzero. The constraint function is now given by

$$g(m, f) = 35m + 30f - I$$

Using the method of Lagrange multipliers, we need to find the points (m, f) such that

$$(U_B)_m = (0.9)m^{-0.1}f^{0.1} = \lambda(35) = \lambda g_m \quad \text{(E1)}$$
$$(U_B)_f = (0.1)m^{0.9}f^{-0.9} = \lambda(30) = \lambda g_f \quad \text{(E2)}$$
$$35m + 30f = I \quad \text{(E3)}$$

For some real number λ. Since m and f are non-zero we can deduce that λ is non-zero. Thus, we can divide (E1) by (E2) to get

$$\frac{\text{E1}}{\text{E2}} \implies 9m^{-1}f = \frac{35}{30} = \frac{7}{6} \implies m = \frac{54}{7}f$$

We put this in (E3):

$$35(\frac{54}{7}f) + 30f = 300f = I \implies f = \frac{I}{300}$$

So

$$m = \frac{54}{7}f \implies m = \frac{9I}{350}$$

Let us denote these points by m_B^* and f_B^*:

$$f_B^* = \frac{I}{300}$$
$$m_B^* = \frac{9I}{350}$$

Note that we can still see that Mr. Blue has a strong bias towards hiring male officers. However, $m_B^* < m_B$ which means that, with the wage gap Mr. Blue hires less male officers. Since $f_B^* = f_B$, Mr. Blue would hire the same amount of female officers.

(ii) For Ms. Reed we used to have that $m_R = f_R$. Let us see if this changes. The new constraint function is given by

$$g(m, f) = 35m + 30f - I$$

The method of Lagrange multipliers tells us that we need to find the points (m, f) such that

$$(U_R)_m = (0.5)m^{-0.5}f^{0.5} = \lambda(35) = \lambda g_m \tag{E1}$$
$$(U_R)_f = (0.5)m^{0.5}f^{-0.5} = \lambda(30) = \lambda g_f \tag{E2}$$
$$35m + 30f = I \tag{E3}$$

For some real number λ. Like before, we from equation (E1) and the fact that m and f are non-zero we can infer that λ is non-zero. To find m in terms of f, we divide (E1) by (E2):

$$\frac{\text{E1}}{\text{E2}} \implies (1)m^{-1}f = \frac{35}{30} = \frac{7}{6} \implies m = \frac{6}{7}f$$

Note that right away we can see that now Ms. Reed prefers to hire female police officers. To find m and f in terms of I, we use (E3):

$$35(\frac{6}{7}f) + 30f = 60f = I \implies f = \frac{I}{60}$$

and so

$$m = \frac{6}{7}f \implies m = \frac{I}{70}$$

Let us denote these points by m_R^* and f_R^*:

$$f_R^* = \frac{I}{60}$$
$$m_R^* = \frac{I}{70}$$

So, because it is cheaper to hire female officers, both hire a higher proportion of female officers. However, Mr. Blue has such a strong bias that he still hires more male officers than female officers. To summarize our findings, we can compare them

$$f_B = f_B^* < m_R^* < f_B^* < f_R = m_R = f_R^* < m_B^* < m_B$$

S-10: Let p_f and p_c be the price of food and coffee respectively at a given time. Moreover, let I denote Luzia's budget at that time (so secretly these parameters depend on time).

Here, unlike question 4, the price of food and coffee as well as the budget may change over time. Therefore, we keep them as parameters and find the general solution for the optimal consumption of coffee and food based on these parameters. Our method is the same, we use the method of Lagrange multipliers. The budget constraint is give by

$$p_c c + p_f f = I$$

which tells us that we can take the constraint function to be

$$g(c, f) = p_c c + p_f f - I$$

Note that our variables are still c and f not p_c, p_f, or I (which we recognize as parameters). Of course, the value of $g(c, f)$ depends on the choice of these parameters but at a given time these parameters will be fixed and have numerical values. Similar to the argument we made in question 4, we can dismiss the case where c or f are zero (see the solution to question 4 to see why). So let us assume that c and f are non-zero here. Lagrange Multipliers Theorem tells us that we need to solve

$$\nabla u(f, c) = \lambda \nabla g(u, c)$$

for some real number λ. This gives the following system of equations

$$u_f = \frac{1}{2\sqrt{f}} = \lambda(p_f) = \lambda g_f \tag{E1}$$

$$u_c = \frac{1}{10} = \lambda(p_c) = \lambda g_c \tag{E2}$$

$$p_c c + p_f f = I \tag{E3}$$

Equation (E2) tells us that λ is nonzero. Furthermore, we may assume the prices of the goods will never be zero (although that would have been great). If we divide equation (E1) by equation (E2) we will get

$$\frac{10}{2\sqrt{f}} = \frac{p_f}{p_c} \implies f = \left(\frac{5p_c}{p_f}\right)^2$$

If we put this into (E3) we find

$$p_c c + p_f \left(\frac{5p_c}{p_f}\right)^2 = I \implies c = \frac{I}{p_c} - \frac{25p_c}{p_f}$$

We may denote the Marshallian demand functions by

$$c^*(p_c, p_f, I) = \frac{Ip_f - 25p_c^2}{p_f p_c}$$

$$f^*(p_c, p_f, I) = \left(\frac{10p_c}{2p_f}\right)^2$$

Note that if we fix $p_c = 1$, $p_f = 3$, and $I = 10$ as in question 4, we will get the same answers as we found earlier!

S-11: let p_k, p_j, and p_n denote the prices of Keitu's, Jorge's and Nefret's packages at a given month, respectively. Moreover, let I be the Alessio's budget at that month. Then the budget constraint is given by

$$p_k k + p_n n = I$$

We take the constraint function to be

$$g(k, j, n) = p_k k + p_n n - I$$

A similar argument as in the solution of question 7 tells us that we may assume k and n are non-zero and that we can apply the Lagrange Multipliers Theorem.

$$\begin{cases} u_k &= \lambda g_k \\ u_n &= \lambda g_n \\ 0 &= g(k, n) \end{cases} \implies \begin{cases} \frac{n^{0.2}}{2k^{0.5}} &= \lambda p_k \\ \frac{k^{0.5}}{5n^{0.8}} &= \lambda p_n \\ 0 &= p_k k + p_n n - I \end{cases}$$

From the first two equations,

$$\begin{aligned} \lambda = \frac{n^{0.2}}{2k^{0.5} p_k} &= \frac{k^{0.5}}{5n^{0.8} p_n} \\ 5np_n &= 2kp_k \\ k &= \frac{5p_n}{2p_k} n \end{aligned}$$

To satisfy the budget constraint,

$$\begin{aligned} 0 = p_k k + p_n n - I &= p_k \left(\frac{5p_n}{2p_k} n \right) + p_n n - I \\ I &= \frac{7}{2} p_n n \\ n &= \frac{2I}{7p_n} \\ k &= \frac{5p_n}{2p_k} n = \frac{5p_n}{2p_k} \left(\frac{2I}{7p_n} \right) = \frac{5I}{7p_k} \end{aligned}$$

So, the optimal consumption is buying $\frac{5I}{7p_k}$ expansion packs for Keitu, and $\frac{2I}{7p_n}$ for Nefret.

S-12:

(a) Recall that logarithm functions can only take positive values as an input. So whenever we have $\ln(y)$, we know that $y > 0$. This tells us that

$$\begin{aligned} \ln(k-1) &\implies k - 1 > 0 \implies k > 1 \\ \ln(50 - c) &\implies 50 - c > 0 \implies c < 50 \end{aligned}$$

Moreover, note that $c > 0$ as we cannot buy a non-positive units of chicken (well, in a way, that is selling chicken which we cannot do here). This means that $0 < c < 50$ and $k > 1$. Since c and k are integers, we have

$$1 \leqslant c \leqslant 49, \qquad k \geqslant 2$$

So the maximum amount of chicken we can buy is 49 and we need to buy at least 1 chicken based on the utility function. We have to buy at least 2 kraft dinners but there is no upper limit (based on the utility function alone, of course our budget is not infinite).

(b) We can apply the Lagrange Multipliers Theorem, keeping in mind the domain restrictions that we found above. The budget is expressed mathematically by

$$p_k k + p_c c = I$$

So we can take the constraint function to be $g(k,c) = p_k k + p_c c - I$. Note that $g(k,c)$ and $u(k,c)$ have continuous partial derivative. We need to find the points (k,c) such that $g(k,c) = 0$ (this describes a line) and

$$\nabla u(k,c) = \lambda \nabla g(k,c)$$

where λ is a real number. This means

$$u_k(k,c) = \frac{1}{k-1} = \lambda p_k \tag{E1}$$

$$u_c(k,c) = \frac{2}{50-c} = \lambda p_c \tag{E2}$$

$$p_k k + p_c c = I \tag{E3}$$

Note that $u_k = \frac{1}{k-1}$ and $u_c = \frac{2}{50-c}$ can never be zero for any value of k and c. This means that λ, p_k, and p_c cannot be zero, by equations (E1) and (E2). So we can divide (E1) by (E2) to get

$$\frac{\text{E1}}{\text{E2}} \implies \frac{50-c}{2(k-1)} = \frac{p_k}{p_c}$$

and so

$$50 - c = \frac{p_k(2k-2)}{p_c} \implies c = 50 - \frac{p_k}{p_c}(2k-2)$$

We put this into (E3) to get

$$\begin{aligned}
p_k k + p_c\left(50 - \frac{p_k(2k-2)}{p_c}\right) = I \implies & \; p_k k - p_k(2k-2) = I - 50p_c \\
\implies & \; -p_k k - 2p_k = I - 50p_c \\
\implies & \; -p_k k = I - 50p_c + 2p_k \\
\implies & \; k = \frac{I - 50p_c + 2p_k}{-p_k} \\
\implies & \; k = \frac{50p_c - I}{p_k} - 2
\end{aligned}$$

Now we put this back in the expression for c to find the Marshallian demand function only in terms of $p_k,\ p_c$ and I (and not k):

$$c = 50 - \frac{p_k}{p_c}(2k-2) \implies c = 50 - \frac{p_k}{p_c}(2\left(\frac{50p_c - I}{p_k} - 2\right) - 2)$$
$$\implies c = \frac{6p_k}{p_c} + \frac{2I}{p_c} - 50$$

So the Marshallian demand functions are as follows:

$$k^*(p_k, p_c, I) = \frac{50p_c - I}{p_k} - 2, \qquad k^*(p_k, p_c, I) \geqslant 2$$
$$c^*(p_k, p_c, I) = \frac{6p_k}{p_c} + \frac{2I}{p_c} - 50, \qquad 1 \leqslant c^*(p_k, p_c, I) \leqslant 49$$

(c) To see whether kraft dinner is a normal or inferior good we need to think what happens to $k^*(p_k, p_c, I)$ as we increase I while we keep p_k and p_c fixed. A nice way to do this is to see what happens we differentiate $k^*(p_k, p_c, I)$ in terms of I[7].

$$\frac{\partial}{\partial I} k^*(p_k, p_c, I) = -\frac{1}{p_k}$$

Note that, $p_k > 0$ which means that $-\frac{1}{p_k}$ is always negative. This means as we increase the income I, the optimal kraft dinner consumption $k^*(p_k, p_c, I)$ decreases. So, kraft dinner is an inferior good!

We do the same for chicken:

$$\frac{\partial}{\partial I} c^*(p_k, p_c, I) = \frac{2}{p_c}$$

Since $p_c > 0$, we must have that $2\frac{1}{p_c}$ is always positive. Therefore, as the income I increases, the optimal chicken consumption $c^*(p_k, p_c, I)$ increases. So, chicken is a normal good!

S-13:

(a) Like the previous examples, we use the method of Lagrange multipliers (Theorem 2.5.2 in the text) to find the Marshallian demand functions. The constraint is given by

$$lp_l + ap_a = D$$

and the constraint function is given by

$$g(l, a) = lp_l + ap_a - D$$

So $g(l, a) = 0$ gives us the constraint. Note that u_l and u_p are not defined when l, a or $4l^{0.5} + 3a^{0.5}$ is zero. This means we have to consider the cases where at least one of l or a is zero and the case where they are non-zero. For now, let us assume that they

7 This is basically directional derivative.

are all non-zero. We will consider the other cases later. In this case, ∇u and ∇g are defined where $g(l,a) = 0$. The Lagrange Multipliers Theorem tells us that we have to find the points (l,a) where $g(l,a) = 0$ and

$$\nabla u(l,a) = \lambda \nabla g(l,a)$$

for some real number λ. This gives us a system of three equations with three unknowns l, a and λ[8].

$$u_l = \left(\frac{1}{2}(4l^{0.5} + 3a^{0.5})^{-0.5}\right)\left(4(\frac{1}{2}l^{-0.5})\right) = \lambda(p_l) = \lambda g_l \tag{E1}$$

$$u_a = \left(\frac{1}{2}(4l^{0.5} + 3a^{0.5})^{-0.5}\right)\left(3(\frac{1}{2}a^{-0.5})\right) = \lambda(p_a) = \lambda g_a \tag{E2}$$

$$lp_l + ap_a = D \tag{E3}$$

Note that for all values of a and l, u_l and u_a are non-zero. This implies that from λ, (p_l), and p_a are non-zero. So, we can divide (E1) by (E2).

$$\frac{\text{E1}}{\text{E2}} \implies \frac{4}{3}\frac{a^{0.5}}{l^{0.5}} = \frac{p_l}{p_a}$$

So

$$\begin{aligned} \frac{a}{l} &= \left(\frac{4}{3} \cdot \frac{p_l}{p_a}\right)^2 \\ \frac{a}{l} &= \frac{9}{16}(\frac{p_l}{p_a})^2 \\ a &= \frac{9}{16}(\frac{p_l}{p_a})^2 l \end{aligned}$$

Now we have that found a in terms of p_l, p_a and l. We put this in (E3):

$$\begin{aligned} lp_l + \left(\frac{9}{16}(\frac{p_l}{p_a})^2 l\right) p_a = D &\implies lp_l + \frac{9}{16}p_l l(\frac{p_l}{p_a}) = D \\ &\implies lp_l\left(1 + \frac{9}{16}(\frac{p_l}{p_a})\right) = D \\ &\implies l = \frac{D}{p_l\left(1 + \frac{9}{16}(\frac{p_l}{p_a})\right)} \\ &\implies l = \frac{16p_a D}{p_l\,(16p_a + 9p_l))} \end{aligned}$$

So now we have found l in term of p_l, p_a, and D which is what we want. Let's do the

8 plus three parameters p_l, p_a, and D

same for a:

$$a = \frac{9}{16}(\frac{p_l}{p_a})^2 l \implies a = \frac{9}{16}(\frac{p_l}{p_a})^2 \left(\frac{D}{p_l \left(1 + \frac{9}{16}(\frac{p_l}{p_a})\right)} \right)$$

$$\implies a = \frac{9 p_l D}{p_a \left(16 p_a + 9 p_l\right)}$$

Note that for these values of a and l, $u(l, a) > 0$. (If $a > 0$ and $l > 0$ then $u(l, a) > 0$).

Now let us explore the cases where at least one of l or a is zero.

(i) If $a = 0$, then the utility function u becomes

$$u(l) = (4l^{0.5})^{0.5} = 2l^{0.25} = 2\sqrt[4]{l}$$

We may optimize now by first finding the critical points. Here we assume $l \neq 0$. Otherwise, if $l = a = 0$ then the utility function u is also zero. Since we are trying to maximize the utility function we can dismiss this case. We have

$$\frac{\mathrm{d}}{\mathrm{d}l} u(l) = 2(\frac{1}{4}) l^{-0.75}, \qquad l > 0$$

So $\frac{\mathrm{d}}{\mathrm{d}l} u(l)$ is zero only when l is zero. This function does not attain a maximum value.

(ii) The case where $l = 0$ is very similar to case (ii).

These cases do not give us a new candidate to find the Marshallian demand functions. So the Marshalian demands are given by

$$l^*(p_l, p_a, D) = \frac{16 p_a D}{p_l \left(16 p_a + 9 p_l)\right)}$$
$$a^*(p_l, p_a, D) = \frac{9 p_l D}{p_a \left(16 p_a + 9 p_l\right)}$$

(b) To categorize Lomanchenko's Marshalian demand function $l^*(p_l, p_a, D)$ and Anthony Joshua's Marshalian demand function $a^*(p_l, p_a, D)$ we need to see what happens to them as we increase D, while we keep p_l and p_a fixed. A nice way to do this is to look at partial derivatives of l^* and a^* in terms of D[9].

$$\frac{\partial}{\partial D} l^*(p_l, p_a, D) = \frac{16 p_a}{p_l \left(16 p_a + 9 p_l)\right)}$$
$$\frac{\partial}{\partial D} a^*(p_l, p_a, D) = \frac{9 p_l}{p_a \left(16 p_a + 9 p_l\right)}$$

Note that since p_l and p_a are always positive, so $\frac{\partial}{\partial D} l^*$ and $\frac{\partial}{\partial D} a^*$ are always positive. Thus, l^* and a^* increase as we increase D. This tells us that l^* and a^* are normal goods.

9 Here, we are basically using the concept of directional derivative. Taking the partial derivative of l^* in terms of D (differentiating in the direction of D) amounts to looking at how l^* changes as we increase D.

(c) Here, we differentiate l^* in terms of p_l

$$\begin{aligned}
\frac{\partial}{\partial p_l} l^*(p_l, p_a, D) &= \frac{\partial}{\partial p_l}\left(\frac{16p_a D}{p_l\,(16p_a + 9p_l))}\right) \\
&= (16p_a D)\frac{\partial}{\partial p_l}\left(\frac{1}{16p_a p_l + 9p_l^2}\right) \\
&= (16p_a D)\frac{\partial}{\partial p_l}(16p_a p_l + 9p_l^2)^{-1} \\
&= (16p_a D)(-1)(16p_a p_l + 9p_l^2)^{-2}\frac{\partial}{\partial p_l}(16p_a p_l + 9p_l^2) \\
&= -(16p_a D)(16p_a p_l + 9p_l^2)^{-2}(16p_a + 18p_l) \\
&= \frac{-(16p_a D)(16p_a + 18p_l)}{(16p_a p_l + 9p_l^2)^2}
\end{aligned}$$

Since p_l, p_a, and D are all positive, $\frac{\partial}{\partial p_l} l^*$ is negative. Which means that as we increase p_l, l^* decreases. Equivalently, l^* increases as p_l decreases. This makes sense with our intuition. If the price for Lomachenko's tickets p_l decreases, then it must be the case that the demand for Lomachenko's tickets l^* increases.

(d) We use the method of Lagrange multipliers to find Liam's Hicksian demand functions. The constraint is given by

$$(4l^{0.5} + 3a^{0.5})^{0.5} = U$$

where u is a fixed parameter. Therefore

$$g(l,a) = (4l^{0.5} + 3a^{0.5})^{0.5} - U$$

Whereas, the budget d (previously D[10]) is now a variable that may change (like u in part (a) when we wanted to find the Marshallian demand functions).

$$d(l,a) = lp_l + ap_a$$

Once again, we assume a and l are non-zero. If say $a = 0$, then the budget function becomes $d(l) = lp_l$ which has global minimum at $l = 0$ and has no local or global maximum. Since we want to find the values of a and l that maximize d, we assumes a is not zero. Similar argument can be made to show that we can safely assume l is non-zero.

Lagrange Multipliers Theorem tells us that in order to find the points (l,a) that optimize d, given the constraint $g(l,a) = 0$, we need to find the points (l,a) such that $g(l,a) = 0$ and

$$\nabla d(l,a) = \lambda \nabla g(l,a)$$

10 The reason for the changing lower cases u into U is just for "bookkeeping", so that we know U is a fixed parameter. Similarly, turning D into d is to remember that now d is a variable.

for some real number λ. So, we need to solve the following system of questions.

$$d_l = p_l = \lambda \left[\left(\frac{1}{2}(4l^{0.5} + 3a^{0.5})^{-0.5} \right) \left(4(\frac{1}{2}l^{-0.5}) \right) \right] = g_l \tag{E1}$$

$$d_a = p_a = \lambda \left[\left(\frac{1}{2}(4l^{0.5} + 3a^{0.5})^{-0.5} \right) \left(3(\frac{1}{2}a^{-0.5}) \right) \right] = g_a \tag{E2}$$

$$U = (4l^{0.5} + 3a^{0.5})^{0.5} \tag{E3}$$

Here, we want to find a and l in terms of p_a, p_l and U. Note that we have seen equation (E1) and (E2) before in part (a), and we know that (E1) divided by (E2) gives

$$\frac{\text{E1}}{\text{E2}} \implies \frac{4}{3}\frac{a^{0.5}}{l^{0.5}} = \frac{p_l}{p_a}$$

and this means

$$a = (\frac{3p_l}{4p_a})^2 l$$

We put this in (E3) to get

$$\begin{aligned}
\left(4l^{0.5} + 3\left((\frac{3p_l}{4p_a})^2 l\right)^{0.5}\right)^{0.5} = U \implies & \left(4l^{0.5} + 3(\frac{3p_l}{4p_a})l^{0.5}\right)^{0.5} = U \\
\implies & \left(l^{0.5}(4 + \frac{9p_l}{4p_a})\right)^{0.5} = U \\
\implies & l^{0.5}(4 + \frac{9p_l}{4p_a}) = U^2 \\
\implies & l^{0.5} = \frac{U^2}{(4 + \frac{9p_l}{4p_a})} \\
\implies & l^{0.5} = \frac{4U^2 p_a}{16p_a + 9p_l} \\
\implies & l = \left(\frac{4U^2 p_a}{16p_a + 9p_l}\right)^2
\end{aligned}$$

Now it remains to find a in terms of p_a, p_l and U.

$$\begin{aligned}
a = (\frac{3p_l}{4p_a})^2 l \implies & a = (\frac{3p_l}{4p_a})^2 \left(\frac{4U^2 p_a}{16p_a + 9p_l}\right)^2 \\
\implies & a = \left(\frac{3U^2 p_l}{16p_a + 9p_l}\right)^2
\end{aligned}$$

We need to check what happens when l or a is zero.

(i) If $l = 0$ then

$$d(l, a) = (0)p_l + ap_a \implies d(a) = ap_a$$

To find the critical points we differentiate d in terms of a

$$\frac{\mathrm{d}}{\mathrm{d}a}d(a) = p_a$$

Note that p_a is always positive. This means that $d(a) = ap_a$ does not achieve a local maximum.

(ii) The case where $a = 0$ is very similar and does not lead to a local maximum.

Therefore, Hicksian demand functions are given by

$$a^h(p_l, p_a, U) = \left(\frac{3U^2 p_l}{16p_a + 9p_l}\right)^2$$
$$l^h(p_l, p_a, U) = \left(\frac{4U^2 p_a}{16p_a + 9p_l}\right)^2$$

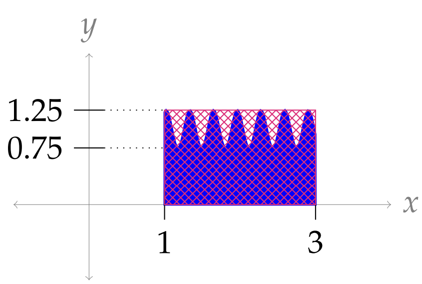
(e) We wish to understand how a^h changes as p_a changes. To do this, as usual, we compute $\frac{\partial a^h}{\partial p_a}$:

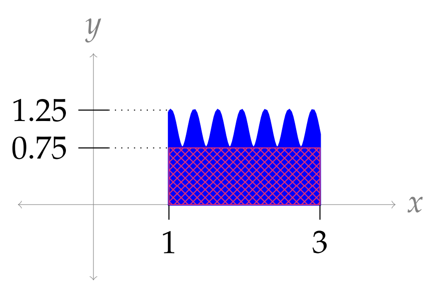
$$\frac{\partial a^h}{\partial p_a} = \frac{\partial}{\partial p_a}\left(\frac{3U^2 p_l}{16p_a + 9p_l}\right)^2$$
$$\frac{\partial a^h}{\partial p_a} = (3U^2 p_l)\frac{\partial}{\partial p_a}\left(\frac{1}{16p_a + 9p_l}\right)^2$$
$$\frac{\partial a^h}{\partial p_a} = (3U^2 p_l)(-2)(16p_a + 9p_l)^{-3}\frac{\partial}{\partial p_a}(16p_a + 9p_l)$$
$$\frac{\partial a^h}{\partial p_a} = (3U^2 p_l)(-2)(16p_a + 9p_l)^{-3}(16)$$
$$\frac{\partial a^h}{\partial p_a} = \frac{-96U^2 p_l}{(16p_a + 9p_l)^3}$$

So $\frac{\partial a^h}{\partial p_a}$ is always negative. This means that as the price for Anthony Joshua's tickets p_a increases, Anthony's Hicksian demand function a^h decreases.

Solutions to Exercises 3.1 — Jump to TABLE OF CONTENTS

S-1:





The diagram on the left shows a rectangle with area $2 \times 1.25 = 2.5$ square units. Since the blue-shaded region is entirely inside this rectangle, the area of the blue-shaded region is no more than 2.5 square units.

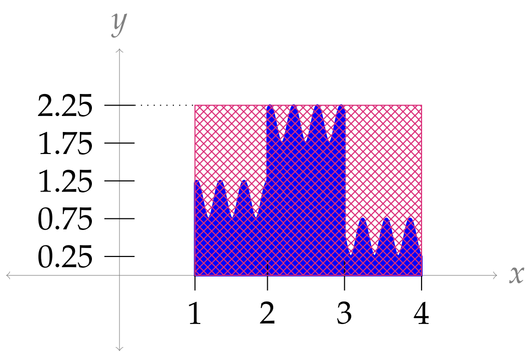
The diagram on the right shows a rectangle with area $2 \times 0.75 = 1.5$ square units. Since the blue-shaded region contains this entire rectangle, the area of the blue region is no less than 1.5 square units.

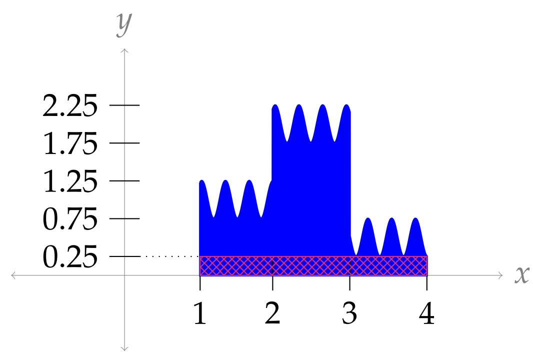
So, the area of the blue-shaded region is between 1.5 and 2.5 square units.

Remark: we could also give an obvious range, like "the shaded area is between zero and one million square units." This would be true, but not very useful or interesting.

S-2:

Solution 1: One naive way to solve this is to simply use the same method as Question 1.

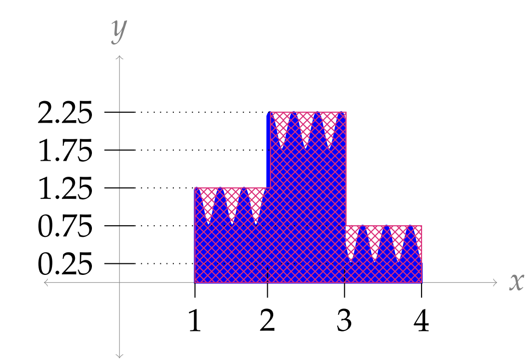


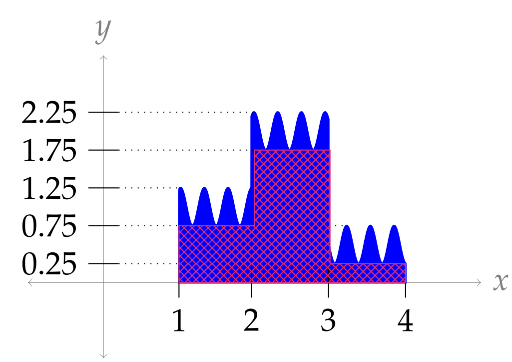


The rectangle on the left has area $3 \times 2.25 = 6.75$ square units, and encompasses the entire shaded region. The rectangle on the right has area $3 \times 0.25 = 0.75$ square units, and is entirely contained inside the blue-shaded region. So, the area of the blue-shaded region is between 0.75 and 6.75 square units.

This is a legitimate approximation, but we can easily do much better. The shape of this graph suggests that using the areas of *three* rectangles would be a natural way to improve our estimate.

Solution 2: Let's use these rectangles instead:

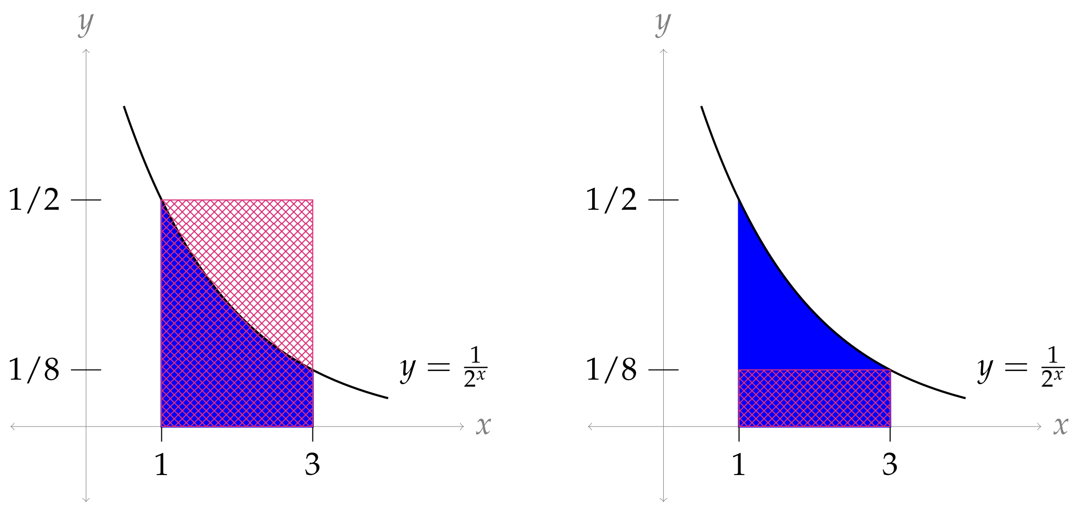




In the left picture, the red area is $(1 \times 1.25) + (1 \times 2.25) + (1 \times 0.75) = 4.25$ square units. In the right picture, the red area is $(1 \times 0.75) + (1 \times 1.75) + (1 \times 0.25) = 2.75$ square units. So, the blue shaded area is between 2.75 and 4.25 square units.

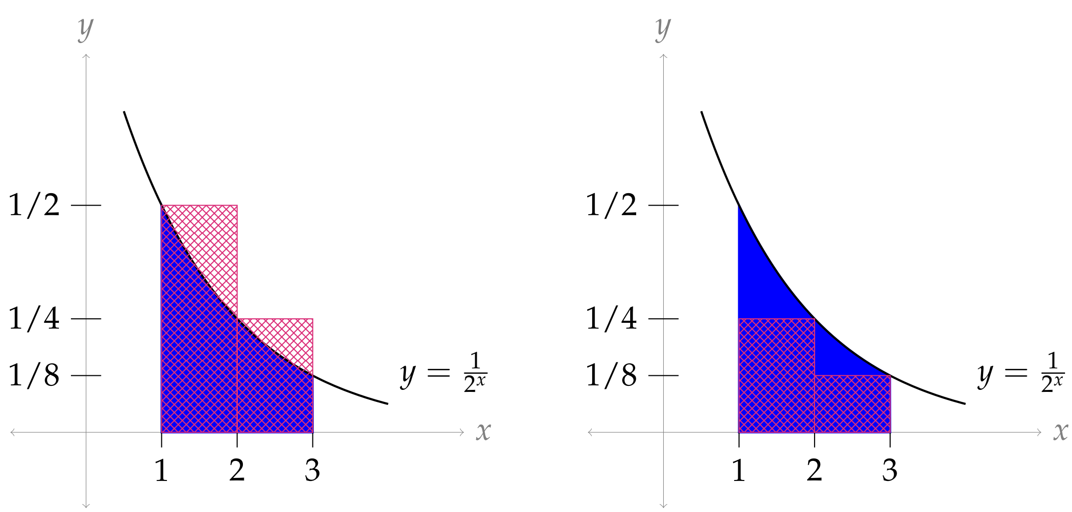
S-3: Remark: in the solution below, we find the appropriate approximation using trial and error. In Question 46, we take a more systematic approach.

Try 1: First, we can try by using a single rectangle as an overestimate, and a single rectangle as an underestimate.



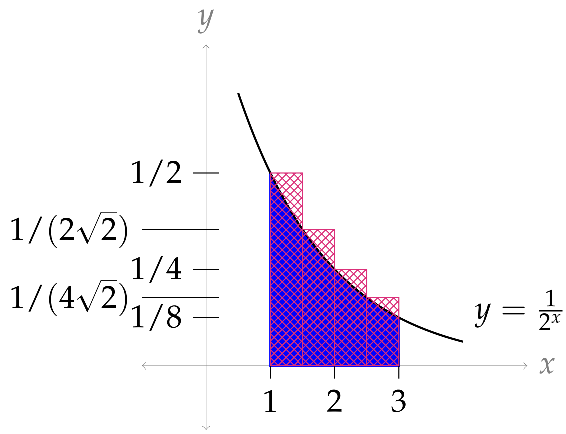
The area under the curve is less than the area of the rectangle on the left ($2 \times \frac{1}{2} = 1$) and greater than the area of the rectangle on the right ($2 \times \frac{1}{8} = \frac{1}{4}$). So, the area is in the range $\left(\frac{1}{4}, 1\right)$. Unfortunately, this range is too big–we need our range to have length at most 0.2. So, we refine our approximation by using more rectangles.

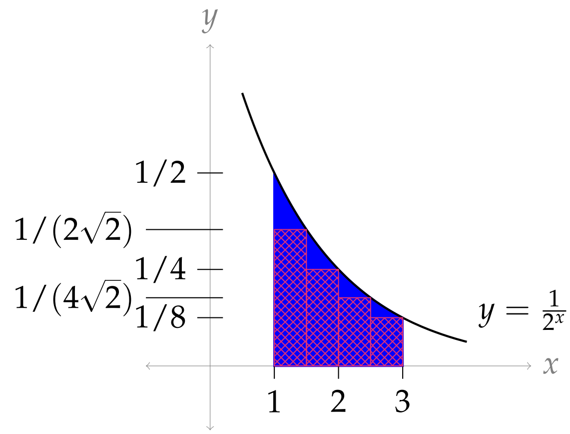
Try 2: Let's try using two rectangles each for the upper and lower bounds.



The rectangles in the left picture have area $\left(1 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) = \frac{3}{4}$, and the rectangles in the right picture have area $\left(1 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{8}\right) = \frac{3}{8}$. So, the area under the curve is in the interval $\left(\frac{3}{8}, \frac{3}{4}\right)$. The length of this interval is $\frac{3}{8}$, and $\frac{3}{8} > \frac{3}{15} = \frac{1}{5} = 0.2$. (Indeed, $\frac{3}{8} = 0.375 > 0.2$.) Since the length of our interval is still bigger than 0.2, we need even more rectangles.

Try 3: Let's go ahead and try four rectangles each for the upper and lower estimates.





The area of the rectangles on the left is:

$$\left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4\sqrt{2}}\right) = \frac{3}{8}\left[1 + \frac{1}{\sqrt{2}}\right],$$

and the area of the rectangles on the right is:

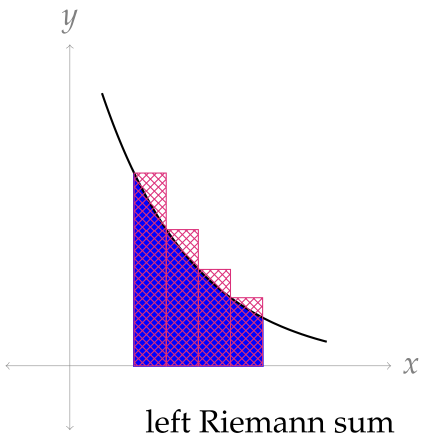
$$\left(\frac{1}{2} \times \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{8}\right) = \frac{3}{8}\left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right].$$

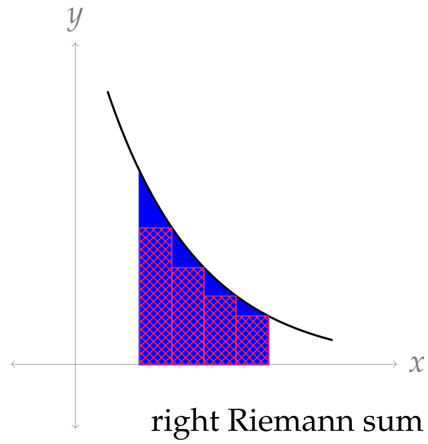
So, the area under the curve is in the interval $\left(\frac{3}{8}\left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right], \frac{3}{8}\left[1 + \frac{1}{\sqrt{2}}\right]\right)$. The length of this interval is $\frac{3}{16}$, and $\frac{3}{16} < \frac{3}{15} = \frac{1}{5} = 0.2$, as desired. (Indeed, $\frac{3}{16} = 0.1875 < 0.2$.)

Note, if we choose any value in the interval $\left(\frac{3}{8}\left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right], \frac{3}{8}\left[1 + \frac{1}{\sqrt{2}}\right]\right)$ as an approximation for the area under the curve, our error is no more than 0.2.

S-4: Since $f(x)$ is decreasing, it is larger on the left endpoint of an interval than on the right endpoint of an interval. So, a *left* Riemann sum gives a larger approximation. Notice this does not depend on n.

Furthermore, the actual area $\displaystyle\int_0^5 f(x)\mathrm{d}x$ is larger than its right Riemann sum, and smaller than its left Riemann sum.

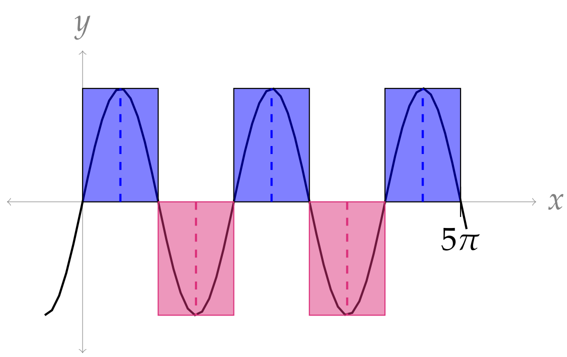




S-5: If $f(x)$ is always increasing or always decreasing, then the midpoint Riemann sum will be between the left and right Riemann sums. So, we need a function that goes up and down. Many examples are possible, but let's work with a familiar one: $\sin x$.

If our intervals have endpoints that are integer multiples of π, then the left and right Riemann sums will be 0, since $\sin(0) = \sin(\pi) = \sin(2\pi) = \cdots = 0$. The midpoints of these intervals will give y-values of 1 and -1. So, for example, we can let $f(x) = \sin x$, $[a, b] = [0, \pi]$, and $n = 1$. Then the right and left Riemann sums are 0, while the midpoint Riemann sum is π.

We can extend the example of $f(x) = \sin x$ to have more intervals. As long as we have more positive terms than negative, the midpoint approximation will be a positive number, and so it will be larger than both the left and right Riemann sums. So, for example, we can let $f(x) = \sin x$, $[a, b] = [0, 5\pi]$, and $n = 5$. Then the midpoint Riemann sum is $\pi - \pi + \pi - \pi + \pi = \pi$, which is strictly larger than 0 and so it is larger than both the left and right Riemann sums.



S-6:

(a) Two possible answers are $\sum_{i=3}^{7} i$ and $\sum_{i=1}^{5}(i+2)$. The first has simpler terms (i versus $i+2$), while the second has simpler indices (we often like to start at $i = 1$). Neither is objectively better than the other, but depending on your purposes you might find one more useful.

(b) The terms of this sum are each double the terms of the sum from part (a), so two possible answers are $\sum_{i=3}^{7} 2i$ and $\sum_{i=1}^{5}(2i+4)$.
We often want to write a sum that involves even numbers: it will be useful for you to remember that the term $2i$ (with index i) generates evens.

(c) The terms of this sum are each one more than the terms of the sum from part (b), so two possible answers are $\sum_{i=3}^{7}(2i+1)$ and $\sum_{i=1}^{5}(2i+5)$.
In the last part, we used the expression $2i$ to generate even numbers; $2i+1$ will generate odds. So will the index $2i+5$, and indeed, $2i+k$ for any odd number k. The choice of what you add will depend on the bounds of i.

(d) This sum adds up the odd numbers from 1 to 15. From Part (c), we know that the formula $2i+1$ is a simple way of generating odd numbers. Since our first term should be 1 and our last term should be 15, if we use $\sum(2i+1)$, then i should run from 0 to 7. So, one way of expressing our sum in sigma notation is $\sum_{i=0}^{7}(2i+1)$.

Sometimes we like our sum to start at $i=1$ instead of $i=0$. If this is our desire, we can use $2i-1$ as our terms, and let i run from 1 to 8. This gives us another way of expressing our sum: $\sum_{i=1}^{8}(2i-1)$.

S-7:

(a) The denominators are successive powers of three, so one way of writing this is $\sum_{i=1}^{4}\frac{1}{3^i}$.
Equivalently, the terms we're adding are powers of $1/3$, so we can also write $\sum_{i=1}^{4}\left(\frac{1}{3}\right)^i$.

(b) This sum is obtained from the sum in (a) by multiplying each term by two, so we can write $\sum_{i=1}^{4}\frac{2}{3^i}$ or $\sum_{i=1}^{4}2\left(\frac{1}{3}\right)^i$.

(c) The difference between this sum and the previous sum is its alternating sign, minus-plus-minus-plus. This behaviour appears when we raise a negative number to successive powers. We can multiply each term by $(-1)^i$, or we can slip a negative into the number that is already raised to the power i: $\sum_{i=1}^{4}(-1)^i\frac{2}{3^i}$, or $\sum_{i=1}^{4}\frac{2}{(-3)^i}$.

(d) This sum is the negative of the sum in part (c), so we can simply multiply each term by negative one: $\sum_{i=1}^{4}(-1)^{i+1}\frac{2}{3^i}$, or $\sum_{i=1}^{4}-\frac{2}{(-3)^i}$.

Be careful with the second form: a common mistake is to think that $-\dfrac{2}{(-3)^i} = \dfrac{2}{3^i}$, but these are not the same.

S-8:

(a) If we re-write the second term as $\frac{3}{9}$ instead of $\frac{1}{3}$, our sum becomes:

$$\frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + \frac{9}{243}$$

The numerators are the first five odd numbers, and the denominators are the first five positive powers of 3. We learned how to generate odd numbers in Question 6, and we learned how to generate powers of three in Question 7. Combining these, we can write our sum as $\displaystyle\sum_{i=1}^{5} \frac{2i-1}{3^i}$.

(b) The denominators of these terms differ from the denominators of part (a) by precisely two, while the numerators are simply 1. So, we can modify our previous answer: $\displaystyle\sum_{i=1}^{5} \frac{1}{3^i+2}$.

(c) Let's re-write the sum to make the pattern clearer.

$$\begin{array}{cccccccccccccc}
 & 1000 & + & 200 & + & 30 & + & 4 & + & \frac{1}{2} & + & \frac{3}{50} & + & \frac{7}{1000} \\
= & 1\cdot 1000 & + & 2\cdot 100 & + & 3\cdot 10 & + & \frac{4}{1} & + & \frac{5}{10} & + & \frac{6}{100} & + & \frac{7}{1000} \\
= & 1\cdot 10^3 & + & 2\cdot 10^2 & + & 3\cdot 10^1 & + & 4\cdot 10^0 & + & 5\cdot 10^{-1} & + & 6\cdot 10^{-2} & + & 7\cdot 10^{-3} \\
= & 1\cdot 10^{4-1} & + & 2\cdot 10^{4-2} & + & 3\cdot 10^{4-3} & + & 4\cdot 10^{4-4} & + & 5\cdot 10^{4-5} & + & 6\cdot 10^{4-6} & + & 7\cdot 10^{4-7}
\end{array}$$

If we let the red numbers be our index i, this gives us the expression $\displaystyle\sum_{i=1}^{7} i\cdot 10^{4-i}$.

Equivalently, we can write $\displaystyle\sum_{i=1}^{7} \frac{i}{10^{i-4}}$.

S-9:

(a) Using Theorem 3.1.6.a in the text, with $a = 1$, $r = \frac{3}{5}$ and $n = 100$:

$$\sum_{i=0}^{100} \left(\frac{3}{5}\right)^i = \frac{1-\left(\frac{3}{5}\right)^{101}}{1-\frac{3}{5}} = \frac{5}{2}\left[1-\left(\frac{3}{5}\right)^{101}\right]$$

(b) We want to use Theorem 3.1.6, part (a) again, but our sum doesn't start at $\left(\frac{3}{5}\right)^0 = 1$. We have two options: factor out the leading term, or use the difference of two sums that start where we want them to.

Solution 1: In this solution, we'll make our sum start at 1 by factoring out the leading term. We wrote our work out the long way (expanding the sigma into "dot-dot-dot" notation) for clarity, but it's faster to do the algebra in sigma notation all the way through.

$$\begin{aligned}\sum_{i=50}^{100}\left(\frac{3}{5}\right)^i &= \left(\frac{3}{5}\right)^{50}+\left(\frac{3}{5}\right)^{51}+\left(\frac{3}{5}\right)^{52}+\cdots+\left(\frac{3}{5}\right)^{100}\\ &= \left(\frac{3}{5}\right)^{50}\left[1+\left(\frac{3}{5}\right)+\left(\frac{3}{5}\right)^{2}+\cdots+\left(\frac{3}{5}\right)^{50}\right]\\ &= \left(\frac{3}{5}\right)^{50}\frac{1-\left(\frac{3}{5}\right)^{51}}{1-\frac{3}{5}}\\ &= \frac{5}{2}\left(\frac{3}{5}\right)^{50}\left[1-\left(\frac{3}{5}\right)^{51}\right].\end{aligned}$$

Solution 2: In this solution, we write our given expression as the difference of two sums, both starting at $i=0$.

$$\begin{aligned}\sum_{i=50}^{100}\left(\frac{3}{5}\right)^i &= \sum_{i=0}^{100}\left(\frac{3}{5}\right)^i-\sum_{i=0}^{49}\left(\frac{3}{5}\right)^i\\ &= \frac{1-\left(\frac{3}{5}\right)^{101}}{1-\frac{3}{5}}-\frac{1-\left(\frac{3}{5}\right)^{50}}{1-\frac{3}{5}}\\ &= \frac{5}{2}\left[\left(\frac{3}{5}\right)^{50}-\left(\frac{3}{5}\right)^{101}\right]\\ &= \frac{5}{2}\left(\frac{3}{5}\right)^{50}\left[1-\left(\frac{3}{5}\right)^{51}\right].\end{aligned}$$

(c) Before we can use the equations in Theorem 3.1.6, we'll need to do a little simplification.

$$\begin{aligned}\sum_{i=1}^{10}\left(i^2-3i+5\right) &= \sum_{i=1}^{10}i^2+\sum_{i=1}^{10}-3i+\sum_{i=1}^{10}5\\ &= \sum_{i=1}^{10}i^2-3\sum_{i=1}^{10}i+5\sum_{i=1}^{10}1\\ &= \frac{1}{6}(10)(11)(21)-3\left(\frac{1}{2}(10\cdot 11)\right)+5\cdot 10\\ &= 270\end{aligned}$$

(d) As in part (c), we'll simplify first. The first part (shown here in red) is a geometric sum, but it does not start at $1 = \left(\frac{1}{e}\right)^0$.

$$\begin{aligned}
\sum_{n=1}^{b}\left[\left(\frac{1}{e}\right)^n + en^3\right] &= \sum_{n=1}^{b}\left(\frac{1}{e}\right)^n + \sum_{n=1}^{b} en^3 \\
&= \sum_{n=0}^{b}\left(\frac{1}{e}\right)^n - 1 + e\sum_{n=1}^{b} n^3 \\
&= \frac{1-\left(\frac{1}{e}\right)^{b+1}}{1-\frac{1}{e}} - 1 + e\left[\frac{1}{2}b(b+1)\right]^2 \\
&= \frac{\frac{1}{e}-\left(\frac{1}{e}\right)^{b+1}}{1-\frac{1}{e}} + e\left[\frac{1}{2}b(b+1)\right]^2 \\
&= \frac{1-\left(\frac{1}{e}\right)^{b}}{e-1} + \frac{e}{4}\left[b(b+1)\right]^2
\end{aligned}$$

S-10:

(a) The two pieces are very similar, which we can see by changing the index, or expanding them out:

$$\begin{aligned}
\sum_{i=50}^{100}(i-50) + \sum_{i=0}^{50} i &= (0+1+2+\cdots+50) + (0+1+2+\cdots+50) \\
&= (1+2+\cdots+50) + (1+2+\cdots+50) \\
&= 2\,(1+2+\cdots+50) \\
&= 2\sum_{i=1}^{50} i \\
&= 2\left(\frac{50\cdot 51}{2}\right) = 50\cdot 51 = 2550
\end{aligned}$$

(b) If we expand $(i-5)^3 = i^3 - 15i^2 + 75i - 225$, we can break the sum into four parts, and evaluate each separately. However, it is much simpler to change the index and make the term $(i-5)^3$ into i^3.

$$\sum_{i=10}^{100}(i-5)^3 = 5^3 + 6^3 + 7^3 + \cdots + 95^3$$

We have a formula to evaluate the sum of cubes if they start at 1, so we turn our expression into the difference of two sums starting at 1:

$$\begin{aligned}
&= \left[1^3+2^3+3^3+4^3+5^3+6^3+7^3+\cdots+95^3\right] - \left[1^3+2^3+3^3+4^3\right] \\
&= \sum_{i=1}^{95} i^3 - \sum_{i=1}^{4} i^3 \\
&= \left[\frac{1}{2}(95)(96)\right]^2 - \left[\frac{1}{2}(4)(5)\right]^2 .
\end{aligned}$$

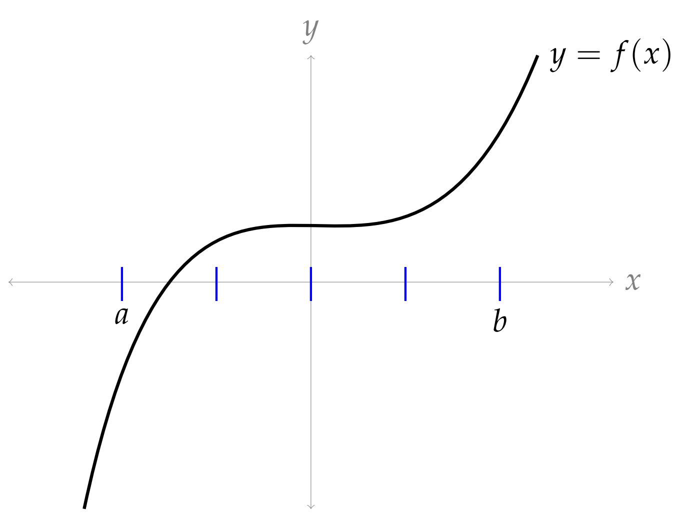
(c) Notice every two terms cancel with each other, since the sum is $(-1)+(+1)$, etc. Then the terms $n=1$ through $n=10$ cancel, and we're left only with the final term, $(-1)^{11} = -1$.

Written out more explicitly:

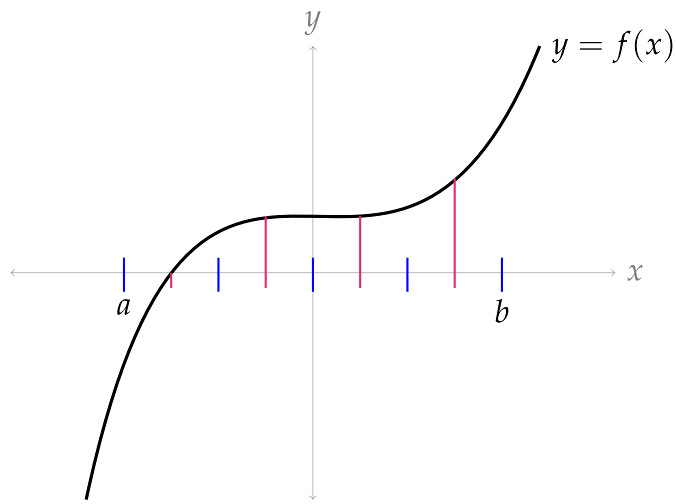
$$\begin{aligned}
\sum_{n=1}^{11}(-1)^n &= -1+1-1+1-1+1-1+1-1+1-1 \\
&= [-1+1]+[-1+1]+[-1+1]+[-1+1]+[-1+1]-1 \\
&= 0+0+0+0+0-1 = -1.
\end{aligned}$$

(d) For every integer n, $2n+1$ is odd, so $(-1)^{2n+1} = -1$. Then $\displaystyle\sum_{n=2}^{11}(-1)^{2n+1} = \sum_{n=2}^{11} -1 = -10.$

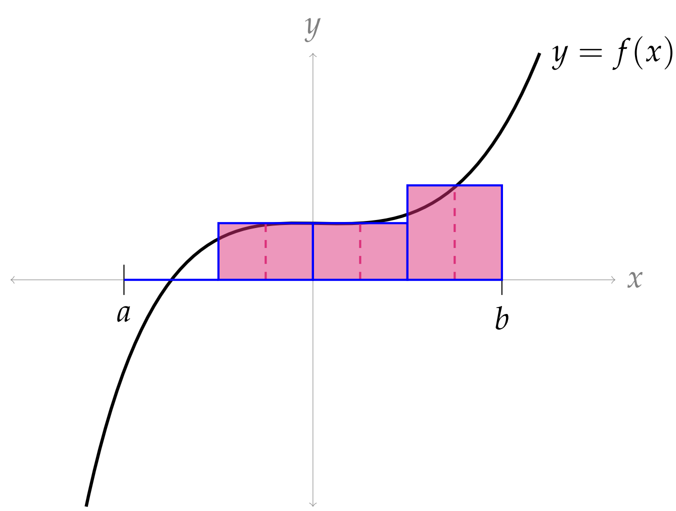
S-11: The index of the sum runs from 1 to 4: the first, second, third, and fourth rectangles. So, we have four rectangles in our Riemann sum. Let's start by drawing in the intervals along the x-axis taken up by these four rectangles. Note each has the same width: $\dfrac{b-a}{4}$.



Since this is a midpoint Riemann sum, the height of each rectangle is given by the y-value of the function in the midpoint of the interval. So, now let's find the height of the function at the midpoints of each of the four intervals.



The left-most interval has a height of about 0, so it gives a "trivial" rectangle with no height and no area. The middle two intervals have rectangles of about the same height, and the right-most interval has the highest rectangle.



S-12: In general, the left Riemann sum for the integral $\int_a^b f(x)\,\mathrm{d}x$ is of the form

$$\sum_{k=1}^{n} f\left(a + (k-1)\frac{b-a}{n}\right)\frac{b-a}{n}$$

- To get the limits of summation to match the given sum, we need $n = 4$.
- Then to get the factor multiplying f to match that in the given sum, we need $\frac{b-a}{n} = 1$, so $b - a = 4$.

- Finally, to get the argument of f to match that in the given sum, we need

$$a+(k-1)\frac{b-a}{n} = a - \frac{b-a}{n} + k\frac{b-a}{n} = 1+k$$

Subbing in $n=4$ and $b-a=4$ gives $a-1+k=1+k$, so $a=2$ and $b=6$.

S-13: The general form of a Riemann sum is $\sum_{i=1}^{n} \Delta x \cdot f(x_i^*)$, where $\Delta x = \frac{b-a}{n}$ is the width of each rectangle, and $f(x_i^*)$ is the height.

There are different ways to interpret the given sum as a Riemann sum. The most obvious is given in Solution 1. You may notice that we make some convenient assumptions in this solution about values for Δx and a, and we assume the sum is a right Riemann sum. Other visualizations of the sum arise from making more exotic choices. Some of these are explored in Solutions 2-4.

All cases have three rectangles, and the three rectangles will have the same areas: 98, 162, and 242 square units, respectively. This is because the terms of the given sum simplify to $98+162+242$.

Solution 1:

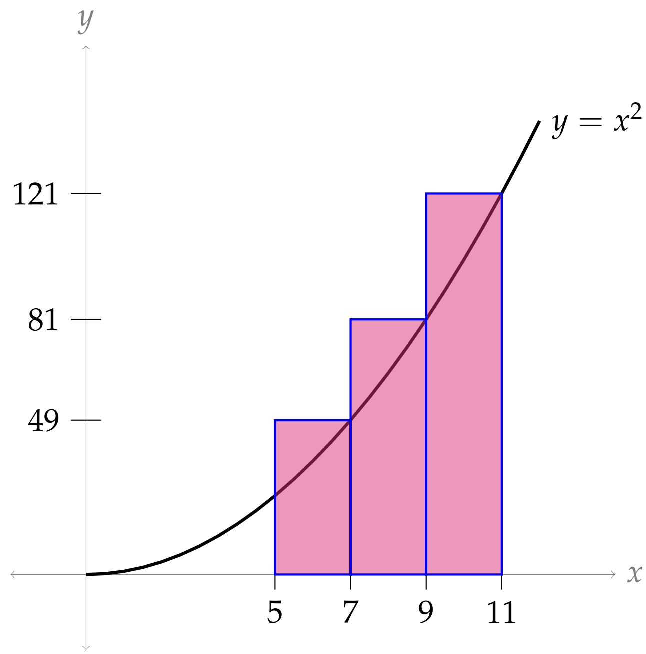
- Because the index runs from 1 to 3, there are three intervals: $n=3$.

- Looking at our sum, it seems reasonable to interpret $\Delta x = 2$. Then, since $n=3$, we conclude $\frac{b-a}{3} = 2$, hence $b-a=6$.

- If $\Delta x = 2$, then $f(x_i^*) = (5+2i)^2$. Recall that x_i^* is the x-coordinate we use to decide the height of the ith rectangle. In a right Riemann sum, $x_i^* = a + i \cdot \Delta x$. So, using $2 = \Delta x$, we can let $f(x_i^*) = f(a+2i) = (5+2i)^2$. This fits with the function $f(x) = x^2$, and $a=5$.

- Since $b-a=6$, and $a=5$, this tells us $b=11$

To sum up, we can interpret the Riemann sum as a right Riemann sum, with three intervals, of the function $f(x)=x^2$ from $x=5$ to $x=11$.

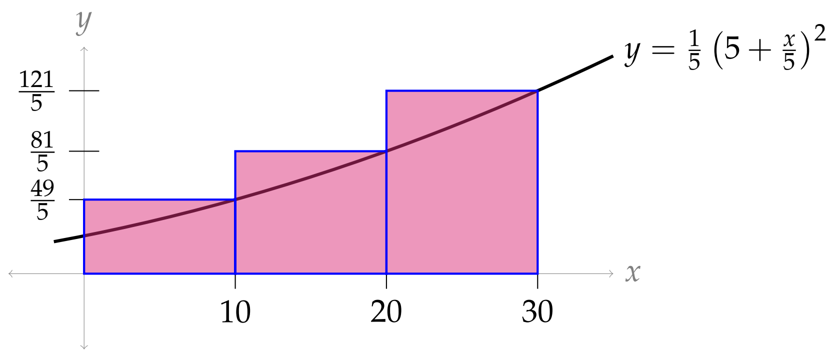


Solution 2: We could have chosen a different value for Δx.

- The index of the sum runs from 1 to 3, so we have $n = 3$.
- We didn't have to interpret Δx as 2–that was just the path of least resistance. We could have chosen it to be any other number–for the sake of argument, let's say $\Delta x = 10$. (Positive numbers are easiest to interpret, but negatives are technically allowed as well.)
- Then $10 = \frac{b-a}{n} = \frac{b-a}{3}$, so $b - a = 30$.
- Let's use the paradigm of a right Riemann sum, and match up the terms of the sum given in the problem to the terms in the definition:

$$\begin{aligned}
\Delta x \cdot f\left(a + i \cdot \Delta x\right) &= 2 \cdot (5 + 2i)^2 \\
10 \cdot f(a + 10i) &= 2 \cdot (5 + 2i)^2 \\
f(a + 10i) &= \frac{1}{5} \cdot (5 + 2i)^2 \\
f(a + 10i) &= \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot 10i\right)^2
\end{aligned}$$

- The easiest value of a in this case is $a = 0$. Then $f(10i) = \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot 10i\right)^2$, so $f(x) = \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot x\right)^2$.
- If $a = 0$ and $b - a = 30$, then $b = 30$.
- To sum up: $n = 3$, $a = 0$, $b = 30$, $\Delta x = 10$, and $f(x) = \frac{1}{5} \cdot \left(5 + \frac{x}{5}\right)^2$.



By changing Δx, we changed the *widths* of the rectangles. The rectangles in this picture are wider and shorter than the rectangles in Solution 1. Their *areas* are the same: 98, 162, and 242.

Solution 3: We could have chosen a different value of a.

- Suppose $\Delta x = 2$, and we interpret our sum as a right Riemann sum, but we didn't assume $a = 5$. We could have chosen a to be any number–say, $a = 1$.

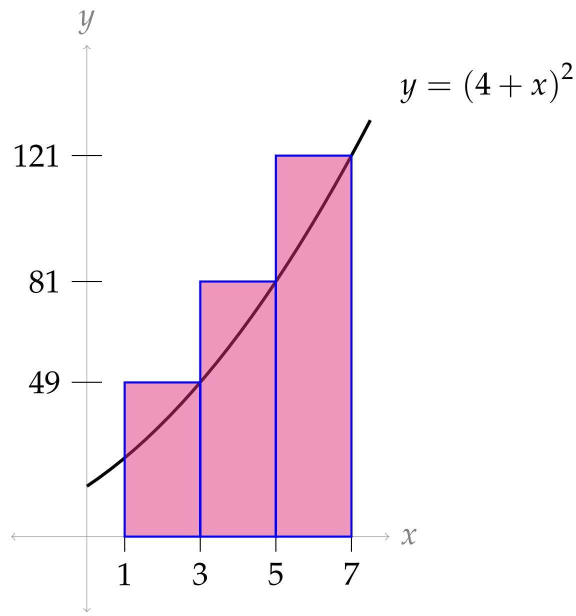
- Let's match up what we're given in the problem to what we're given as a definition:

$$\begin{aligned}
\Delta x \cdot f\left(a + i \cdot \Delta x\right) &= 2 \cdot (5 + 2i)^2 \\
2 \cdot f\left(1 + 2i\right) &= 2 \cdot (5 + 2i)^2 \\
f\left(1 + 2i\right) &= (5 + 2i)^2 \\
f\left(1 + 2i\right) &= (4 + 1 + 2i)^2
\end{aligned}$$

- Since $f(1 + 2i) = (4 + 1 + 2i)^2$, we have $f(x) = (4 + x)^2$

- Since $a = 1$ and $\frac{b-a}{3} = 2$, in this case $b = 7$.

- To sum up: $n = 3$, $a = 1$, $b = 7$, $\Delta x = 2$, and $f(x) = (4 + x)^2$.



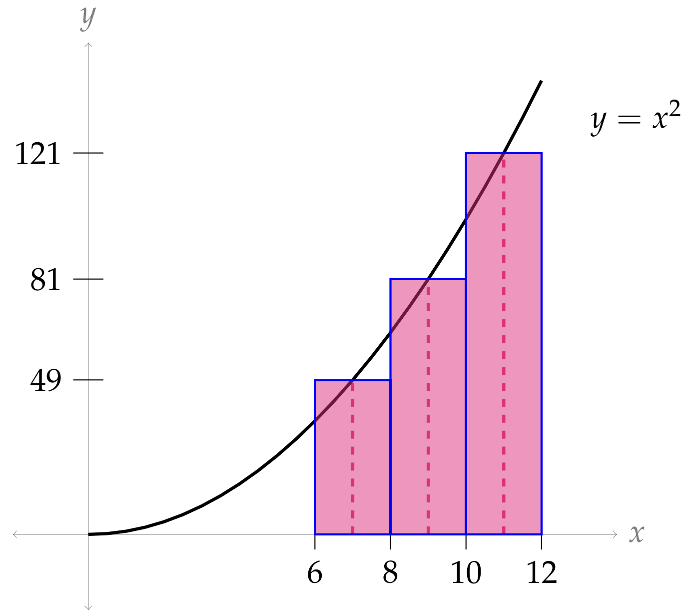
This picture is a lot like the picture in Solution 1, but shifted to the left. By changing a, we changed the *left endpoint* of our region.

Solution 4: We could have chosen a different kind of Riemann sum.

- We didn't have to assume that we were dealing with a *right* Riemann sum. Suppose $\Delta x = 2$, and we have a *midpoint* Riemann sum.
- Let's match up what we're given in the problem with what we're given in the definition:

$$\begin{aligned}
\Delta x \cdot f\left(a + \left(i - \tfrac{1}{2}\right)\Delta x\right) &= 2 \cdot (5+2i)^2 \\
2 \cdot f\left(a + \left(i - \tfrac{1}{2}\right)2\right) &= 2 \cdot (5+2i)^2 \\
f\left(a + \left(i - \tfrac{1}{2}\right)2\right) &= (5+2i)^2 \\
f\left(a + 2i - 1\right) &= (5+2i)^2 \\
f\left((a-1) + 2i\right) &= (5+2i)^2
\end{aligned}$$

- It is now convenient to set $a - 1 = 5$, hence $a = 6$.
- Then $f(5+2i) = (5+2i)^2$, so $f(x) = x^2$
- Since $2 = \frac{b-a}{3}$ and $a = 6$, we see $b = 12$.
- To sum up: $n = 3$, $a = 6$, $b = 12$, $\Delta x = 2$, and $f(x) = x^2$.



By choosing to interpret our sum as a midpoint Riemann sum instead of a right Riemann sum, we changed where our rectangles intersect the graph $y = f(x)$: instead of the graph hitting the right corner of the rectangle, it hits in the middle.

S-14: Many interpretations are possible–see the solution to Question 13 for a more thorough discussion–but the most obvious is given below. Recall the definition of a left Riemann sum:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + (i-1)\Delta x\right)$$

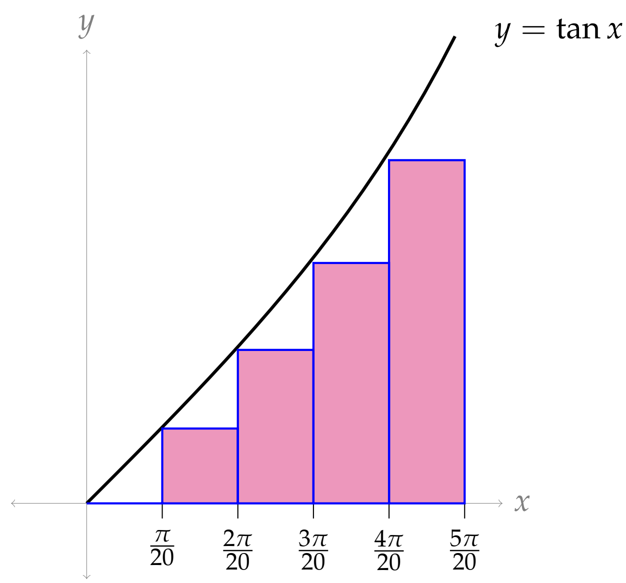
We chose a left Riemann sum instead of right or midpoint because our given sum has $(i-1)$ in it, rather than $(i - \frac{1}{2})$ or simply i.

- Since the sum has five terms (i runs from 1 to 5), there are 5 rectangles. That is, $n = 5$.
- In the definition of the Riemann sum, note that the term Δx appears twice: once multiplied by the entire term, and once multiplied by $i-1$. So, a convenient choice for Δx is $\frac{\pi}{20}$, because this is the constant that is both multiplied at the start of the term, and multiplied by $i-1$.
- Since $\dfrac{\pi}{20} = \Delta x = \dfrac{b-a}{n} = \dfrac{b-a}{5}$, we see $b - a = \dfrac{5\pi}{20} = \dfrac{\pi}{4}$.
- We match the terms in the definition with the terms in the problem:

$$f(a + (i-1)\Delta x) = \tan\left(\frac{\pi(i-1)}{20}\right)$$
$$f\left(a + (i-1)\frac{\pi}{20}\right) = \tan\left((i-1)\frac{\pi}{20}\right)$$

So, we choose $a = 0$ and $f(x) = \tan x$.

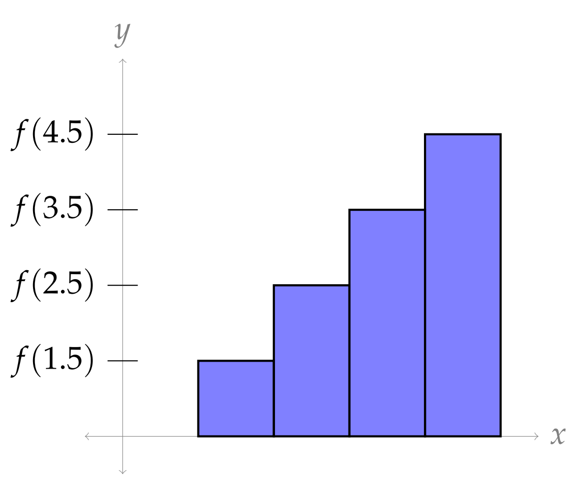
- Since $a = 0$ and $b - a = \frac{\pi}{4}$, we see $b = \frac{\pi}{4}$.



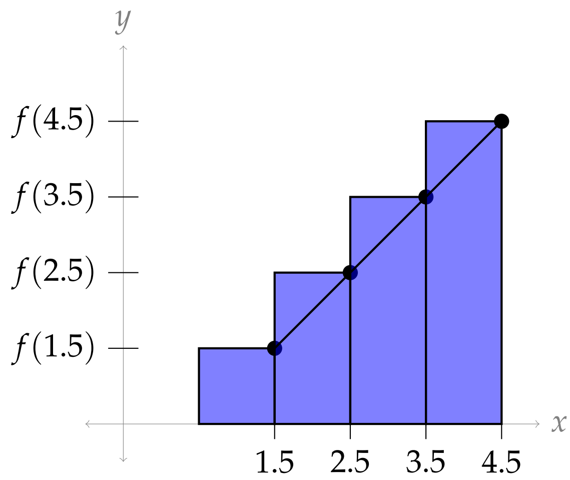
We note that the first rectangle of the five is a "trivial" rectangle, with height (and area) 0.

S-15: Since there are four terms in the sum, $n = 4$. (Note the sum starts at $k = 0$, instead of $k = 1$.) Since the function is multiplied by 1, $1 = \Delta x = \dfrac{b-a}{n} = \dfrac{b-a}{4}$, hence $b - a = 4$.

We can choose to view the given sum as a left, right, or midpoint Riemann sum. The choice we make determines the interval. Note that the heights of the rectangles are determined when $x = 1.5$, 2.5, 3.5, and 4.5.

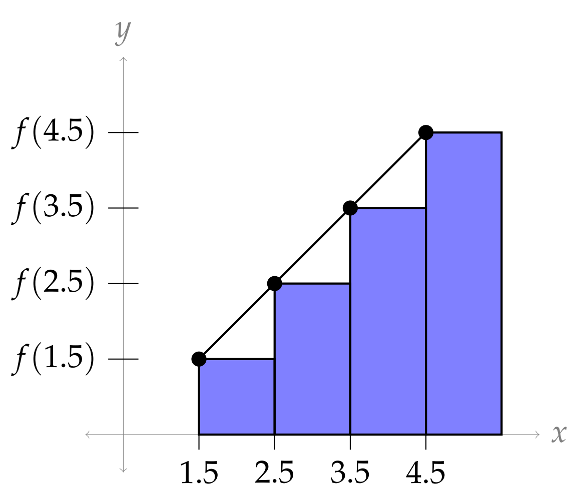


Option 1: right Riemann sum If our sum is a right Riemann sum, then we take the heights of the rectangles from the right endpoint of each interval.



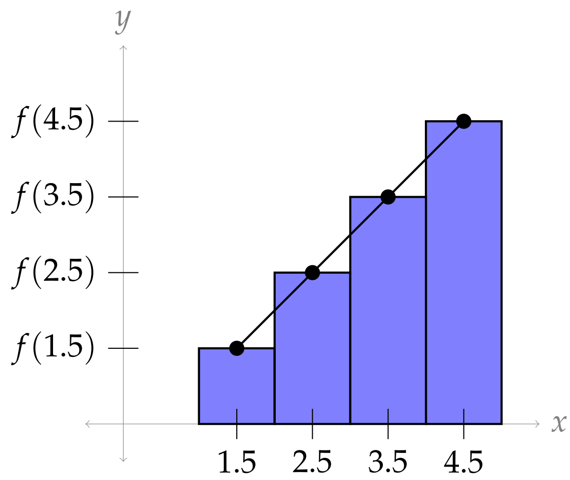
Then $a = 0.5$ and $b = 4.5$. Therefore: $\sum_{k=0}^{3} f(1.5+k) \cdot 1$ is a right Riemann sum on the interval $[0.5, 4.5]$ with $n = 4$.

Option 2: left Riemann sum If our sum is a left Riemann sum, then we take the heights of the rectangles from the left endpoint of each interval.



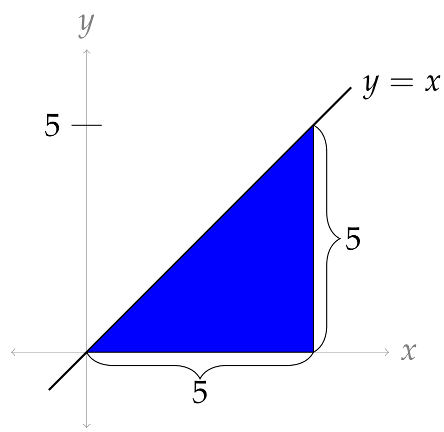
Then $a = 1.5$ and $b = 5.5$. Therefore: $\sum_{k=0}^{3} f(1.5+k) \cdot 1$ is a left Riemann sum on the interval $[1.5, 5.5]$ with $n = 4$.

Option 3: midpoint Riemann sum If our sum is a midpoint Riemann sum, then we take the heights of the rectangles from the midpoint of each interval.



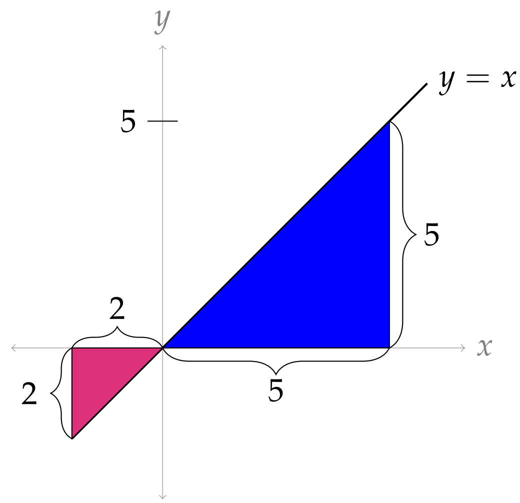
Then $a = 1$ and $b = 5$. Therefore: $\sum_{k=0}^{3} f(1.5 + k) \cdot 1$ is a midpoint Riemann sum on the interval $[1, 5]$ with $n = 4$.

S-16: The area in question is a triangle with base 5 and height 5, so its area is $\frac{25}{2}$.



S-17:

There is a positive and a negative portion of this area. The positive area is a triangle with base 5 and height 5, so area $\frac{25}{2}$ square units. The negative area is a triangle with base 2 and height 2, so negative area $\frac{4}{2} = 2$ square units. So, the net area is $\frac{25}{2} - \frac{4}{2} = \frac{21}{2}$ square units.



S-18: In general, the midpoint Riemann sum is given by

$$\sum_{i=1}^{n} f\Big(a + \big(i - 1/2\big)\Delta x\Big)\ \Delta x\,, \qquad \text{where } \Delta x = \frac{b-a}{n}.$$

In this problem we are told that $f(x) = x^8$, $a = 5$, $b = 15$ and $n = 50$, so that $\Delta x = \frac{b-a}{n} = \frac{1}{5}$ and the desired Riemann sum is:

$$\sum_{i=1}^{50} \left(5 + \big(i - 1/2\big)\frac{1}{5}\right)^8 \frac{1}{5}$$

S-19: The given integral has interval of integration going from $a = -1$ to $b = 5$. So when we use three approximating rectangles, all of the same width, the common width is $\Delta x = \frac{b-a}{n} = 2$. The first rectangle has left endpoint $x_0 = a = -1$, the second has left hand endpoint $x_1 = a + \Delta x = 1$, and the third has left hand end point $x_2 = a + 2\Delta x = 3$. So

$$\int_{-1}^{5} x^3\ \mathrm{d}x \approx \big[f(x_0) + f(x_1) + f(x_2)\big]\Delta x = \big[(-1)^3 + 1^3 + 3^3\big] \times 2 = 54$$

S-20: In the given integral, the domain of integration runs from $a = -1$ to $b = 7$. So, we have $\Delta x = \frac{(b-a)}{n} = \frac{(7-(-1))}{n} = \frac{8}{n}$. The left-hand end of the first subinterval is at $x_0 = a = -1$. So, the right-hand end of the i^{th} interval is at $x_i^* = -1 + \frac{8i}{n}$. So:

$$\int_{-1}^{7} f(x)\ \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1 + \frac{8i}{n}\right) \frac{8}{n}$$

S-21: We identify the given sum as the right Riemann sum $\sum\limits_{i=1}^{n} f(a+i\Delta x)\Delta x$, with $a = 0$ (that's specified in the statement of the question). Since $\frac{4}{n}$ is multiplied in every term, and is also multiplied by i, we let $\Delta x = \frac{4}{n}$. Then $x_i^* = a + i\Delta x = \frac{4i}{n}$ and $f(x) = \sin^2(2+x)$. So, $b = a + n\Delta x = 0 + n \cdot \frac{4}{n} = 4$.

S-22: The given sum is of the form

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2}\sqrt{1-\frac{k^2}{n^2}} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n}\right) \frac{k}{n}\sqrt{1-\left(\frac{k}{n}\right)^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x f(x_k^*)$$

with $\Delta x = \frac{1}{n}$, $a = 0$, $x_k^* = \frac{k}{n} = a + k\Delta x$ and $f(x) = x\sqrt{1-x^2}$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of $\int_0^1 f(x)\, \mathrm{d}x$.

S-23: As i ranges from 1 to n, $3i/n$ range from $3/n$ to 3 with jumps of $\Delta x = 3/n$, so this is

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} e^{-i/n} \cos(3i/n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\, \mathrm{d}x$$

where $x_i^* = 3i/n$, $f(x) = e^{-x/3}\cos(x)$, $a = x_0 = 0$ and $b = x_n = 3$. Thus

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} e^{-i/n} \cos(3i/n) = \int_0^3 e^{-x/3}\cos(x)\, \mathrm{d}x$$

S-24: As i ranges from 1 to n, the exponent $\frac{i}{n}$ ranges from $\frac{1}{n}$ to 1 with jumps of $\Delta x = \frac{1}{n}$. So let's try $x_i^* = \frac{i}{n}$, $\Delta x = \frac{1}{n}$. Then:

$$R_n = \sum_{i=1}^{n} \frac{ie^{i/n}}{n^2} = \sum_{i=1}^{n} \frac{i}{n} e^{i/n} \frac{1}{n} = \sum_{i=1}^{n} x_i^* e^{x_i^*} \Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

with $f(x) = xe^x$, and the limit

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\, \mathrm{d}x$$

Since we chose $x_i^* = \frac{i}{n} = 0 + i\Delta x$, we let $a = 0$. Then $\frac{1}{n} = \Delta x = \frac{b-a}{n} = \frac{b}{n}$ tells us $b = 1$. Thus,

$$\lim_{n\to\infty} R_n = \int_0^1 xe^x\, \mathrm{d}x\,.$$

S-25:

Choice #1: If we set $\Delta x = \frac{2}{n}$ and $x_i^* = \frac{2i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 0$, then

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-x_i^*}\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) && \text{with } f(x) = e^{-1-x} \\
&= \int_a^b f(x)\ \mathrm{d}x && \text{with } a = x_0 = 0 \text{ and } b = x_n = 2 \\
&= \int_0^2 e^{-1-x}\ \mathrm{d}x
\end{aligned}$$

Choice #2: If we set $\Delta x = \frac{2}{n}$ and $x_i^* = 1 + \frac{2i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 1$, then

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-x_i^*}\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) && \text{with } f(x) = e^{-x} \\
&= \int_a^b f(x)\ \mathrm{d}x && \text{with } a = x_0 = 1 \text{ and } b = x_n = 3 \\
&= \int_1^3 e^{-x}\ \mathrm{d}x
\end{aligned}$$

Choice #3: If we set $\Delta x = \frac{1}{n}$ and $x_i^* = \frac{i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 0$, then

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2x_i^*}\ 2\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) && \text{with } f(x) = 2e^{-1-2x} \\
&= \int_a^b f(x)\ \mathrm{d}x && \text{with } a = x_0 = 0 \text{ and } b = x_n = 1 \\
&= 2\int_0^1 e^{-1-2x}\ \mathrm{d}x
\end{aligned}$$

Choice #4: If we set $\Delta x = \frac{1}{n}$ and $x_i^* = \frac{1}{2} + \frac{i}{n}$, i.e. $x_i = a + i\Delta x$ with $a = \frac{1}{2}$, then

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-2x_i}\, 2\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) \quad \text{with } f(x) = 2e^{-2x} \\
&= \int_a^b f(x)\,\mathrm{d}x \qquad\qquad \text{with } a = x_0 = \frac{1}{2} \text{ and } b = x_n = \frac{3}{2} \\
&= 2\int_{1/2}^{3/2} e^{-2x}\,\mathrm{d}x
\end{aligned}$$

S-26: This is similar to the familiar form of a geometric sum, but the powers go up by threes. So, we make a subsitution. If $x = r^3$, then:

$$1 + r^3 + r^6 + r^9 + \cdots + r^{3n} = 1 + x + x^2 + x^3 + \cdots + x^n$$

Now, using Equation 3.1.3 in the text,

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$$

Substituting back in $x = r^3$, we find our sum is equal to $\dfrac{(r^3)^{n+1}-1}{r^3-1}$, or $\dfrac{r^{3n+3}-1}{r^3-1}$.

S-27: The sum does not start at 1, so we need to do some algebra. We can either factor out the first term, or subtract off the initial terms that are missing.

Solution 1: If we factor out r^5, then what's left fits the form of Equation 3.1.3 in the text:

$$r^5 + r^6 + r^7 + \cdots + r^{100} = r^5\left[1 + r + r^2 + \cdots + r^{95}\right] = r^5\left(\frac{r^{96}-1}{r-1}\right).$$

Solution 2: We know how to evaluate sums of this form if they start at 1, so we re-write our sum as follows:

$$\begin{aligned}
r^5 + r^6 + r^7 + \cdots + r^{100} &= \left(1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^{100}\right) - \left(1 + r + r^2 + r^3 + r^4\right) \\
&= \frac{r^{101}-1}{r-1} - \frac{r^5-1}{r-1} \\
&= \frac{r^{101}-1-r^5+1}{r-1} = \frac{r^{101}-r^5}{r-1} = r^5\left(\frac{r^{96}-1}{r-1}\right).
\end{aligned}$$

S-28: Recall that

$$|x| = \begin{cases} -x & \text{if } x \leqslant 0 \\ x & \text{if } x \geqslant 0 \end{cases}$$

so that

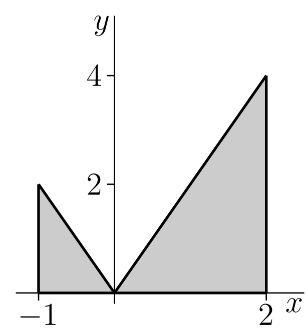
$$|2x| = \begin{cases} -2x & \text{if } x \leqslant 0 \\ 2x & \text{if } x \geqslant 0 \end{cases}$$

To picture the geometric figure whose area the integral represents observe that

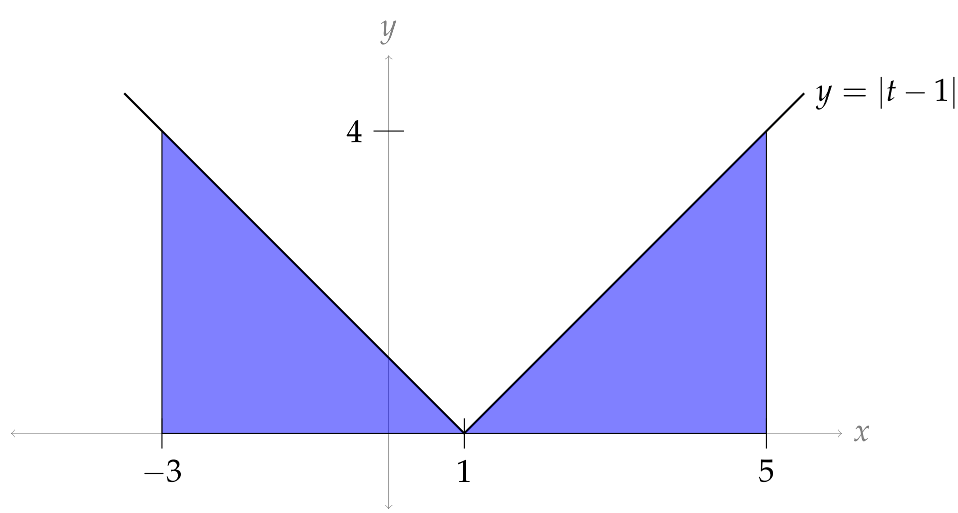
- at the left hand end of the domain of integration $x = -1$ and the integrand $|2x| = |-2| = 2$ and
- as x increases from -1 towards 0, the integrand $|2x| = -2x$ decreases linearly, until
- when x hits 0 the integrand hits $|2x| = |0| = 0$ and then
- as x increases from 0, the integrand $|2x| = 2x$ increases linearly, until
- when x hits $+2$, the right hand end of the domain of integration, the integrand hits $|2x| = |4| = 4$.

So the integral $\int_{-1}^{2} |2x| \, \mathrm{d}x$ is the area of the union of the two shaded triangles (one of base 1 and of height 2 and the other of base 2 and height 4) in the figure on the right below and

$$\int_{-1}^{2} |2x| \, \mathrm{d}x = \frac{1}{2} \times 1 \times 2 + \frac{1}{2} \times 2 \times 4 = 5$$



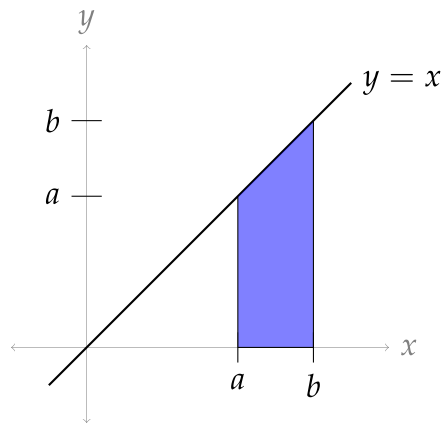
S-29: The area we want is two triangles, both above the x-axis. Each triangle has base 4 and height 4, so the total area is $2 \cdot \left(\frac{4 \cdot 4}{2}\right) = 16$.



If you had a hard time sketching the function, recall that the absolute value of a number leaves it unchanged if it is positive or zero, and flips the sign if it is negative. So, when

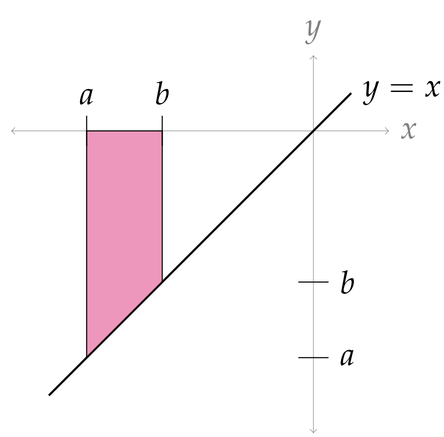
$t - 1 \geqslant 0$ (that is, when $t \geqslant 1$), our function is simply $f(t) = |t-1| = t-1$. On the other hand, when $t = 1$ is negative (that is, when $t < 1$), the absolute value changes the sign, so $f(t) = |t-1| = -(t-1) = -t+1$.

S-30: The area we want is a trapezoid with base $(b-a)$ and heights a and b, so its area is $\dfrac{(b-a)(b+a)}{2} = \dfrac{b^2-a^2}{2}$.



Instead of using a formula for the area of a trapezoid, you can find the blue area as the area of a triangle with base and height b, minus the area of a triangle with base and height a.

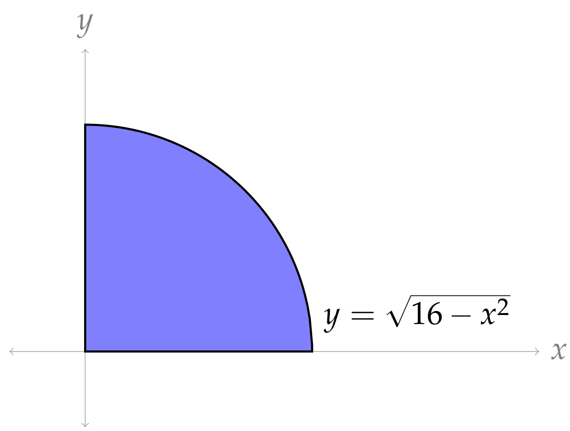
S-31: The area is negative. The shape is a trapezoid with base length $(b-a)$ and heights $0-a=-a$ and $0-b=-b$ (note: those are nonnegative numbers), so its area is $\dfrac{(b-a)(-b-a)}{2} = \dfrac{-b^2+a^2}{2}$. Since the shape is below the x-axis, we change its sign. Thus, the integral evaluates to $\dfrac{b^2-a^2}{2}$.



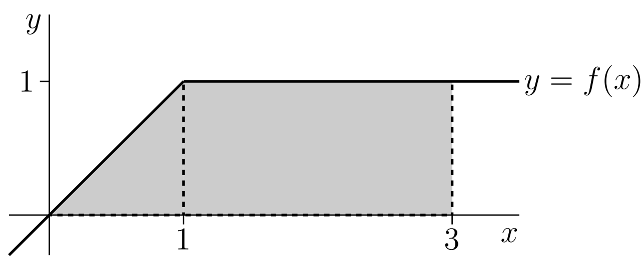
The signs can be a little hard to keep track of. The base of our trapezoid is $|a-b|$; since $b > a$, this is $b-a$. The heights of the trapezoid are $|a|$ and $|b|$; since these are both negative, $|a| = -a$ and $|b| = -b$.

We note that this is the same result as in Question 30.

S-32: If $y = \sqrt{16 - x^2}$, then y is nonnegative, and $y^2 + x^2 = 16$. So, the graph $y = \sqrt{16 - x^2}$ is the upper half of a circle of radius 4. Since x only runs from 0 to 4, we have a quarter of a circle of radius 4. Then the area under the curve is $\frac{1}{4}\left[\pi \cdot 4^2\right] = 4\pi$.



S-33: Here is a sketch the graph of $f(x)$.



There is a linear increase from $x = 0$ to $x = 1$, followed by a constant. Using the interpretation of $\int_0^3 f(x)\ \mathrm{d}x$ as the area between $y = f(x)$ and the x–axis with x between 0 and 3, we can break this area into:

- $\int_0^1 f(x)\ \mathrm{d}x$: a right-angled triangle of height 1 and base 1 and hence area 0.5.
- $\int_1^3 f(x)\ \mathrm{d}x$: a rectangle of height 1 and base 2 and hence area 2.

Summing up: $\int_0^3 f(x)\ \mathrm{d}x = 2.5$.

S-34: The car's speed increases with time. So its highest speed on any time interval occurs at the right hand end of the interval and the best possible upper estimate for the distance traveled is given by the *right* Riemann sum with $\Delta x = 0.5$, which is

$$\left[v(0.5) + v(1.0) + v(1.5) + v(2.0)\right] \times 0.5 = \left[14 + 22 + 30 + 40\right] \times 0.5 = 53 \text{ m}$$

S-35: There is a key detail in the statement of Question 34: namely, that the car is continuously accelerating. So, although we don't know exactly what's going on in between our brief snippets of information, we know that the car is not going any faster during an interval than at the end of that interval. Therefore, the car certainly travelled no farther than our estimation.

We ask this question in order to point out an important detail. If we did not have the information that the car was continuously accelerating, we would not be able to give a certain upper bound on its distance travelled. It would be possible that, when the car is not being observed (for example, when $t = 0.25$), it is going much faster than when it is being observed.

S-36: First, note that the distance travelled by the plane is equal to the area under the curve of its speed.

We need to know the speed of the plane at the midpoints of our intervals. So (for example) noon to 1pm *is not* one of your intervals–we don't know the speed at 12:30. (A common idea is to average the two end values, 700 and 800. This is a fine approximation, but it is not a Riemann sum.) So, we use the two intervals 12:00 to 2:00, and 2:00 to 4:00. Then our intervals have length 2 hours, and at the midpoints of the intervals the speed of the plane is 700 kph and 900 kph, respectively. So, our midpoint Riemann sum gives us:

$$700(2) + 900(2) = 3200$$

an approximation of 3200 km travelled by the plane from noon to 4:00 pm.

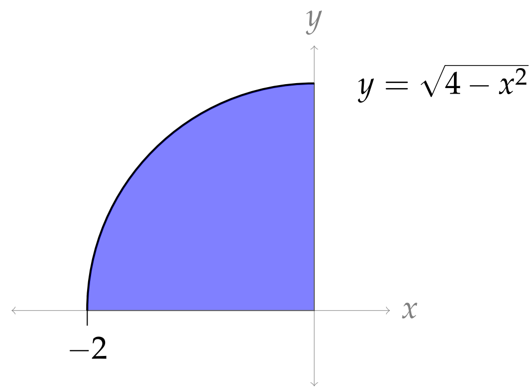
Remark: if we had been asked to approximate the distance travelled from 11:30 am to 4:30 pm, then we could have used the midpoint rule with five intervals and made use of every entry in the data table. With the question as stated, however, we ignore three out of five entries in the table because they are not the midpoints of our intervals.

S-37:

Solution #1: Set $x_i^* = -2 + \frac{2i}{n}$. Then $a = x_0 = -2$ and $b = x_n = 0$ and $\Delta x = \frac{2}{n}$. So

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{4 - \left(-2 + \frac{2i}{n}\right)^2} &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \text{with } f(x) = \sqrt{4 - x^2} \text{ and } \Delta x = \frac{2}{n} \\
&= \int_{-2}^{0} \sqrt{4 - x^2}\, \mathrm{d}x
\end{aligned}$$

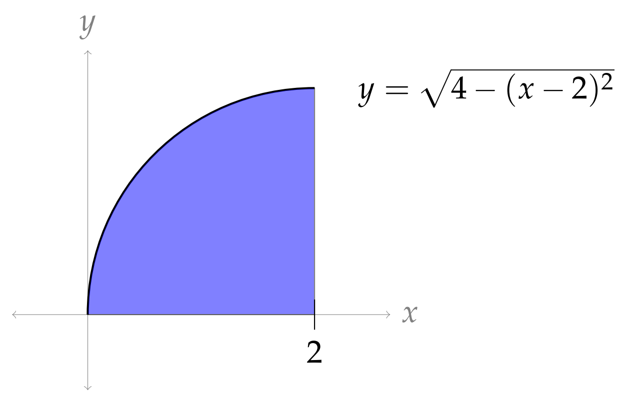
For the integral $\int_{-2}^{0} \sqrt{4 - x^2}\, \mathrm{d}x$, $y = \sqrt{4 - x^2}$ is equivalent to $x^2 + y^2 = 4$, $y \geqslant 0$. So the integral represents the area between the upper half of the circle $x^2 + y^2 = 4$ (which has radius 2) and the x-axis with $-2 \leqslant x \leqslant 0$, which is a quarter circle with area $\frac{1}{4} \cdot \pi\, 2^2 = \pi$.



Solution #2: Set $x_i^* = \frac{2i}{n}$. Then $a = x_0 = 0$ and $b = x_n = 2$ and $\Delta x = \frac{2}{n}$. So

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{4-\left(-2+\frac{2i}{n}\right)^2} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{ with } f(x)=\sqrt{4-(-2+x)^2}, \Delta x = \frac{2}{n}\\ &= \int_0^2 \sqrt{4-(-2+x)^2}\,\mathrm{d}x\end{aligned}$$

For the integral $\int_0^2 \sqrt{4-(-2+x)^2}\,\mathrm{d}x$, $y = \sqrt{4-(x-2)^2}$ is equivalent to $(x-2)^2+y^2 = 4$, $y \geqslant 0$. So the integral represents the area between the upper half of the circle $(x-2)^2+y^2=4$ (which is centered at $(2,0)$ and has radius 2) and the x-axis with $0 \leqslant x \leqslant 2$, which is a quarter circle with area $\frac{1}{4}\cdot\pi 2^2 = \pi$.



S-38: (a) The left Riemann sum is defined as

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad \text{with } x_i = a + i\Delta x$$

We subdivide into $n=3$ intervals, so that $\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$, $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. The function $f(x) = 7+x^3$ has the values $f(x_0) = 7+0^3 = 7$, $f(x_1) = 7+1^3 = 8$, and $f(x_2) = 7+2^3 = 15$, from which we evaluate

$$L_3 = \big[f(x_0)+f(x_1)+f(x_2)\big]\Delta x = \big[7+8+15\big]\times 1 = 30$$

(b) We divide into n intervals so that $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = a + i\Delta x = \frac{3i}{n}$. The right Riemann sum is therefore:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left[7 + \frac{(3i)^3}{n^3}\right] \frac{3}{n} = \sum_{i=1}^{n} \left[\frac{21}{n} + \frac{81\, i^3}{n^4}\right]$$

To calculate the sum:

$$\begin{aligned} R_n &= \left(\frac{21}{n}\sum_{i=1}^{n} 1\right) + \left(\frac{81}{n^4}\sum_{i=1}^{n} i^3\right) \\ &= \left(\frac{21}{n} \times n\right) + \left(\frac{81}{n^4} \times \frac{n^4 + 2n^3 + n^2}{4}\right) \\ &= 21 + \frac{81}{4}(1 + 2/n + 1/n^2) \end{aligned}$$

To evaluate the limit exactly, we take $n \to \infty$. The expressions involving $1/n$ vanish leaving:

$$\int_0^3 (7 + x^3)\ \mathrm{d}x = \lim_{n\to\infty} R_n = 21 + \frac{81}{4} = 41\frac{1}{4}$$

S-39: In general, the right–endpoint Riemann sum approximation to the integral $\int_a^b f(x)\ \mathrm{d}x$ using n rectangles is

$$\sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

where $\Delta x = \frac{b-a}{n}$. In this problem, $a = 2$, $b = 4$, and $f(x) = x^2$, so that $\Delta x = \frac{2}{n}$ and the right–endpoint Riemann sum approximation becomes

$$\begin{aligned} \sum_{i=1}^{n} f\Big(2 + \frac{2i}{n}\Big)\frac{2}{n} &= \sum_{i=1}^{n} \Big(2 + \frac{2i}{n}\Big)^2 \frac{2}{n} \\ &= \sum_{i=1}^{n} \left(4 + \frac{8i}{n} + \frac{4i^2}{n^2}\right)\frac{2}{n} \\ &= \sum_{i=1}^{n} \left(\frac{8}{n} + \frac{16i}{n^2} + \frac{8i^2}{n^3}\right) \\ &= \sum_{i=1}^{n} \frac{8}{n} + \sum_{i=1}^{n} \frac{16i}{n^2} + \sum_{i=1}^{n} \frac{8i^2}{n^3} \\ &= \frac{8}{n}\sum_{i=1}^{n} 1 + \frac{16}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{8}{n}n + \frac{16}{n^2} \cdot \frac{n(n+1)}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= 8 + 8\Big(1 + \frac{1}{n}\Big) + \frac{4}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) \end{aligned}$$

So

$$\int_2^4 x^2 \, dx = \lim_{n\to\infty} \left[8 + 8\left(1+\frac{1}{n}\right) + \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 8+8+\frac{4}{3}\times 2 = \frac{56}{3}$$

S-40: We'll use right Riemann sums with $a = 0$ and $b = 2$. When there are n rectangles, $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and $x_i = a + i\Delta x = 2i/n$. So we need to evaluate

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left((x_i)^3 + x_i\right) \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right)^3 + \frac{2i}{n}\right) \frac{2}{n} \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\frac{8i^3}{n^3} + \frac{2i}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{16}{n^4} \sum_{i=1}^{n} i^3 + \frac{4}{n^2} \sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \left(\frac{16(n^4+2n^3+n^2)}{n^4\cdot 4} + \frac{4(n^2+n)}{n^2\cdot 2}\right) \\
&= \lim_{n\to\infty} \left(\frac{16}{4}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) + \frac{4}{2}\left(1+\frac{1}{n}\right)\right) \\
&= \frac{16}{4} + \frac{4}{2} = 6.
\end{aligned}$$

S-41: We'll use right Riemann sums with $a = 1$, $b = 4$ and $f(x) = 2x - 1$. When there are n rectangles, $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = a + i\Delta x = 1 + 3i/n$. So we need to evaluate

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i - 1)\, \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{6i}{n} - 1\right) \frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{6i}{n} + 1\right) \\
&= \lim_{n\to\infty} \left(\frac{18}{n^2} \sum_{i=1}^{n} i + \frac{3}{n} \sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty} \left(\frac{18\cdot n(n+1)}{n^2\cdot 2} + \frac{3}{n}n\right) \\
&= \lim_{n\to\infty} \left(9\left(1+\frac{1}{n}\right) + 3\right) \\
&= 9 + 3 = 12.
\end{aligned}$$

S-42: Using the definition of a right Riemann sum,

$$\sum_{i=1}^{10} 3(7+2i)^2 \sin(4i) = \sum_{i=1}^{10} \Delta x f(a+i\Delta x)$$

Since $\Delta x = 10$ and $a = -5$,

$$\sum_{i=1}^{10} 3(7+2i)^2 \sin(4i) = \sum_{i=1}^{10} 10 f(-5+10i)$$

Dividing both expressions by 10,

$$\sum_{i=1}^{10} \frac{3}{10}(7+2i)^2 \sin(4i) = \sum_{i=1}^{10} f(-5+10i)$$

So, we have an expression for $f(-5+10i)$:

$$f(-5+10i) = \frac{3}{10}(7+2i)^2 \sin(4i)$$

In order to find $f(x)$, let $x = -5+10i$. Then $i = \frac{x}{10} + \frac{1}{2}$.

$$\begin{aligned} f(x) &= \frac{3}{10}\left(7+2\left(\frac{x}{10}+\frac{1}{2}\right)\right)^2 \sin\left(4\left(\frac{x}{10}+\frac{1}{2}\right)\right) \\ &= \frac{3}{10}\left(\frac{x}{5}+8\right)^2 \sin\left(\frac{2x}{5}+2\right) . \end{aligned}$$

S-43: As in the text, we'll set up a Riemann sum for the given integral. Right Riemann sums have the simplest form, so we use a right Riemann sum, but we could equally well use left or midpoint.

$$\begin{aligned} \int_0^1 2^x \mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(a+i\Delta x) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot 2^{i/n} \\ &= \lim_{n\to\infty} \frac{1}{n}\left(2^{1/n} + 2^{2/n} + 2^{3/n} + \cdots + 2^{n/n}\right) \\ &= \lim_{n\to\infty} \frac{2^{1/n}}{n}\left(1 + 2^{1/n} + 2^{2/n} + \cdots + 2^{\frac{n-1}{n}}\right) \\ &= \lim_{n\to\infty} \frac{2^{1/n}}{n}\left(1 + 2^{1/n} + \left(2^{1/n}\right)^2 + \cdots + \left(2^{1/n}\right)^{n-1}\right) \end{aligned}$$

The sum in parenthesis has the form of a geometric sum, with $r = 2^{1/n}$:

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{2^{1/n}}{n}\left(\frac{\left(2^{1/n}\right)^n - 1}{2^{1/n}-1}\right)\\
&= \lim_{n\to\infty} \frac{2^{1/n}}{n}\left(\frac{2-1}{2^{1/n}-1}\right)\\
&= \lim_{n\to\infty} \frac{2^{1/n}}{n(2^{1/n}-1)}
\end{aligned}$$

Note as $n \to \infty$, $1/n \to 0$, so the numerator has limit 1, while the denominator has indeterminate form $\infty \cdot 0$. So, we'll do a little algebra to get this into a l'Hôpital-style indeterminate form:

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{1}{n}\cdot 2^{1/n}}{2^{1/n}-1}\\
&= \lim_{n\to\infty} \underbrace{\frac{\frac{1}{n}}{1-2^{-1/n}}}_{\substack{\text{num}\to 0\\ \text{den}\to 0}}
\end{aligned}$$

Now we can use l'Hôpital's rule. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{2^x\} = 2^x \log x$, where $\log x$ is the natural logarithm of x, also sometimes written $\ln x$. We'll need to use the chain rule when we differentiate the denominator.

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{-1}{n^2}}{-2^{-1/n}\log 2\cdot\frac{1}{n^2}}\\
&= \lim_{n\to\infty} \frac{2^{1/n}}{\log 2}\\
&= \frac{1}{\log 2}
\end{aligned}$$

Using a calculator, we see this is about 1.44 square units.

S-44: As in the text, we'll set up a Riemann sum for the given integral. Right Riemann sums have the simplest form:

$$\begin{aligned}
\int_a^b 10^x \mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(a + i\Delta x) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot 10^{a+i\frac{b-a}{n}} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot 10^{a} \cdot \left(10^{\frac{b-a}{n}}\right)^i \\
&= \lim_{n\to\infty} \frac{b-a}{n} \cdot 10^{a} \left(\left(10^{\frac{b-a}{n}}\right)^1 + \left(10^{\frac{b-a}{n}}\right)^2 + \left(10^{\frac{b-a}{n}}\right)^3 + \cdots + \left(10^{\frac{b-a}{n}}\right)^n\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \cdot 10^{a} \cdot 10^{\frac{b-a}{n}} \left(1 + \left(10^{\frac{b-a}{n}}\right) + \left(10^{\frac{b-a}{n}}\right)^2 + \cdots + \left(10^{\frac{b-a}{n}}\right)^{n-1}\right)
\end{aligned}$$

Now the sum in parentheses has the form of a geometric sum, with $r = 10^{\frac{b-a}{n}}$:

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{b-a}{n} \cdot 10^{a} \cdot 10^{\frac{b-a}{n}} \left(\frac{\left(10^{\frac{b-a}{n}}\right)^n - 1}{10^{\frac{b-a}{n}} - 1}\right) \\
&= \lim_{n\to\infty} \frac{{\color{blue}b-a}}{n} \cdot {\color{magenta}10^{a}} \cdot 10^{\frac{b-a}{n}} \left(\frac{{\color{red}10^{b-a} - 1}}{10^{\frac{b-a}{n}} - 1}\right)
\end{aligned}$$

The coloured parts do not depend on n, so for simplicity we can move them outside the limit.

$$\begin{aligned}
&= {\color{blue}(b-a)} \cdot {\color{magenta}10^{a}} \left({\color{red}10^{b-a} - 1}\right) \lim_{n\to\infty} \frac{1}{n} \cdot \left(\frac{10^{\frac{b-a}{n}}}{10^{\frac{b-a}{n}} - 1}\right) \\
&= (b-a) \cdot \left(10^{b} - 10^{a}\right) \lim_{n\to\infty} \underbrace{\left(\frac{1/n}{1 - 10^{-\frac{b-a}{n}}}\right)}_{\substack{\text{num}\to 0 \\ \text{den}\to 0}}
\end{aligned}$$

Now we can use l'Hôpital's rule. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{10^x\} = 10^x \log x$, where $\log x$ is the natural logarithm of x, also sometimes written $\ln x$. For the denominator, we will have to use the chain rule.

$$\begin{aligned}
&= (b-a)\cdot\left(10^b - 10^a\right)\lim_{n\to\infty}\left(\frac{-1/n^2}{-10^{-\frac{b-a}{n}}\cdot\log 10\cdot\frac{b-a}{n^2}}\right)\\
&= (b-a)\cdot\left(10^b - 10^a\right)\lim_{n\to\infty}\left(\frac{1}{10^{-\frac{b-a}{n}}\cdot\log 10\cdot(b-a)}\right)\\
&= (b-a)\cdot\left(10^b - 10^a\right)\left(\frac{1}{\log 10\cdot(b-a)}\right)\\
&= \frac{1}{\log 10}\left(10^b - 10^a\right)
\end{aligned}$$

For part (b), we can guess that if 10 were changed to c, our answer would be

$$\int_a^b c^x\,\mathrm{d}x = \frac{1}{\log c}\left(c^b - c^a\right)$$

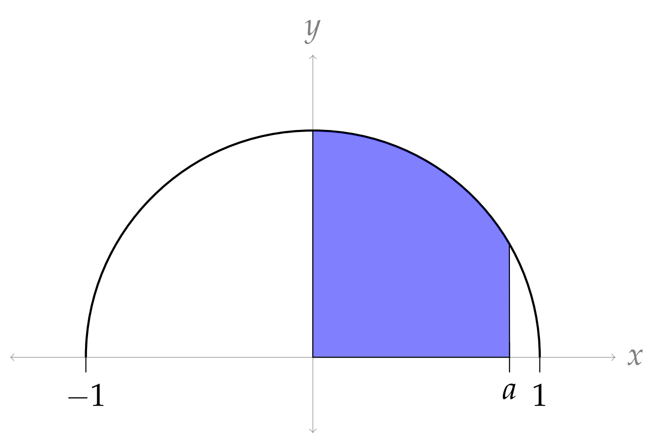
In Question 43, we had $a = 0$, $b = 1$, and $c = 2$. In this case, the formula we guessed above gives

$$\int_0^1 2^x\,\mathrm{d}x = \frac{1}{\log 2}\left(2^1 - 2^0\right) = \frac{1}{\log 2}$$

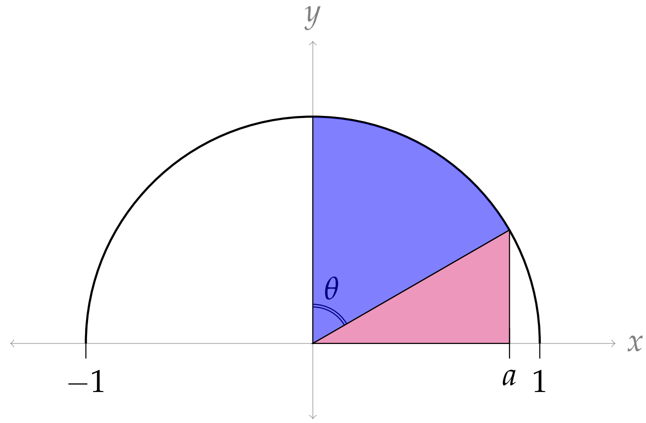
This does indeed match the answer we calculated.

(In fact, we can directly show $\displaystyle\int_a^b c^x\,\mathrm{d}x = \frac{1}{\log c}\left(c^b - c^a\right)$ using the method of this problem.)

S-45: First, we note $y = \sqrt{1-x^2}$ is the upper half of a circle of radius 1, centred at the origin. We're taking the area under the curve from 0 to a, so the area in question is as shown in the picture below.



In order to use geometry to find this area, we break it up into two pieces: a sector of a circle, and a triangle, shown below.



Area of sector: The sector is a portion of a circle with radius 1, with inner angle θ. So, its area is $\frac{\theta}{2\pi}$ (area of circle) $= \frac{\theta}{2\pi}(\pi) = \frac{\theta}{2}$.

Our job now is to find θ in terms of a. Note $\frac{\pi}{2} - \theta$ is the inner angle of the red triangle, which lies in the unit circle. So, $\cos\left(\frac{\pi}{2} - \theta\right) = a$. Then $\frac{\pi}{2} - \theta = \arccos(a)$, and so $\theta = \frac{\pi}{2} - \arccos(a)$.

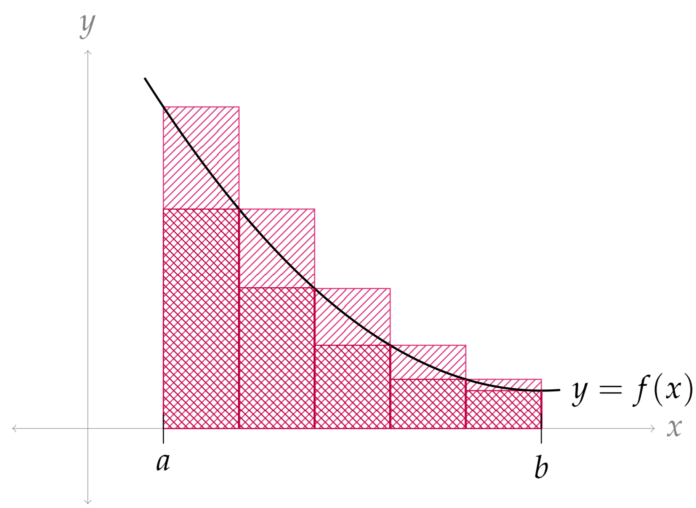
Then the area of the sector is $\frac{\pi}{4} - \frac{1}{2}\arccos(a)$ square units.

Area of triangle: The triangle has base a. Its height is the y-value of the function when $x = a$, so its height is $\sqrt{1-a^2}$. Then the area of the triangle is $\frac{1}{2}a\sqrt{1-a^2}$.

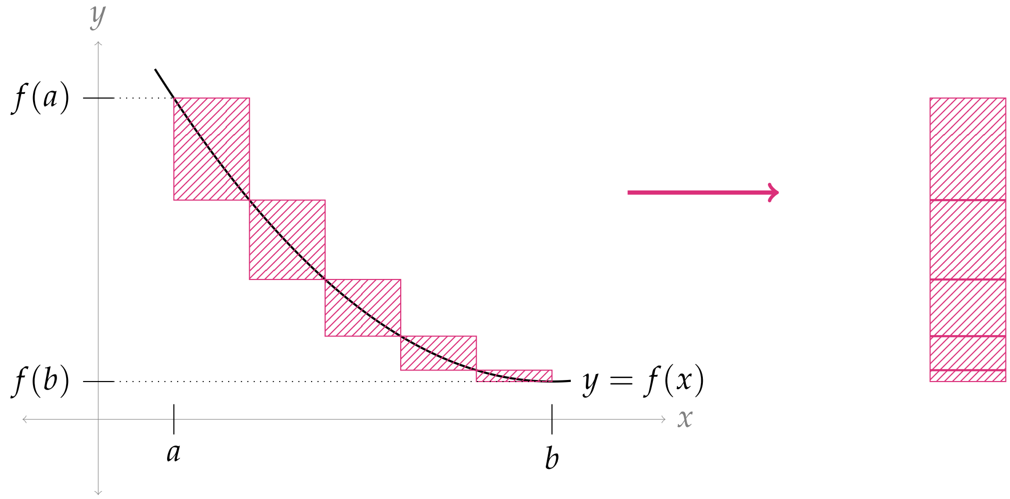
We conclude $\displaystyle\int_0^a \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}$.

S-46:

(a) The difference between our upper and lower bounds is the difference in areas between the larger set of rectangles and the smaller set of rectangles. Drawing them on a single picture makes this a little clearer.



Each of the rectangles has width $\frac{b-a}{n}$, since we took a segment of the x-axis with length $b-a$ and chopped it into n pieces. We could calculate the height of each rectangle, but it would be a little complicated, since it differs for each of them. An easier method is to notice that the area we want to calculate can be imagined as a single rectangle:



The rectangle has base $\frac{b-a}{n}$. Its highest coordinate is $f(a)$, and its lowest is $f(b)$, so its height is $f(b) - f(a)$. Therefore, the difference in area between our lower bound and our upper bound is:

$$[f(b) - f(a)] \cdot \frac{b-a}{n}$$

(b) We want to give a range with length at most 0.01, and guarantee that the area under the curve $y = f(x)$ is inside that range. In the previous part, we figured out that when we use n rectangles, the length of our range is $[f(b) - f(a)] \cdot \frac{b-a}{n}$. So, all we have to do is set this to be less than or equal to 0.01, and solve for n:

$$[f(b) - f(a)] \cdot \frac{b-a}{n} \leqslant 0.01$$
$$100\,[f(b) - f(a)] \cdot (b-a) \leqslant n$$

We can choose n to be an integer that is greater than or equal to $100\,[f(b) - f(a)] \cdot (b-a)$. Using that many rectangles, we find an upper and lower bound for the area under the curve. If we choose any number between our upper and lower bound as an approximation for the area under the curve, our error is no more than 0.01.

Remark: this question depends on the fact that f is decreasing and positive from a to b. In general, bounding errors on approximations like this is not so straightforward.

S-47: Since $f(x)$ is linear, there exist real numbers m and c such that $f(x) = mx + c$. Now we can do some calculations. Suppose we have a rectangle in our Riemann sum that takes up the interval $[x, x+w]$.

- If we are using a left Riemann sum, our rectangle has height $f(x) = mx + c$. Then it has area $w(mx + c)$.
- If we are using a right Riemann sum, our rectangle has height $f(x+w) = m(x+w) + c = mx + c + mw$. Then it has area $w(mx + c + mw)$.

- If we are using a midpoint Riemann sum, our rectangle has height $f(x+\frac{1}{2}w) = m(x+\frac{1}{2}w)+c = mx+c+\frac{1}{2}mw$. Then it has area $w\left(mx+c+\frac{1}{2}w\right)$.

So, for each rectangle in our sums, the midpoint rectangle has the same area as the average of the left and right rectangles:

$$w\left(mx+c+\frac{1}{2}mw\right) = \frac{w(mx+c)+w(mx+c+mw)}{2}$$

It follows that the midpoint Riemann sum has a value equal to the average of the values of the left and right Riemann sums. To see this, let the rectangles in the midpoint Riemann sum have areas $M_1, M_2, \ldots, M_n$, let the rectangles in the left Riemann sum have areas $L_1, L_2, \ldots, L_n$, and let the rectangles in the right Riemann sum have areas $R_1, R_2, \ldots, R_n$. Then the midpoint Riemann sum evaluates to $M_1 + M_2 + \cdots + M_n$, and:

$$\begin{aligned}\frac{[L_1+L_2+\ldots+L_n]+[R_1+R_2+\ldots+R_n]}{2} &= \frac{L_1+R_1}{2}+\frac{L_2+R_2}{2}+\cdots+\frac{L_n+R_n}{2}\\ &= M_1+M_2+\cdots+M_n\end{aligned}$$

So, the statement is true.

(Note, however, it is false for many non-linear functions $f(x)$.)

S-48:

1. Let $f(n)$ be the function giving the number of stitches on one side of round n. We're told $f(n)$ increases by two every time n increases by one – that means $f(n)$ has a constant slope of two, so $f(n) = 2n + c$ for some appropriate c. Since $f(1) = 1$, we see $f(n) = 2n - 1$.

   The last round has a side length of 299, so we solve $f(n) = 299$ for n:

   $$\begin{aligned}299 &= 2n-1\\ n &= 150\end{aligned}$$

   There are 150 rounds in the blanket.

2. We can count the number of stitches using sigma notation and Theorem 3.1.5. Each round n has $4(2n-1)$ stitches, which we sum over 150 rounds.

   $$\begin{aligned}\sum_{n=1}^{150} 4(2n-1) &= \left(8\sum_{n=1}^{150} n\right) - 4\left(\sum_{n=1}^{150} 1\right)\\ &= 8\frac{150\cdot 151}{2} - 4\cdot 150 = 4(150\cdot 151 - 150) = 4\cdot 150^2\end{aligned}$$

3. The halfway point is when $2\cdot 150^2$ stitches have been made. At the end of round N,

the number of stitches that have been made is

$$\sum_{n=1}^{N} 4(2n-1) = \left(8\sum_{n=1}^{N} n\right) - 4\left(\sum_{n=1}^{N} 1\right)$$
$$= 4N(N+1) - 4N = 4N^2$$

So, we solve: $4N^2 = 2 \cdot 150^2$

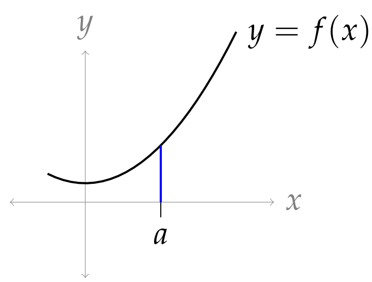
$$N = \frac{150}{\sqrt{2}} \approx 106.06$$

The halfway mark is reached *slightly after* the end of the 106th round. That is, the crocheter is halfway finished some time near the start of the 107th round.

Solutions to Exercises 3.2 — Jump to TABLE OF CONTENTS

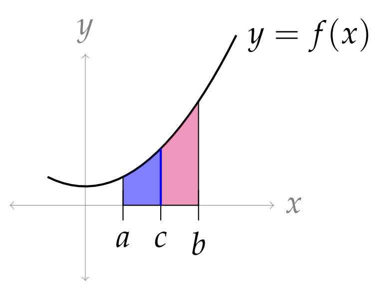
S-1:

(a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$



The area under the curve is zero, because it's a region with no width.

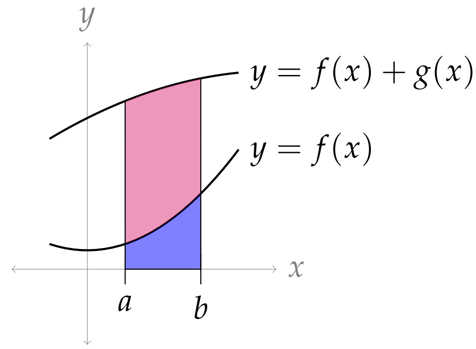
(b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$



If we assume $a \leqslant c \leqslant b$, then this identity simply tells us that if we add up the area under the curve from a to c, and from c to b, then we get the whole area under the curve from a to b.

(The situation is slightly more complicated when c is not between a and b, but it still works out.)

(c) $\displaystyle\int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$



The blue-shaded area in the picture above is $\displaystyle\int_a^b f(x)\,\mathrm{d}x$. The area under the curve $f(x)+g(x)$ but above the curve $f(x)$ (shown in red) is $\displaystyle\int_a^b g(x)\,\mathrm{d}x$.

S-2: Using the identity

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x\,,$$

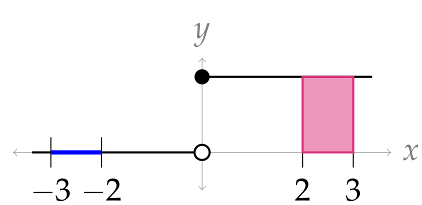
we see

$$\begin{aligned}\int_a^b \cos x\,\mathrm{d}x &= \int_a^0 \cos x\,\mathrm{d}x + \int_0^b \cos x\,\mathrm{d}x \\ &= -\int_0^a \cos x\,\mathrm{d}x + \int_0^b \cos x\,\mathrm{d}x \\ &= -\sin a + \sin b \\ &= \sin b - \sin a\end{aligned}$$

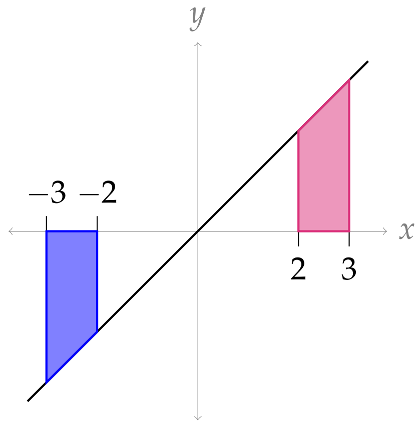
S-3: (a) False. For example if

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geqslant 0 \end{cases}$$

then $\int_{-3}^{-2} f(x)\mathrm{d}x = 0$ and $-\int_3^2 f(x)\mathrm{d}x = -1$.



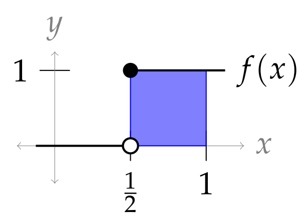
(b) False. For example, if $f(x) = x$, then $\int_{-3}^{-2} f(x)\,\mathrm{d}x$ is negative while $\int_2^3 f(x)\,\mathrm{d}x$ is positive, so they cannot be the same.

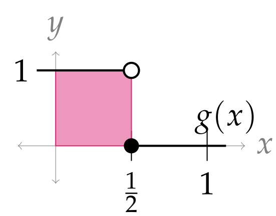


(c) False. For example, consider the functions

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} \\ 1 & \text{for } x \geqslant \frac{1}{2} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0 & \text{for } x \geqslant \frac{1}{2} \\ 1 & \text{for } x < \frac{1}{2} \end{cases}$$

Then $f(x) \cdot g(x) = 0$ for all x, so $\int_0^1 f(x) \cdot g(x)\mathrm{d}x = 0$. However, $\int_0^1 f(x)\mathrm{d}x = \frac{1}{2}$ and $\int_0^1 g(x)\mathrm{d}x = \frac{1}{2}$, so $\int_0^1 f(x)\mathrm{d}x \cdot \int_0^1 g(x)\mathrm{d}x = \frac{1}{4}$.





S-4:

(a) $\Delta x = \dfrac{b-a}{n} = \dfrac{0-5}{100} = -\dfrac{1}{20}$

Note: if we were to use the Riemann-sum definition of a definite integral, this is how we would justify the identity $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$.

(b) The heights of the rectangles are given by $f(x_i)$, where $x_i = a + i\Delta x = 5 - \frac{i}{20}$. Since $f(x)$ only gives positive values, $f(x_i) > 0$, so the heights of the rectangles are positive.

(c) Our Riemann sum is the sum of the signed areas of individual rectangles. Each rectangle has a negative base (Δx) and a positive height ($f(x_i)$). So, each term of our sum is negative. If we add up negative numbers, the sum is negative. So, the Riemann sum is negative.

(d) Since $f(x)$ is always above the x-axis, $\int_0^5 f(x)\mathrm{d}x$ is positive.

S-5: The definite integral tallies up the signed area under the curve. All together, that makes $A_1 - A_2 + A_3 - A_4$.

S-6: The operation of integration is linear (that's part (d) of the "arithmetic of integration" Theorem 3.2.1 in the text), so that:

$$\begin{aligned}\int_2^3 [6f(x) - 3g(x)]\,\mathrm{d}x &= \int_2^3 6f(x)\,\mathrm{d}x - \int_2^3 3g(x)\,\mathrm{d}x \\ &= 6\int_2^3 f(x)\,\mathrm{d}x - 3\int_2^3 g(x)\,\mathrm{d}x = (6 \times (-1)) - (3 \times 5) = -21\end{aligned}$$

S-7: The operation of integration is linear (that's part (d) of the "arithmetic of integration" Theorem 3.2.1 in the text), so that:

$$\begin{aligned}\int_0^2 [2f(x) + 3g(x)]\,\mathrm{d}x &= \int_0^2 2f(x)\,\mathrm{d}x + \int_0^2 3g(x)\,\mathrm{d}x \\ &= 2\int_0^2 f(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x = (2 \times 3) + (3 \times (-4)) = -6\end{aligned}$$

S-8: Using part (d) of the "arithmetic of integration" Theorem 3.2.1, followed by parts (c) and (b) of the "arithmetic for the domain of integration" Theorem 3.2.3 in the in the text,

$$\begin{aligned}\int_{-1}^2 \big[3g(x) - f(x)\big]\,\mathrm{d}x &= 3\int_{-1}^2 g(x)\,\mathrm{d}x - \int_{-1}^2 f(x)\,\mathrm{d}x \\ &= 3\int_{-1}^0 g(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x - \int_{-1}^0 f(x)\,\mathrm{d}x - \int_0^2 f(x)\,\mathrm{d}x \\ &= 3\int_{-1}^0 g(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x + \int_0^{-1} f(x)\,\mathrm{d}x - \int_0^2 f(x)\,\mathrm{d}x \\ &= 3 \times 3 + 3 \times 4 + 1 - 2 = 20\end{aligned}$$

S-9:

(a) Since $\sqrt{1-x^2}$ is an even function,

$$\begin{aligned}\int_a^0 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{|a|} \sqrt{1-x^2}\,\mathrm{d}x &= \frac{\pi}{4} - \frac{1}{2}\arccos(|a|) + \frac{1}{2}|a|\sqrt{1-|a|^2} \\ &= \frac{\pi}{4} - \frac{1}{2}\arccos(-a) - \frac{1}{2}a\sqrt{1-a^2}\end{aligned}$$

(b) Note $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{4}$, since the area under the curve represents one-quarter of the unit circle. Then,

$$\begin{aligned}
\int_a^1 \sqrt{1-x^2}\,\mathrm{d}x &= \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x - \int_0^a \sqrt{1-x^2}\,\mathrm{d}x \\
&= \frac{\pi}{4} - \left(\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}\right) \\
&= \frac{1}{2}\arccos(a) - \frac{1}{2}a\sqrt{1-a^2}
\end{aligned}$$

S-10: Recall that

$$|x| = \begin{cases} -x & \text{if } x \leqslant 0 \\ x & \text{if } x \geqslant 0 \end{cases}$$

so that

$$|2x| = \begin{cases} -2x & \text{if } x \leqslant 0 \\ 2x & \text{if } x \geqslant 0 \end{cases}$$

Also recall, from Example 3.2.5 in the text that

$$\int_a^b x\,\mathrm{d}x = \frac{b^2-a^2}{2}$$

So

$$\begin{aligned}
\int_{-1}^2 |2x|\,\mathrm{d}x &= \int_{-1}^0 |2x|\,\mathrm{d}x + \int_0^2 |2x|\,\mathrm{d}x = \int_{-1}^0 (-2x)\,\mathrm{d}x + \int_0^2 2x\,\mathrm{d}x \\
&= -2\int_{-1}^0 x\,\mathrm{d}x + 2\int_0^2 x\,\mathrm{d}x = -2\cdot\frac{0^2-(-1)^2}{2} + 2\cdot\frac{2^2-0^2}{2} \\
&= 1+4 = 5
\end{aligned}$$

S-11: If $x^2 \leqslant x$, then $e^{x^2} \leqslant e^x$. Using Part c of Theorem 3.2.12:

$$\int_0^1 e^{x^2}\mathrm{d}x \leqslant \int_0^1 e^x\mathrm{d}x = e^x\Big|_0^1 = e-1$$

S-12: We note that the integrand $f(x) = x|x|$ is an odd function, because $f(-x) = -x|-x| = -x|x| = -f(x)$. Then, by Theorem 3.2.11.b in the text, $\int_{-5}^5 x|x|\,\mathrm{d}x = 0$.

S-13: Using Theorem 3.2.11.a in the text,

$$10 = \int_{-2}^{2} f(x)\mathrm{d}x = 2\int_{0}^{2} f(x)\mathrm{d}x$$
$$5 = \int_{0}^{2} f(x)\mathrm{d}x$$

Also,

$$\int_{-2}^{2} f(x)\mathrm{d}x = \int_{-2}^{0} f(x)\mathrm{d}x + \int_{0}^{2} f(x)\mathrm{d}x$$

So,

$$\int_{-2}^{0} f(x)\mathrm{d}x = \int_{-2}^{2} f(x)\mathrm{d}x - \int_{0}^{2} f(x)\mathrm{d}x$$
$$= 10 - 5 = 5$$

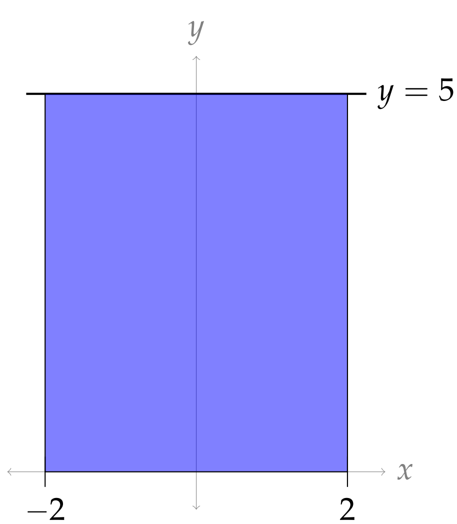
Indeed, for any even function $f(x)$, $\int\limits_{-a}^{0} f(x)\mathrm{d}x = \int\limits_{0}^{a} f(x)\mathrm{d}x$.

S-14: We first use additivity:

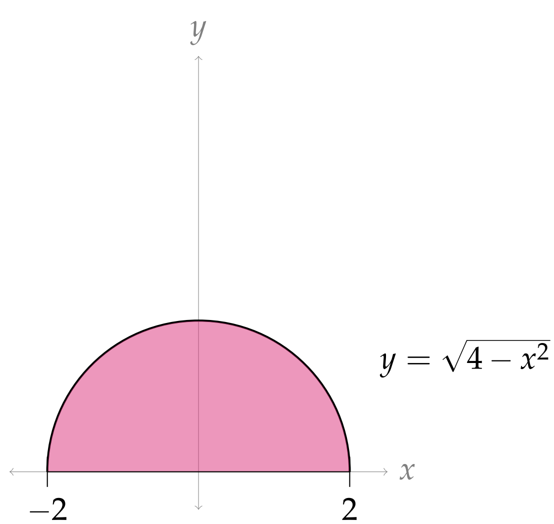
$$\int_{-2}^{2} \left(5 + \sqrt{4 - x^2}\right) \mathrm{d}x = \int_{-2}^{2} 5\,\mathrm{d}x + \int_{-2}^{2} \sqrt{4 - x^2}\,\mathrm{d}x$$

The first integral represents the area of a rectangle of height 5 and width 4 and so equals 20. The second integral represents the area above the x–axis and below the curve $y = \sqrt{4 - x^2}$ or $x^2 + y^2 = 4$. That is a semicircle of radius 2, which has area $\frac{1}{2}\pi 2^2$. So

$$\int_{-2}^{2} \left(5 + \sqrt{4 - x^2}\right) \mathrm{d}x = 20 + 2\pi$$



$\int\limits_{-2}^{2} 5\mathrm{d}x = 20$



$\int\limits_{-2}^{2} \sqrt{4 - x^2}\mathrm{d}x = 2\pi$

S-15: **Solution 1:** On the interval $[0,1]$:

$$\sin^2 x \leqslant x\sin x \leqslant \sin x$$

So, using Theorem 3.2.12,

$$\int_0^1 \sin^2 x \mathrm{d}x \leqslant \int_0^1 x\sin x \mathrm{d}x \leqslant \int_0^1 \sin x \mathrm{d}x$$

So, whatever bound we get from the inequality $\sin^2 x \leqslant \sin x$ is not going to be more useful than the bound we get from the inequality $\sin^2 x \leqslant x\sin x$. That is, once we know the bound from $\sin^2 x \leqslant x\sin x$, the other bound will not tell us anything new.

So, the inequality $\sin^2 x \leqslant x\sin x$ will give a more useful bound.

Solution 2:

Let's find the upper bounds on $\int_0^1 \sin^2 x \mathrm{d}x$ using Theorem 3.2.12 and both inequalities.

(a) Using $\sin^2 x \leqslant x\sin x$:

$$\int_0^1 \sin^2 x \mathrm{d}x \leqslant \int_0^1 x\sin x \mathrm{d}x$$

We can evaluate this integral using integration by parts with $u = x$, $\mathrm{d}v = \sin x \mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = -\cos x$

$$\begin{aligned} &= -x\cos x\big|_0^1 - \int_0^1 -\cos x \mathrm{d}x \\ &= -\cos 1 - [-\sin]_0^1 \\ &= -\cos 1 + \sin 1 = \sin 1 - \cos 1 \end{aligned}$$

(b) Using $\sin^2 x \leqslant \sin x$:

$$\begin{aligned} \int_0^1 \sin^2 x \mathrm{d}x \leqslant \int_0^1 \sin x \mathrm{d}x &= -\cos x\big|_0^1 \\ &= -\cos 1 + \cos 0 = 1 - \cos 1 \end{aligned}$$

Since $\sin 1 < 1$, we see $\sin 1 - \cos 1 < 1 - \cos 1$. So, the inequality $\sin^2 x \leqslant x\sin x$ gives the more useful approximation.

To be even more clear, note $\sin 1 - \cos 1 \approx 0.301$ and $1 - \cos 1 \approx 0.460$. If we already know that our desired value is not any larger than 0.301, then we could have automatically said that it was also not any larger than 0.46 — so who needs the bound 0.46?

S-16: Note that the integrand $f(x) = \frac{\sin x}{\log(3+x^2)}$ is an odd function, because:

$$f(-x) = \frac{\sin(-x)}{\log(3+(-x)^2)} = \frac{-\sin x}{\log(3+x^2)} = -f(x)$$

The domain of integration $-2012 \leqslant x \leqslant 2012$ is symmetric about $x = 0$. So, by Theorem 3.2.11 of the text,

$$\int_{-2012}^{+2012} \frac{\sin x}{\log(3+x^2)} \mathrm{d}x = 0$$

S-17: Note that the integrand $f(x) = x^{1/3}\cos x$ is an odd function, because:

$$f(-x) = (-x)^{1/3}\cos(-x) = -x^{1/3}\cos x = -f(x)$$

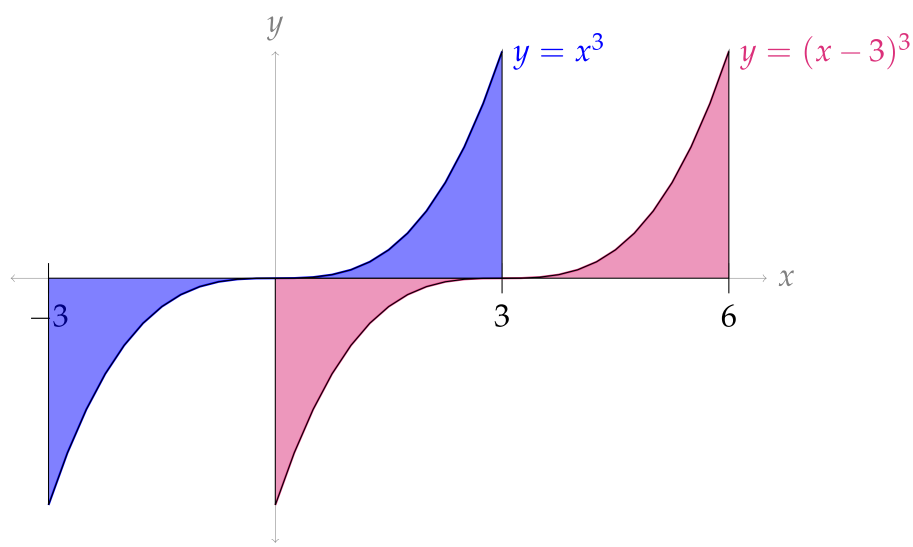
The domain of integration $-2012 \leqslant x \leqslant 2012$ is symmetric about $x = 0$. So, by Theorem 3.2.11 of the text,

$$\int_{-2012}^{+2012} x^{1/3}\cos x \,\mathrm{d}x = 0$$

S-18: Our integrand $f(x) = (x-3)^3$ is neither even nor odd. However, it does have a similar symmetry. Namely, $f(3+x) = -f(3-x)$. So, f is "negatively symmetric" across the line $x = 3$. This suggests that the integral should be 0: the positive area to the right of $x = 3$ will be the same as the negative area to the left of $x = 3$.

Another way to see this is to notice that the graph of $f(x) = (x-3)^3$ is equivalent to the graph of $g(x) = x^3$ shifted three units to the right, and $g(x)$ is an odd function. So,

$$\int_0^6 (x-3)^3 \,\mathrm{d}x = \int_{-3}^{3} x^3 \,\mathrm{d}x = 0$$



S-19:

(a)

$$\begin{aligned}(ax)^2 + (by)^2 &= 1\\ by &= \sqrt{1-(ax)^2}\\ y &= \frac{1}{b}\sqrt{1-(ax)^2}\end{aligned}$$

(b) The values of x in the domain of the function above are those that satisfy $1-(ax)^2 \geqslant 0$. That is, $-\frac{1}{a} \leqslant x \leqslant \frac{1}{a}$. Therefore, the upper half of the ellipse has area

$$\frac{1}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{1-(ax)^2}\,\mathrm{d}x$$

The upper half of a circle has equation $y = \sqrt{r^2 - x^2}$.

$$\begin{aligned}&= \frac{1}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{a^2\left(\frac{1}{a^2} - x^2\right)}\,\mathrm{d}x\\ &= \frac{1}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} a\sqrt{\frac{1}{a^2} - x^2}\,\mathrm{d}x\\ &= \frac{a}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{\frac{1}{a^2} - x^2}\,\mathrm{d}x\end{aligned}$$

(c) The function $y = \sqrt{\frac{1}{a^2} - x^2}$ is the upper-half of the circle centred at the origin with radius $\frac{1}{a}$. So, the expression from (b) evaluates to $\left(\frac{a}{b}\right)\frac{\pi}{2a^2} = \frac{\pi}{2ab}$.

The expression from (b) was half of the ellipse, so the area of the ellipse is $\frac{\pi}{ab}$.

Remark: this was a slightly long-winded way of getting the result. The reasoning is basically this:

- The area of the unit circle $x^2 + y^2 = 1$ is π .
- The ellipse $(ax)^2 + y^2 = 1$ is obtained by shrinking the unit circle horizontally by a factor of a. So, its area is $\frac{\pi}{a}$.
- Further, the ellipse $(ax)^2 + (by)^2 = 1$ is obtained from the previous ellipse by shrinking it vertically by a factor of b. So, its area is $\frac{\pi}{ab}$.

S-20: Let's recall the definitions of even and odd functions: $f(x)$ is *even* if $f(-x) = f(x)$ for every x in its domain, and $f(x)$ is *odd* if $f(-x) = -f(x)$ for every x in its domain.

Let $h(x) = f(x) \cdot g(x)$.

even $\times$ **even:** If f and g are both even, then
$h(-x) = f(-x) \cdot g(-x) = f(x) \cdot g(x) = h(x)$, so their product is even.

odd $\times$ **odd:** If f and g are both odd, then
$h(-x) = f(-x) \cdot g(-x) = [-f(x)] \cdot [-g(x)] = f(x) \cdot g(x) = h(x)$, so their product is even.

even $\times$ **odd:** If f is even and g is odd, then
$h(-x) = f(-x) \cdot g(-x) = f(x) \cdot [-g(x)] = -[f(x) \cdot g(x)] = -h(x)$, so their product is odd. Because multiplication is commutative, the order we multiply the functions in doesn't matter.

We note that the table would be the same as if we were *adding* (not multiplying) even and odd *numbers* (not functions).

S-21: Since $f(x)$ is odd, $f(0) = -f(-0) = -f(0)$. So, $f(0) = 0$.

However, this restriction does not apply to $g(x)$. For example, for any constant c, let $g(x) = c$. Then $g(x)$ is even and $g(0) = c$. So, $g(0)$ can be any real number.

S-22: Let x be any real number.

- $f(x) = f(-x)$ (since $f(x)$ is even), and
- $f(x) = -f(-x)$ (since $f(x)$ is odd).
- So, $f(x) = -f(x)$.
- Then (adding $f(x)$ to both sides) we see $2f(x) = 0$, so $f(x) = 0$.

So, $f(x) = 0$ for every x.

S-23:

Solution 1: Suppose $f(x)$ is an odd function. We investigate $f'(x)$ using the chain rule:

$$\begin{aligned} f(-x) &= -f(x) && \text{(odd function)} \\ \frac{\mathrm{d}}{\mathrm{d}x}\{f(-x)\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{-f(x)\} \\ -f'(-x) &= -f'(x) && \text{(chain rule)} \\ f'(-x) &= f'(x) \end{aligned}$$

So, when $f(x)$ is odd, $f'(x)$ is even.

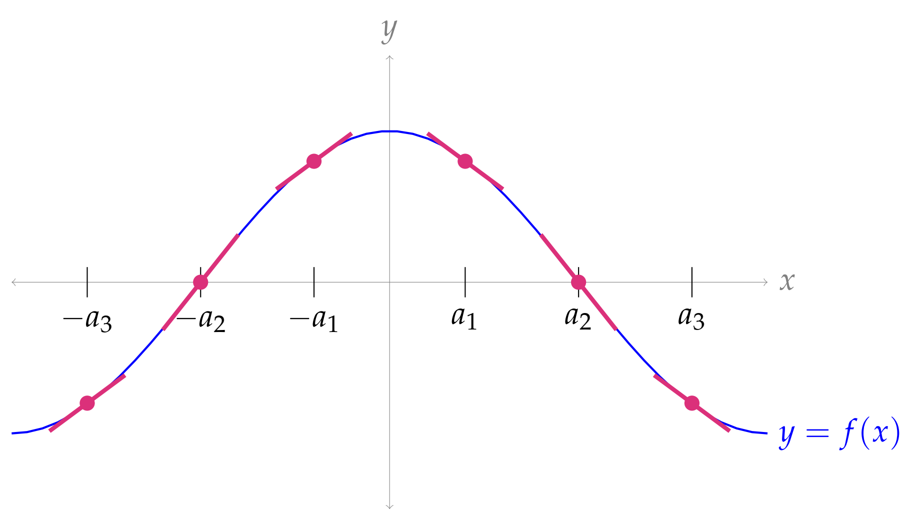
Similarly, suppose $f(x)$ is even.

$$
\begin{aligned}
f(-x) &= f(x) && \text{(even function)}\\
\frac{\mathrm{d}}{\mathrm{d}x}\{f(-x)\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)\}\\
-f'(-x) &= f'(x) && \text{(chain rule)}\\
f'(-x) &= -f'(x)
\end{aligned}
$$

So, when $f(x)$ is even, $f'(x)$ is odd.

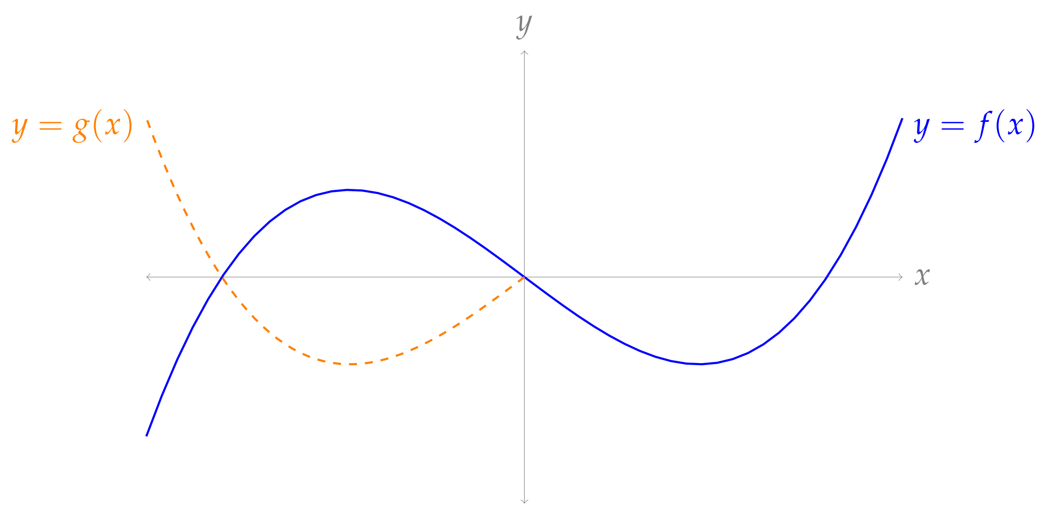
Solution 2: Another way to think about this problem is to notice that "mirroring" a function changes the sign of its derivative. Then since an even function is "mirrored once" (across the y-axis), it should have $f'(x) = -f'(-x)$, and so the derivative of an even function should be an odd function. Since an odd function is "mirrored twice" (across the y-axis and across the x-axis), it should have $f'(x) = -(-f'(-x)) = f'(-x)$. So the derivative of an odd function should be even. These ideas are presented in more detail below.

First, we consider the case where $f(x)$ is even, and investigate $f'(x)$.

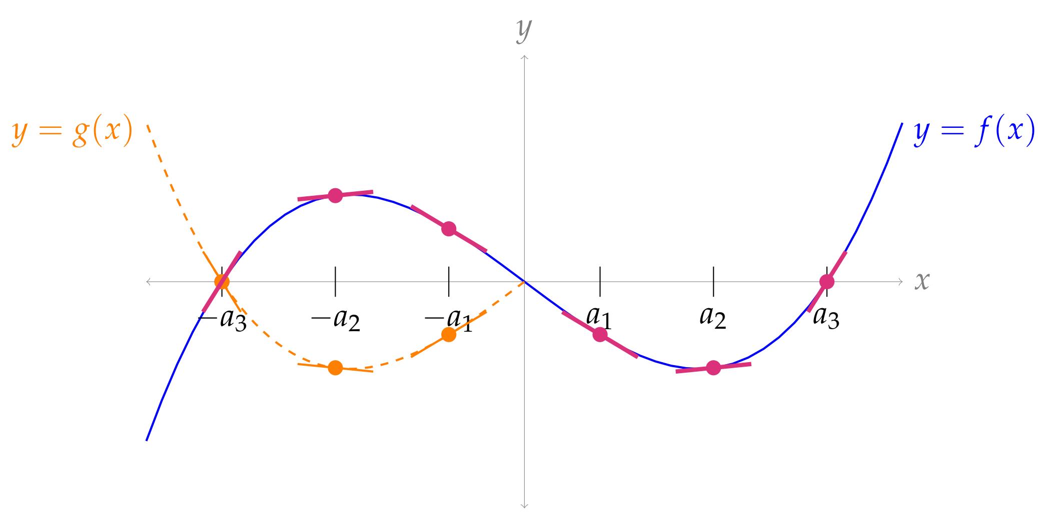


The whole function has a mirror-like symmetry across the y-axis. So, at x and $-x$, the function will have the same "steepness," but if one is increasing then the other is decreasing. That is, $f'(-x) = -f'(x)$. (In the picture above, compare the slope at some point a_i with its corresponding point $-a_i$.) So, $f'(x)$ is odd when $f(x)$ is even.

Second, let's consider the case where $f(x)$ is odd, and investigate $f'(x)$. Suppose the blue graph below is $y = f(x)$. If $f(x)$ were *even*, then to the left of the y-axis, it would look like the orange graph, which we'll call $y = g(x)$.



From our work above, we know that, for every $x > 0$, $-f'(x) = g'(-x)$. When $x < 0$, $f(x) = -g(x)$. So, if $x > 0$, then $-f'(x) = g'(-x) = -f'(-x)$. In other words, $f'(x) = f'(-x)$. Similarly, if $x < 0$, then $f'(x) = -g'(x) = f'(-x)$. Therefore $f'(x)$ is even. (In the graph below, you can anecdotally verify that $f'(a_i) = f'(-a_i)$.)



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Solutions to Exercises 3.3 — Jump to TABLE OF CONTENTS

S-1: The Fundamental Theorem of Calculus Part 2 (Theorem 3.3.1 in the text) tells us that

$$\begin{aligned}\int_1^{\sqrt{5}} f(x)\,\mathrm{d}x &= F(\sqrt{5}) - F(1) \\ &= \left(e^{(\sqrt{5}^2-3)}+1\right) - \left(e^{(1^2-3)}+1\right) \\ &= e^{5-3} - e^{1-3} = e^2 - e^{-2}\end{aligned}$$

S-2: First, let's find a general antiderivative of $x^3 - \sin(2x)$.

- One function with derivative x^3 is $\frac{x^4}{4}$.
- To find an antiderivative of $\sin(2x)$, we might first guess $\cos(2x)$; checking, we see $\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(2x)\} = -2\sin(2x)$. So, we only need to multiply by $-\frac{1}{2}$: $\frac{\mathrm{d}}{\mathrm{d}x}\left\{-\frac{1}{2}\cos 2x\right\} = \sin(2x)$.

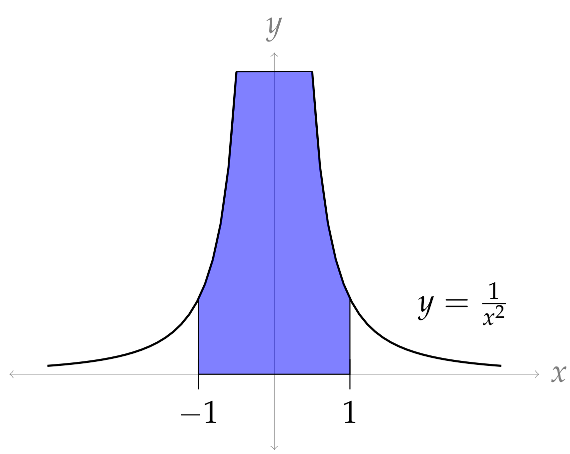
So, the general antiderivative of $f(x)$ is $\frac{x^4}{4} + \frac{1}{2}\cos 2x + C$. To satisfy $F(0) = 1$, we need[11]

$$\left[\frac{x^4}{4} + \frac{1}{2}\cos 2x + C\right]_{x=0} = 1 \iff \frac{1}{2} + C = 1 \iff C = \frac{1}{2}$$

So $F(x) = \frac{x^4}{4} + \frac{1}{2}\cos 2x + \frac{1}{2}$.

S-3: (a) This is true, by part 2 of the Fundamental Theorem of Calculus, Thereom 3.3.1 in the text with $G(x) = f(x)$ and $f(x)$ replaced by $f'(x)$.

(b) This is not only false, but it makes no sense at all. The integrand is strictly positive so the integral has to be strictly positive. In fact it's $+\infty$. The Fundamental Theorem of Calculus does not apply because the integrand has an infinite discontinuity at $x = 0$.



11 The symbol $\iff$ is read "if and only if". This is used in mathematics to express the logical equivalence of two statements. To be more precise, the statement $P \iff Q$ tells us that P is true whenever Q is true and Q is true whenever P is true.

(c) This is not only false, but it makes no sense at all, unless $\int_a^b f(x)\,\mathrm{d}x = \int_a^b xf(x)\,\mathrm{d}x = 0$. The left hand side is a number. The right hand side is a number times x.

$$\underbrace{\int_a^b xf(x)\,\mathrm{d}x}_{\text{area}} \qquad \text{vs} \qquad \underbrace{x}_{\text{variable}} \cdot \underbrace{\int_a^b f(x)\,\mathrm{d}x}_{\text{area}}$$

For example, if $a = 0$, $b = 1$ and $f(x) = 1$, then the left hand side is $\int_0^1 x\,\mathrm{d}x = \frac{1}{2}$ and the right hand side is $x\int_0^1 \mathrm{d}x = x$.

S-4: This is a tempting thought:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

so perhaps similarly

$$\int \frac{1}{x^2}\,\mathrm{d}x \stackrel{?}{=} \ln|x^2| + C = \ln(x^2) + C$$

We check by differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\ln(x^2)\} = \frac{\mathrm{d}}{\mathrm{d}x}\{2\ln x\} = \frac{2}{x} \neq \frac{1}{x^2}$$

So, it wasn't so easy: false.

When we're guessing antiderivatives, we often need to adjust our original guesses a little. Changing constants works well; changing functions usually does not.

S-5: This is tempting:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(e^x)\} = e^x\cos(e^x)$$

so perhaps

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin(e^x)}{e^x}\right\} \stackrel{?}{=} \cos(e^x)$$

We check by differentiating:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin(e^x)}{e^x}\right\} &= \frac{e^x\left(\cos(e^x)\cdot e^x\right) - \sin(e^x)e^x}{e^{2x}} && \text{(quotient rule)}\\ &= \cos(e^x) - \frac{\sin(e^x)}{e^x}\\ &\neq \cos(e^x)\end{aligned}$$

So, the statement is false.

When we're guessing antiderivatives, we often need to adjust our original guesses a little. Dividing by constants works well; dividing by functions usually does not.

S-6: "The instantaneous rate of change of $F(x)$ with respect to x" is another way of saying "$F'(x)$". From the Fundamental Theorem of Calculus Part 1, we know this is $\sin(x^2)$.

S-7: The slope of the tangent line to $y = F(x)$ when $x = 3$ is exactly $F'(3)$. By the Fundamental Theorem of Calculus Part 1, $F'(x) = e^{1/x}$. Then $F'(3) = e^{1/3} = \sqrt[3]{e}$.

S-8: For any constant C, $F(x) + C$ is an antiderivative of $f(x)$, because $\frac{\mathrm{d}}{\mathrm{d}x}\{F(x) + C\} = \frac{\mathrm{d}}{\mathrm{d}x}\{F(x)\} = f(x)$. So, for example, $F(x)$ and $F(x) + 1$ are both antiderivatives of $f(x)$.

S-9:

(a) We differentiate with respect to a. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$. To differentiate $\frac{1}{2}a\sqrt{1-a^2}$, we use the product and chain rules.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\left\{\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}\right\} &= 0 - \frac{1}{2}\cdot\frac{-1}{\sqrt{1-a^2}} + \left(\frac{1}{2}a\right)\cdot\frac{-2a}{2\sqrt{1-a^2}} + \frac{1}{2}\sqrt{1-a^2}\\ &= \frac{1}{2\sqrt{1-a^2}} - \frac{a^2}{2\sqrt{1-a^2}} + \frac{1-a^2}{2\sqrt{1-a^2}}\\ &= \frac{1-a^2+1-a^2}{2\sqrt{1-a^2}}\\ &= \frac{2(1-a^2)}{2\sqrt{1-a^2}}\\ &= \sqrt{1-a^2}\end{aligned}$$

(b) Let $G(x) = \frac{\pi}{4} - \frac{1}{2}\arccos(x) + \frac{1}{2}x\sqrt{1-x^2}$. We showed in part (a) that $G(x)$ is an antiderivative of $\sqrt{1-x^2}$. Since $F(x)$ is also an antiderivative of $\sqrt{1-x^2}$, $F(x) = G(x) + C$ for some constant C (this is Lemma 3.3.8 in the text).

Note $G(0) = \displaystyle\int_0^0 \sqrt{1-x^2}\,\mathrm{d}x = 0$, so if $F(0) = \pi$, then $F(x) = G(x) + \pi$. That is,

$$F(x) = \frac{5\pi}{4} - \frac{1}{2}\arccos(x) + \frac{1}{2}x\sqrt{1-x^2}\,.$$

S-10:

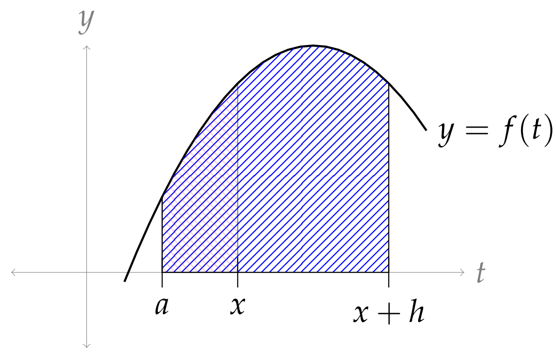
(a) The antiderivative of $\cos x$ is $\sin x$, and $\cos x$ is continuous everywhere, so $\displaystyle\int_{-\pi}^{\pi}\cos x\,\mathrm{d}x = \sin(\pi) - \sin(-\pi) = 0.$

(b) Since $\sec^2 x$ is discontinuous at $x = \pm\frac{\pi}{2}$, the Fundamental Theorem of Calculus Part 2 does not apply to $\displaystyle\int_{-\pi}^{\pi}\sec^2 x\,\mathrm{d}x$.

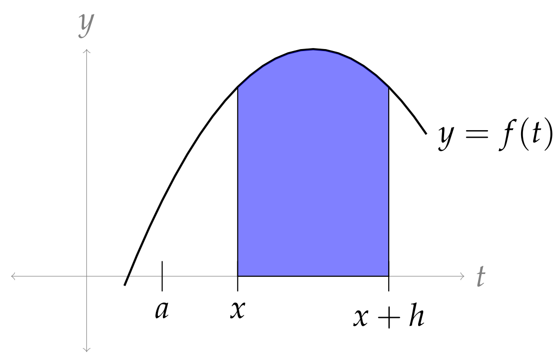
(c) Since $\frac{1}{x+1}$ is discontinuous at $x = -1$, the Fundamental Theorem of Calculus Part 2 does not apply to $\displaystyle\int_{-2}^{0} \frac{1}{x+1}\,dx$.

S-11:

Using the definition of F, $F(x)$ is the area under the curve from a to x, and $F(x+h)$ is the area under the curve from a to $x+h$. These are shown on the same diagram, below.



Then the area represented by $F(x+h) - F(x)$ is the area that is outside the red, but inside the blue. Equivalently, it is $\int\limits_{x}^{x+h} f(t)\,dt$.



S-12: We evaluate $F(0)$ using the definition: $F(0) = \int_0^0 f(t)\,dt = 0$. Although $f(0) > 0$, the *area* from $t = 0$ to $t = 0$ is zero.
As x moves along, $F(x)$ adds bits of signed area. If it's adding positive area, it's increasing, and if it's adding negative area, it's decreasing. So, $F(x)$ is increasing when $0 < x < 1$ and $3 < x < 4$, and $F(x)$ is decreasing when $1 < x < 3$.

S-13: This question is nearly identical to Question 12, with

$$G(x) = \int_x^0 f(t)\,dt = -\int_0^x f(t)\,dt = -F(x).$$

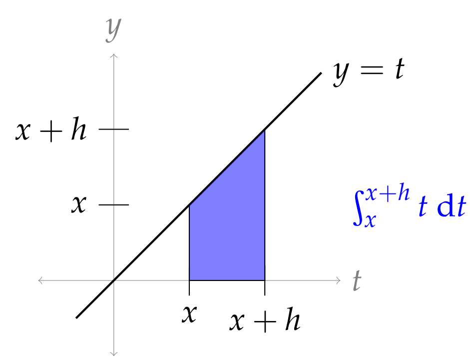
So, $G(x)$ increases when $F(x)$ decreases, and vice-versa. Therefore: $G(0) = 0$, $G(x)$ is increasing when $1 < x < 3$, and $G(x)$ is decreasing when $0 < x < 1$ and when $3 < x < 4$.

S-14: Using the definition of the derivative,

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{\int_a^{x+h} t\,\mathrm{d}t - \int_a^x t\,\mathrm{d}t}{h} \\ &= \lim_{h\to 0} \frac{\int_x^{x+h} t\,\mathrm{d}t}{h} \end{aligned}$$

The numerator describes the area of a trapezoid with base h and heights x and $x+h$.

$$\begin{aligned} &= \lim_{h\to 0} \frac{\frac{1}{2}h(x+x+h)}{h} \\ &= \lim_{h\to 0} \left(x + \frac{1}{2}h\right) \\ &= x \end{aligned}$$



So, $F'(x) = x$.

S-15: If $F(x)$ is constant, then $F'(x) = 0$. By the Fundamental Theorem of Calculus Part 1, $F'(x) = f(x)$. So, the only possible continuous function fitting the question is $f(x) = 0$.

This makes intuitive sense: if moving x doesn't add or subtract area under the curve, then there must not be any area under the curve–the curve should be the same as the x-axis.

As an aside, we mention that there are other, *non-continuous* functions $f(t)$ such that $\int_0^x f(t)\,\mathrm{d}t = 0$ for all x. For example, $f(t) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$. These kinds of removable discontinuities will not factor heavily in our discussion of integrals.

S-16:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\{x\ln(ax) - x\} &= x\left(\frac{a}{ax}\right) + \ln(ax) - 1 && \text{(product rule, chain rule)} \\ &= \ln(ax) \end{aligned}$$

So, we know

$$\int \ln(ax)\,\mathrm{d}x = x\ln(ax) - x + C \qquad \text{where } a \text{ is a given constant, and } C \text{ is any constant.}$$

Remark: $\int \ln(ax)\,\mathrm{d}x$ can be calculated using the method of Integration by Parts, which you will learn in Section 3.5 of the text.

S-17:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^x\left(x^3-3x^2+6x-6\right)\right\} &= e^x\left(3x^2-6x+6\right)+e^x\left(x^3-3x^2+6x-6\right) \quad \text{(product rule)}\\ &= e^x\left(3x^2-6x+6+x^3-3x^2+6x-6\right)\\ &= x^3e^x\end{aligned}$$

So,

$$\int x^3e^x\,\mathrm{d}x = e^x\left(x^3-3x^2+6x-6\right)+C$$

Remark: $\int x^3e^x\,\mathrm{d}x$ can be calculated using the method of Integration by Parts, which you will learn in Section 3.5 of the text.

S-18:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\ln\left|x+\sqrt{x^2+a^2}\right|\right\} &= \frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(1+\frac{1}{2\sqrt{x^2+a^2}}\cdot 2x\right) \qquad \text{(chain rule)}\\ &= \frac{1+\frac{x}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}} = \frac{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}\\ &= \frac{1}{\sqrt{x^2+a^2}}\end{aligned}$$

So,

$$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2+a^2}\right|+C$$

Remark: $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x$ can be calculated using the method of Trigonometric Substitution, which you will learn in Section 3.7 of the text.

S-19: Using the chain rule:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{x(a+x)} - a\ln\left(\sqrt{x}+\sqrt{a+x}\right)\right\} \\
&\quad= \frac{x+(a+x)}{2\sqrt{x(a+x)}} - a\left(\frac{1}{\sqrt{x}+\sqrt{a+x}}\cdot\left(\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{a+x}}\right)\right) \\
&\quad= \frac{2x+a}{2\sqrt{x(a+x)}} - a\left(\frac{1}{\sqrt{x}+\sqrt{a+x}}\cdot\left(\frac{\sqrt{a+x}+\sqrt{x}}{2\sqrt{x(a+x)}}\right)\right) \\
&\quad= \frac{2x+a}{2\sqrt{x(a+x)}} - a\left(\frac{1}{2\sqrt{x(a+x)}}\right) \\
&\quad= \frac{2x}{2\sqrt{x(a+x)}} = \frac{x}{\sqrt{x(a+x)}}
\end{aligned}$$

So,

$$\int \frac{x}{\sqrt{x(a+x)}}\,\mathrm{d}x = \sqrt{x(a+x)} - a\ln\left(\sqrt{x}+\sqrt{a+x}\right) + C$$

Remark: $\int \frac{x}{\sqrt{x(a+x)}}\,\mathrm{d}x$ can be calculated using the method of Trigonometric Substitution, which you will learn in Section 3.7 of the text.

S-20: By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_0^2 \left(x^3+\sin x\right)\mathrm{d}x &= \left[\frac{x^4}{4}-\cos x\right]_0^2 \\
&= \left(\frac{2^4}{4}-\cos 2\right) - (0-\cos 0) \\
&= 4-\cos 2+1 = 5-\cos 2.
\end{aligned}$$

S-21: By part (d) of our "Arithmetic of Integration" theorem, Theorem 3.2.1 in the text,

$$\int_1^2 \frac{x^2+2}{x^2}\,\mathrm{d}x = \int_1^2 \left[1+\frac{2}{x^2}\right]\mathrm{d}x = \int_1^2 \mathrm{d}x + 2\int_1^2 \frac{1}{x^2}\,\mathrm{d}x$$

Then by the Fundamental Theorem of Calculus Part 2,

$$\int_1^2 \mathrm{d}x + 2\int_1^2 \frac{1}{x^2}\,\mathrm{d}x = \Big[x\Big]_1^2 + 2\Big[-\frac{1}{x}\Big]_1^2 = [2-1] + 2\Big[-\frac{1}{2}+1\Big] = 2$$

S-22: The integrand is similar to $\dfrac{1}{1+x^2}$, which is the derivative of arctangent. Indeed, we have

$$\int \frac{1}{1+25x^2}\mathrm{d}x = \int \frac{1}{1+(5x)^2}\mathrm{d}x.$$

So, a reasonable first guess for the antiderivative might be

$$F(x) \stackrel{?}{=} \arctan(5x).$$

However, because of the chain rule,

$$F'(x) = \frac{5}{1+(5x)^2}.$$

In order to "fix" the numerator, we make a second guess:

$$F(x) = \frac{1}{5}\arctan(5x)$$

$$F'(x) = \frac{1}{5}\left(\frac{5}{1+(5x)^2}\right) = \frac{1}{1+25x^2}$$

So, $$\int \frac{1}{1+25x^2}\mathrm{d}x = \frac{1}{5}\arctan(5x) + C.$$

S-23: The integrand is similar to $\frac{1}{\sqrt{1-x^2}}$. In order to formulate a guess for the antiderivative, let's factor out $\sqrt{2}$ from the denominator:

$$\begin{aligned}
\int \frac{1}{\sqrt{2-x^2}}\mathrm{d}x &= \int \frac{1}{\sqrt{2\left(1-\frac{x^2}{2}\right)}}\mathrm{d}x \\
&= \int \frac{1}{\sqrt{2}\sqrt{1-\frac{x^2}{2}}}\mathrm{d}x \\
&= \int \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\mathrm{d}x
\end{aligned}$$

At this point, we might guess that our antiderivative is something like $F(x) = \arcsin\left(\frac{x}{\sqrt{2}}\right)$. To explore this possibility, we can differentiate, and see what we get.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\arcsin\left(\frac{x}{\sqrt{2}}\right)\right\} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}$$

This is exactly what we want! So,

$$\int \frac{1}{\sqrt{2-x^2}}\mathrm{d}x = \arcsin\left(\frac{x}{\sqrt{2}}\right) + C$$

S-24: We know that $\int \sec^2 x \, dx = \tan x + C$, and $\sec^2 x = \tan^2 x + 1$, so

$$\begin{aligned}
\int \tan^2 x \, dx &= \int \sec^2 x - 1 \, dx \\
&= \int \sec^2 x \, dx - \int 1 \, dx \\
&= \tan x - x + C
\end{aligned}$$

S-25:

Solution 1: This might not obviously look like the derivative of anything familiar, but it does look like half of a familiar trig identity: $2 \sin x \cos x = \sin(2x)$.

$$\begin{aligned}
\int 3 \sin x \cos x \, dx &= \int \frac{3}{2} \cdot 2 \sin x \cos x \, dx \\
&= \int \frac{3}{2} \sin(2x) \, dx
\end{aligned}$$

So, we might guess that the antiderivative is something like $-\cos(2x)$. We only need to figure out the constants.

$$\begin{aligned}
& \frac{d}{dx}\{-\cos(2x)\} = 2 \sin(2x) \\
\text{So,} \qquad & \frac{d}{dx}\left\{-\frac{3}{4}\cos(2x)\right\} = \frac{3}{2}\sin(2x) \\
\text{Therefore,} \qquad & \int 3 \sin x \cos x \, dx = -\frac{3}{4}\cos(2x) + C
\end{aligned}$$

Solution 2: You might notice that the integrand looks like it came from the chain rule, since $\cos x$ is the derivative of $\sin x$. Using this observation, we can work out the antideriative:

$$\begin{aligned}
& \frac{d}{dx}\left\{\sin^2 x\right\} = 2 \sin x \cos x \\
& \frac{d}{dx}\left\{\frac{3}{2}\sin^2 x\right\} = 3 \sin x \cos x \\
\text{So,} \qquad & \int 3 \sin x \cos x \, dx = \frac{3}{2}\sin^2 x + C
\end{aligned}$$

These two answers look different. Using the identity $\cos(2x) = 1 - 2\sin^2(x)$, we reconcile them:

$$\begin{aligned}
-\frac{3}{4}\cos(2x) + C &= -\frac{3}{4}\left(1 - 2\sin^2 x\right) + C \\
&= \frac{3}{2}\sin^2 x + \left(C - \frac{3}{4}\right)
\end{aligned}$$

The $\frac{3}{4}$ here is not significant. Remember that C is used to designate a constant that can take any value between $-\infty$ and $+\infty$. So $C - \frac{3}{4}$ is also just a constant that can take any value between $-\infty$ and $+\infty$. As the two answers we found differ by a constant, they are equivalent.

S-26: It's not immediately obvious which function has $\cos^2 x$ as its derivative, but we can make the situation a little clearer by using the identity $\cos^2 x = \dfrac{1+\cos(2x)}{2}$:

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{1}{2} \cdot (1+\cos(2x)) \, dx \\ &= \int \frac{1}{2} \, dx + \int \frac{1}{2} \cos(2x) \, dx \\ &= \frac{1}{2}x + C + \int \frac{1}{2} \cos(2x) \, dx\end{aligned}$$

For the remaining integral, we might guess something like $F(x) = \sin(2x)$. Let's figure out the appropriate constant:

$$\begin{aligned} & \frac{d}{dx}\{\sin(2x)\} = 2\cos(2x) \\ & \frac{d}{dx}\left\{\frac{1}{4}\sin(2x)\right\} = \frac{1}{2}\cos(2x) \\ \text{So,} \quad & \int \frac{1}{2}\cos(2x)\, dx = \frac{1}{4}\sin(2x) + C \\ \text{Therefore,} \quad & \int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C \end{aligned}$$

S-27: By the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_0^x \ln(2+\sin t)\, dt &&= \ln(2+\sin x) \\ G'(y) &= \frac{d}{dy}\left[-\int_0^y \ln(2+\sin t)\, dt\right] &&= -\ln(2+\sin y) \end{aligned}$$

So,

$$F'\left(\frac{\pi}{2}\right) = \ln 3 \qquad G'\left(\frac{\pi}{2}\right) = -\ln(3)$$

S-28: By the Fundamental Theorem of Calculus Part 1,

$$f'(x) = 100(x^2 - 3x + 2)e^{-x^2} = 100(x-1)(x-2)e^{-x^2}$$

As $f(x)$ is increasing whenever $f'(x) > 0$ and $100e^{-x^2}$ is always strictly bigger than 0, we have $f(x)$ increasing if and only if $(x-1)(x-2) > 0$, which is the case if and only if

$(x-1)$ and $(x-2)$ are of the same sign. Both are positive when $x>2$ and both are negative when $x<1$. So $f(x)$ is increasing when $-\infty<x<1$ and when $2<x<\infty$.

Remark: even without the Fundamental Theorem of Calculus, since $f(x)$ is the area under a curve from 1 to x, $f(x)$ is increasing when the curve is above the x-axis (because we're adding positive area), and it's decreasing when the curve is below the x-axis (because we're adding negative area).

S-29: Write $G(x)=\displaystyle\int_0^x \frac{1}{t^3+6}\,\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $G'(x)=\dfrac{1}{x^3+6}$. Since $F(x)=G(\cos x)$, the chain rule gives us

$$F'(x)=G'(\cos x)\cdot(-\sin x)=-\frac{\sin x}{\cos^3 x+6}$$

S-30: Define $g(x)=\displaystyle\int_0^x e^{t^2}\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $g'(x)=e^{x^2}$. As $f(x)=g(1+x^4)$ the chain rule gives us

$$f'(x)=4x^3g'(1+x^4)=4x^3e^{(1+x^4)^2}$$

S-31: Define $g(x)=\int_0^x(t^6+8)\mathrm{d}t$. By the fundamental theorem of calculus, $g'(x)=x^6+8$. We are to compute the derivative of $f(x)=g(\sin x)$. The chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_0^{\sin x}(t^6+8)\mathrm{d}t\right\}=g'(\sin x)\cdot\cos x=\big(\sin^6 x+8\big)\cos x$$

S-32: Let $G(x)=\displaystyle\int_0^x e^{-t}\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $G'(x)=e^{-x}\sin\left(\frac{\pi x}{2}\right)$ and, since $F(x)=G(x^3)$, $F'(x)=3x^2G'(x^3)=3x^2e^{-x^3}\sin\left(\frac{\pi x^3}{2}\right)$. Then $F'(1)=3e^{-1}\sin\left(\frac{\pi}{2}\right)=3e^{-1}$.

S-33: Define $G(x)=\displaystyle\int_x^0\frac{\mathrm{d}t}{1+t^3}=-\int_0^x\frac{1}{1+t^3}\,\mathrm{d}t$, so that $G'(x)=-\dfrac{1}{1+x^3}$ by the Fundamental Theorem of Calculus Part 1. Then by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}u}\left\{\int_{\cos u}^0\frac{\mathrm{d}t}{1+t^3}\right\}=\frac{\mathrm{d}}{\mathrm{d}u}G(\cos u)=G'(\cos u)\cdot\frac{\mathrm{d}}{\mathrm{d}u}\cos u=-\frac{1}{1+\cos^3 u}\cdot(-\sin u).$$

S-34: Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $x^2 = 1 + \int_1^x f(t)\,\mathrm{d}t$ gives, by the Fundamental Theorem of Calculus Part 1, $2x = f(x)$.

S-35: Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $x\sin(\pi x) = \int_0^x f(t)\,\mathrm{d}t$. Then, by the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\{x\sin(\pi x)\} \\ \implies \qquad f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\{x\sin(\pi x)\} = \sin(\pi x) + \pi x\cos(\pi x) \\ \implies \qquad f(4) &= \sin(4\pi) + 4\pi\cos(4\pi) = 4\pi \end{aligned}$$

S-36: (a) Write

$$F(x) = G(x^2) - H(-x) \qquad \text{with} \quad G(y) = \int_0^y e^{-t}\,\mathrm{d}t,\ H(y) = \int_0^y e^{-t^2}\,\mathrm{d}t$$

By the Fundamental Theorem of Calculus Part 1,

$$G'(y) = e^{-y}, \qquad H'(y) = e^{-y^2}$$

Hence, by the chain rule,

$$F'(x) = 2xG'(x^2) - (-1)H'(-x) = 2xe^{-(x^2)} + e^{-(-x)^2} = (2x+1)e^{-x^2}$$

(b) Observe that $F'(x) < 0$ for $x < -1/2$ and $F'(x) > 0$ for $x > -1/2$. Hence $F(x)$ is decreasing for $x < -1/2$ and increasing for $x > -1/2$, and $F(x)$ must take its minimum value when $x = -1/2$.

S-37: Define $G(y) = \displaystyle\int_0^y e^{\sin t}\,\mathrm{d}t$. Then:

$$\begin{aligned} F(x) &= \int_0^x e^{\sin t}\,\mathrm{d}t + \int_{x^4-x^3}^0 e^{\sin t}\,\mathrm{d}t = \int_0^x e^{\sin t}\,\mathrm{d}t - \int_0^{x^4-x^3} e^{\sin t}\,\mathrm{d}t \\ &= G(x) - G(x^4 - x^3) \end{aligned}$$

By the Fundamental Theorem of Calculus Part 1,

$$G'(y) = e^{\sin y}$$

Hence, by the chain rule,

$$\begin{aligned} F'(x) &= G'(x) - G'(x^4 - x^3)\,\frac{\mathrm{d}}{\mathrm{d}x}\{x^4 - x^3\} \\ &= G'(x) - G'(x^4 - x^3)\,(4x^3 - 3x^2) \\ &= e^{\sin x} - e^{\sin(x^4 - x^3)}\big(4x^3 - 3x^2\big) \end{aligned}$$

S-38: Define with $G(y) = \int_0^y \cos\left(e^t\right)\,\mathrm{d}t$. Then:

$$\begin{aligned}
F(x) &= \int_{x^5}^{-x^2} \cos\left(e^t\right)\,\mathrm{d}t = \int_0^{-x^2} \cos\left(e^t\right)\,\mathrm{d}t + \int_{x^5}^{0} \cos\left(e^t\right)\,\mathrm{d}t \\
&= \int_0^{-x^2} \cos\left(e^t\right)\,\mathrm{d}t - \int_0^{x^5} \cos\left(e^t\right)\,\mathrm{d}t \\
&= G(-x^2) - G(x^5)
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$G'(y) = \cos\left(e^y\right)$$

Hence, by the chain rule,

$$\begin{aligned}
F'(x) &= G'(-x^2)\,\frac{\mathrm{d}}{\mathrm{d}x}\{-x^2\} - G'(x^5)\,\frac{\mathrm{d}}{\mathrm{d}x}\{x^5\} \\
&= G'(-x^2)\,(-2x) - G'(x^5)\,(5x^4) \\
&= -2x\cos\left(e^{-x^2}\right) - 5x^4\cos\left(e^{x^5}\right)
\end{aligned}$$

S-39: Define with $G(y) = \int_0^y \sqrt{\sin t}\,\mathrm{d}t$ Then:

$$\begin{aligned}
F(x) &= \int_x^{e^x} \sqrt{\sin t}\,\mathrm{d}t \\
&= \int_0^{e^x} \sqrt{\sin t}\,\mathrm{d}t + \int_x^0 \sqrt{\sin t}\,\mathrm{d}t = \int_0^{e^x} \sqrt{\sin t}\,\mathrm{d}t - \int_0^x \sqrt{\sin t}\,\mathrm{d}t \\
&= G(e^x) - G(x)
\end{aligned}$$

By the Fundamental Theorem of Calculus Part 1,

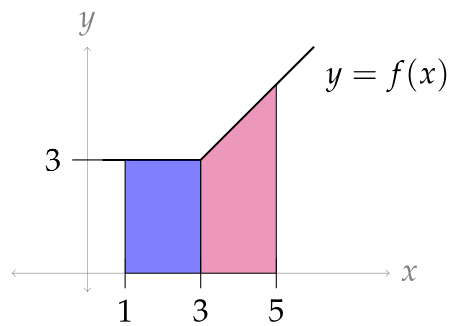
$$G'(y) = \sqrt{\sin y}$$

Hence, by the chain rule,

$$\begin{aligned}
F'(x) &= G'(e^x)\,\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\} - G'(x) \\
&= e^x G'(e^x) - G'(x) \\
&= e^x\sqrt{\sin(e^x)} - \sqrt{\sin(x)}
\end{aligned}$$

S-40: Splitting up the domain of integration,

$$\begin{aligned}
\int_1^5 f(x)\,\mathrm{d}x &= \int_1^3 f(x)\,\mathrm{d}x + \int_3^5 f(x)\,\mathrm{d}x \\
&= \int_1^3 3\,\mathrm{d}x + \int_3^5 x\,\mathrm{d}x \\
&= 3x\Big|_{x=1}^{x=3} + \frac{x^2}{2}\Big|_{x=3}^{x=5} \\
&= 14
\end{aligned}$$



S-41: By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\, (f'(x))^2 \,\right\} = 2f'(x)\,f''(x)$$

so $\frac{1}{2}f'(x)^2$ is an antiderivative for $f'(x)\,f''(x)$ and, by the Fundamental Theorem of Calculus Part 2,

$$\int_1^2 f'(x)f''(x)\,\mathrm{d}x = \left[\frac{1}{2}\,(f'(x))^2\right]_{x=1}^{x=2} = \frac{1}{2}f'(2)^2 - \frac{1}{2}f'(1)^2 = \frac{5}{2}$$

Remark: evaluating antiderivatives of this type will occupy the next section, Section 3.4 of the text.

S-42: The car stops when $v(t) = 30 - 10t = 0$, which occurs at time $t = 3$. The distance covered up to that time is

$$\int_0^3 v(t)\,\mathrm{d}t = (30t - 5t^2)\Big|_0^3 = (90 - 45) - 0 = 45\,\mathrm{m}.$$

S-43: Define $g(x) = \displaystyle\int_0^x \ln\left(1 + e^t\right)\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $g'(x) = \ln\left(1 + e^x\right)$. But $f(x) = g(2x - x^2)$, so by the chain rule,

$$f'(x) = g'(2x - x^2)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\{2x - x^2\} = (2 - 2x)\cdot\ln\left(1 + e^{2x - x^2}\right)$$

Observe that $e^{2x-x^2} > 0$ for all x so that $1 + e^{2x-x^2} > 1$ for all x and $\ln\left(1 + e^{2x-x^2}\right) > 0$ for all x. Since $2 - 2x$ is positive for $x < 1$ and negative for $x > 1$, $f'(x)$ is also positive for $x < 1$ and negative for $x > 1$. That is, $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$. So $f(x)$ achieves its absolute maximum at $x = 1$.

S-44: Let $f(x) = \int_0^{x^2-2x} \frac{dt}{1+t^4}$ and $g(x) = \int_0^x \frac{dt}{1+t^4}$. Then $g'(x) = \frac{1}{1+x^4}$ and, since $f(x) = g(x^2 - 2x)$, $f'(x) = (2x-2)g'(x^2-2x) = 2\frac{x-1}{1+(x^2-2x)^4}$. This is zero for $x = 1$, negative for $x < 1$ and positive for $x > 1$. Thus as x runs from $-\infty$ to ∞, $f(x)$ decreases until x reaches 1 and then increases all $x > 1$. So the minimum of $f(x)$ is achieved for $x = 1$. At $x = 1$, $x^2 - 2x = -1$ and $f(1) = \int_0^{-1} \frac{dt}{1+t^4}$.

S-45: Define $G(x) = \displaystyle\int_0^x \sin(\sqrt{t})\,dt$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = \sin(\sqrt{x})$. Since $F(x) = G(x^2)$, and since $x > 0$, we have

$$F'(x) = 2xG'(x^2) = 2x\sin|x| = 2x\sin x.$$

Thus F increases as x runs from to 0 to π (since $F'(x) > 0$ there) and decreases as x runs from π to 4 (since $F'(x) < 0$ there). Thus F achieves its maximum value at $x = \pi$.

S-46: The given sum is of the form

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\pi}{n} \sin\left(\frac{j\pi}{n}\right) = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$$

with $\Delta x = \frac{\pi}{n}$, $x_j^* = \frac{j\pi}{n}$ and $f(x) = \sin(x)$. Since $x_0^* = 0$ and $x_n^* = \pi$, the right hand side is the definition (using the right Riemann sum) of

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} \sin(x)\,dx = [-\cos(x)]_0^{\pi} = 2$$

where we evaluate the definite integral using the Fundamental Theorem of Calculus Part 2.

S-47: The given sum is of the form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{1+\frac{j}{n}} = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j = \frac{j}{n}$ and $f(x) = \frac{1}{1+x}$. The right hand side is the definition (using the right Riemann sum) of

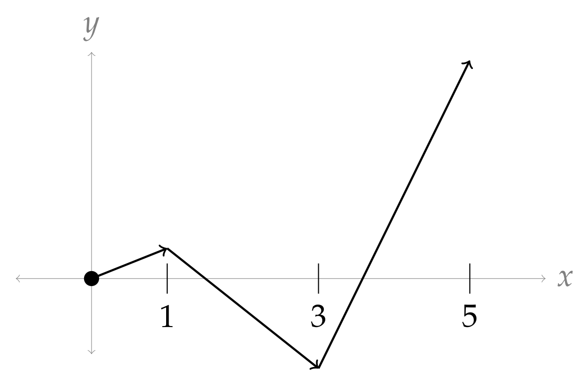
$$\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x}\,dx = \ln|1+x|\Big|_0^1 = \ln 2$$

S-48:

F(x), x ⩾ 0 We learned quite a lot last semester about curve sketching. We can use those techniques here. We have to be quite careful about the sign of x, though. We can only directly apply the Fundamental Theorem of Calculus Part 1 (as it's written in your text) when $x \geqslant 0$. So first, let's graph the right-hand portion. Notice $f(x)$ has even symmetry–so, if we know one half of $F(x)$, we should be able to figure out the other half with relative ease.

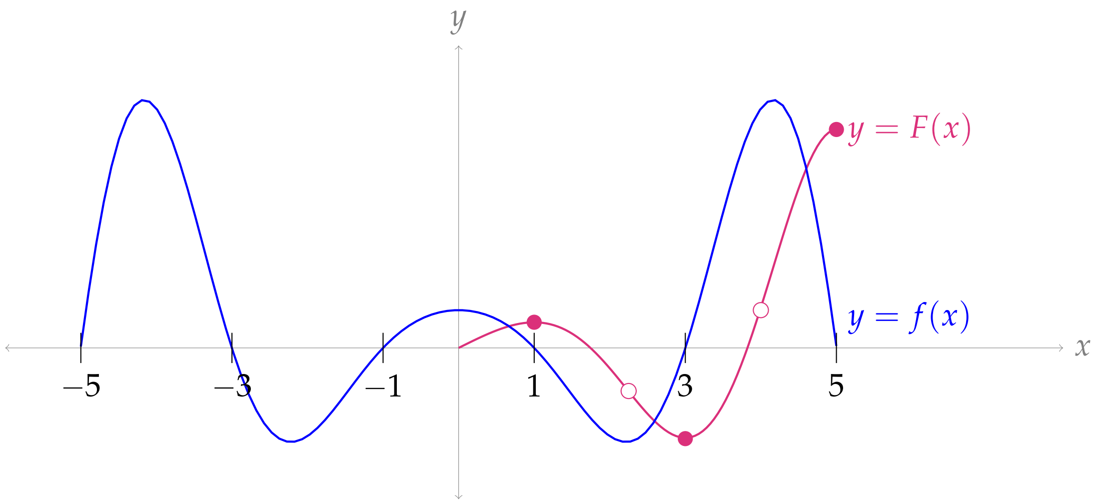
- $F(0) = \int_0^0 f(t)\,\mathrm{d}t = 0$ (so, $F(x)$ passes through the origin)

- Using the Fundamental Theorem of Calculus Part 1, $F'(x) > 0$ when $0 < x < 1$ and when $3 < x < 5$; $F'(x) < 0$ when $1 < x < 3$. So, $F(x)$ is decreasing from 1 to 3, and increasing from 0 to 1 and also from 3 to 5. That gives us a skeleton to work with.



We get the relative sizes of the maxes and mins by eyeballing the area under $y = f(t)$. The first lobe (from $x = 0$ to $x = 1$ has a small positive area, so $F(1)$ is a small positive number. The next lobe (from $x = 1$ to $x = 3$) has a larger absolute area than the first, so $F(3)$ is negative. Indeed, the second lobe seems to have more than twice the area of the first, so $|F(3)|$ should be larger than $F(1)$. The third lobe is larger still, and even after subtracting the area of the second lobe it looks much larger than the first or second lobe, so $|F(3)| < F(5)$.

- We can use $F''(x)$ to get the concavity of $F(x)$. Note $F''(x) = f'(x)$. We observe $f(x)$ is decreasing on (roughly) $(0, 2.5)$ and $(4, 5)$, so $F(x)$ is concave down on those intervals. Further, $f(x)$ is increasing on (roughly) $(2.5, 4)$, so $F(x)$ is concave up there, and has inflection points at about $x = 2.5$ and $x = 4$.

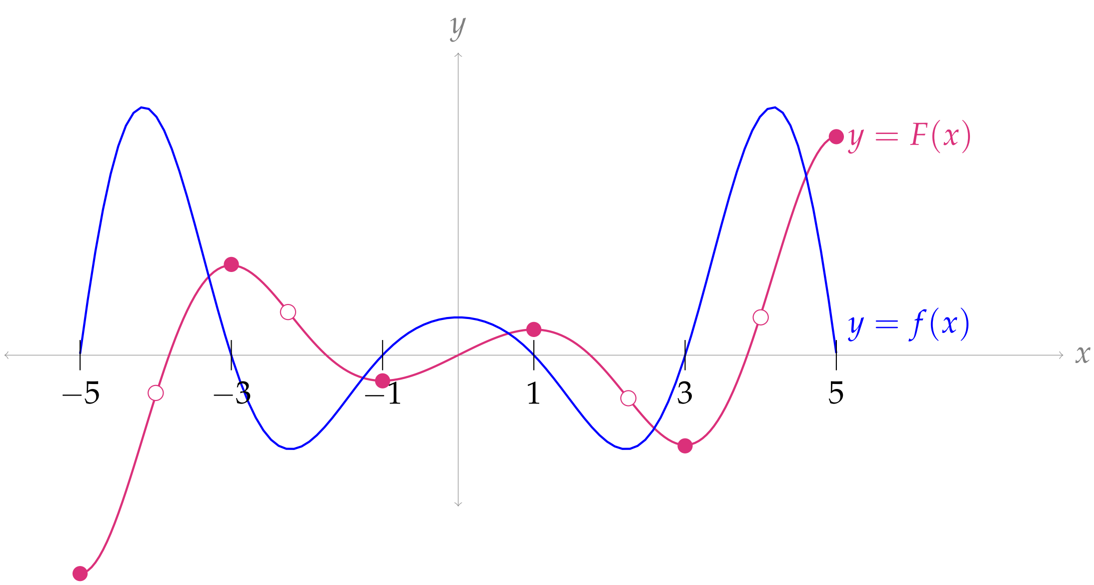


In the sketch above, closed dots are extrema, and open dots are inflection points.

F(**x**), **x** < **0** Now we can consider the left half of the graph. If you stare at it long enough, you might convince yourself that $F(x)$ is an odd function. We can also show this with the following calculation:

$$\begin{aligned} F(-x) &= \int_0^{-x} f(t)\,\mathrm{d}t && \text{As in Example 3.2.9 of the text, since } f(t) \text{ is even,} \\ &= \int_x^0 f(t)\,\mathrm{d}t = -\int_0^x f(t)\,\mathrm{d}t \\ &= -F(x) \end{aligned}$$

Knowing that $F(x)$ is odd allows us to finish our sketch.



S-49: (a) Using the product rule, followed by the chain rule, followed by the

Fundamental Theorem of Calculus Part 1,

$$\begin{aligned}
f'(x) &= 3x^2 \int_0^{x^3+1} e^{t^3} \mathrm{d}t + x^3 \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{x^3+1} e^{t^3} \mathrm{d}t \\
&= 3x^2 \int_0^{x^3+1} e^{t^3} \mathrm{d}t + x^3 \left[3x^2\right] \left[\frac{d}{dy} \int_0^{y} e^{t^3} \mathrm{d}t\right]_{y=x^3+1} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3} \mathrm{d}t + x^3 \left[3x^2\right] \left[e^{y^3}\right]_{y=x^3+1} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3} \mathrm{d}t + x^3 \left[3x^2\right] e^{(x^3+1)^3} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3} \mathrm{d}t + 3x^5 e^{(x^3+1)^3}
\end{aligned}$$

(b) In general, the equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)\,(x - a)$$

Substituting in the given $f(x)$ and $a = -1$:

$$\begin{aligned}
f(a) = f(-1) &= (-1)^3 \int_0^0 e^{t^3}\, \mathrm{d}t = 0 \\
f'(a) = f'(-1) &= 3(-1)^2 \int_0^0 e^{t^3}\, \mathrm{d}t + 3(-1)^5 e^0 \\
&= 0 - 3 = -3 \\
(x - a) = x - (-1) &= x + 1
\end{aligned}$$

So, the equation of the tangent line is

$$y = -3(x + 1)\,.$$

S-50: Recall that "+C" means that we can add any constant to the function. Since $\tan^2 x = \sec^2 x - 1$, Students A and B have equivalent answers: they only differ by a constant.

So, if one is right, both are right; if one is wrong, both are wrong. We check Student A's work:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan^2 x + x + C\} = \frac{\mathrm{d}}{\mathrm{d}x}\{\tan^2 x\} + 1 + 0 = f(x) - 1 + 1 = f(x)$$

So, Student A's answer is indeed an anditerivative of $f(x)$. Therefore, *both students* ended up with the correct answer.

Remark: it is a frequent occurrence that equivalent answers might look quite different. As you are comparing your work to others', this is a good thing to keep in mind!

S-51:

(a) When $x = 3$,

$$F(3) = \int_0^3 3^3 \sin(t)\, \mathrm{d}t = 27 \int_0^3 \sin t\, \mathrm{d}t$$

Using the Fundamental Theorem of Calculus Part 2,

$$\begin{aligned} &= 27\left[-\cos t\right]_{t=0}^{t=3} = 27\left[-\cos 3 - (-\cos 0)\right] \\ &= 27(1 - \cos 3) \end{aligned}$$

(b) Since the integration is with respect to t, the x^3 term can be moved outside the integral. That is: for the purposes of the integral, x^3 is a constant (although for the purposes of the derivative, it certainly is not).

$$F(x) = \int_0^x x^3 \sin(t)\, \mathrm{d}t = x^3 \int_0^x \sin(t)\, \mathrm{d}t$$

Using the product rule and the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned} F'(x) &= x^3 \cdot \sin(x) + 3x^2 \int_0^x \sin(t)\, \mathrm{d}t \\ &= x^3 \sin(x) + 3x^2 \left[-\cos(t)\right]_{t=0}^{t=x} \\ &= x^3 \sin(x) + 3x^2[-\cos(x) - (-\cos(0))] \\ &= x^3 \sin(x) + 3x^2[1 - \cos(x)] \end{aligned}$$

Remark: Since x and t play different roles in our problem, it's crucial that they have different names. This is one reason why we should avoid the common mistake of writing $\int_a^x f(x)\mathrm{d}x$ when we mean $\int_a^x f(t)\mathrm{d}t$.

S-52: If $F(x)$ is even, then $f(x)$ is odd (by the result of Question 23 in Section 1.2). So, $F(x)$ can only be even if $f(x)$ is both even and odd. By the result in Question 22, Section 1.2, this means $F(x)$ is only even if $f(x) = 0$ for all x. Note if $f(x) = 0$, then $F(x)$ is a constant function. So, it is certainly even, and it might be odd as well if $F(x) = f(x) = 0$.

Therefore, if $f(x) \neq 0$ for some x, then $F(x)$ is not even. It could be odd, or it could be neither even nor odd. We can come up with examples of both types: if $f(x) = 1$, then $F(x) = x$ is an odd antiderivative, and $F(x) = x + 1$ is an antiderivative that is neither even nor odd.

Interestingly, the antiderivative of an odd function is always even. The proof is a little beyond what we might ask you, but is given below for completeness. The proof goes like this: First, we'll show that if $g(x)$ is odd, then there is *some* antiderivative of $g(x)$ that is even. Then, we'll show that *every* antiderivative of $g(x)$ is even.

So, suppose $g(x)$ is odd and define $G(x) = \displaystyle\int_0^x g(t)dt$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = g(x)$, so $G(x)$ is an antiderivative of $g(x)$. Since $g(x)$ is odd, for any $x \geqslant 0$, the net signed area under the curve along $[0, x]$ is the *negative* of the net signed area under the curve along $[-x, 0]$. So,

$$\int_0^x g(t)\,\mathrm{d}t = -\int_{-x}^0 g(t)\,\mathrm{d}t \quad \text{(See Example 3.2.10 in the text)}$$
$$= \int_0^{-x} g(t)\,\mathrm{d}t$$

By the definition of $G(x)$,

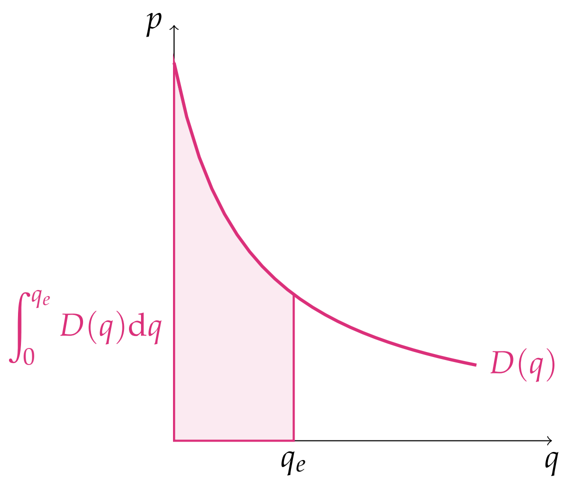
$$G(x) = G(-x)$$

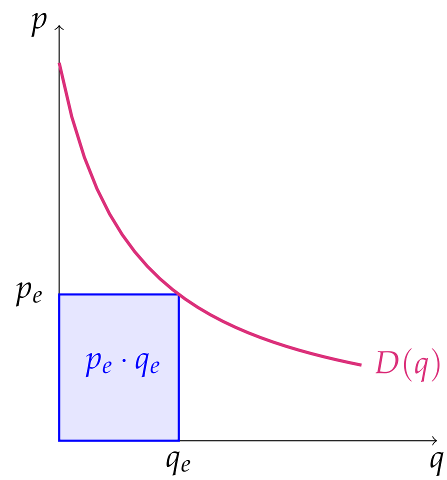
That is, $G(x)$ is even. We've shown that there exists *some* antiderivative of $g(x)$ that is even; it remains to show that *all of them* are even.

Recall that every antiderivative of $g(x)$ differs from $G(x)$ by some constant. So, any antiderivative of $g(x)$ can be written as $G(x) + C$, and $G(-x) + C = G(x) + C$. So, every antiderivative of an odd function is even.

S-53:

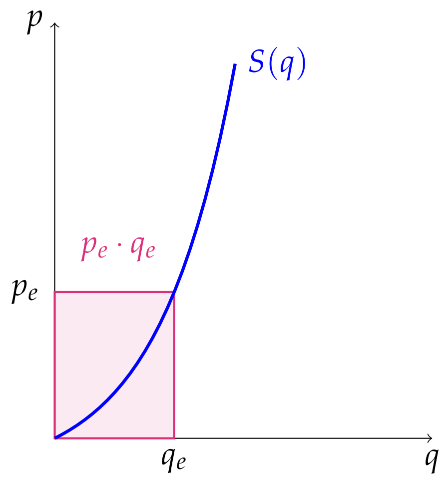
(a) To find the area C, we'll first find the area under the demand curve, then subtract the area under the constant curve $p = p_e$.

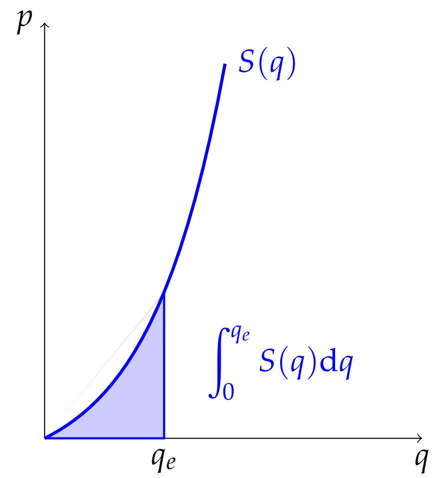




$$\mathrm{CS} = \left(\int_0^{q_e} D(q)\mathrm{d}q\right) - p_e \cdot q_e = \left(\int_0^{q_e} \frac{10}{q+1}\mathrm{d}q\right) - p_e \cdot q_e$$
$$= \left(10\int_0^{q_e} \frac{1}{q+1}\mathrm{d}q\right) - p_e \cdot q_e = 10\ln|q+1|\Big|_0^{q_e} - p_e \cdot q_e$$
$$= 10\ln|q_e+1| - 10\ln|1| - p_e \cdot q_e = 10\ln(q_e+1) - p_e \cdot q_e$$

(b) To find the area P, we'll first find the area under the constant curve $p = p_e$, then subtract the area under the supply curve.





$$\begin{aligned}\text{PS} &= p_e \cdot q_e - \int_0^{q_e} S(q)\mathrm{d}q = p_e \cdot q_e - \int_0^{q_e} (e^q - 1)\mathrm{d}q \\ &= p_e \cdot q_e - \Big[e^q - q\Big]_0^{q_e} = p_e \cdot q_e - \Big[(e^{q_e} - q_e) - (e^0 - 0)\Big] \\ &= p_e \cdot q_e - e^{q_e} + q_e + 1\end{aligned}$$

(c)

$$\begin{aligned}\text{TS} = \text{CS} + \text{PS} &= (10\ln(q_e + 1) - p_e \cdot q_e) + (p_e \cdot q_e - e^{q_e} + q_e + 1) \\ &= 10\ln(q_e + 1) - e^{q_e} + q_e + 1\end{aligned}$$

S-54:

(a) B is the area under the Lorenz curve, so $B = \int_0^1 L(x)\mathrm{d}x$. To find A, rather than use an integral, we note that it's the area of a triangle minus the area of B. The triangle is half of a unit square, so $A = \frac{1}{2} - B$. All together,

$$\frac{A}{A+B} = \frac{\frac{1}{2} - B}{\frac{1}{2} - B + B} = 1 - 2B = 1 - 2\int_0^1 L(x)\mathrm{d}x$$

This isn't the only correct answer, but it's probably the easiest to understand and calculate. For example, since $A + B = \frac{1}{2}$, we can also say $\frac{A}{A+B} = 2A$; however, the integral to find A looks mildly more complicated.

(b) If $L(x) = x$, then $A = 0$, so the Gini coefficient is 0.

(c) First, let's calculate B.

$$\begin{aligned} B = \int_0^2 L(x)\mathrm{d}x &= \int_0^2 \frac{x^6+x^2}{2}\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \left(x^6+x^2\right)\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{1}{7}x^7+\frac{1}{3}x^3\right]_0^1 \\ &= \frac{1}{2}\left[\frac{1}{7}+\frac{1}{3}\right] = \frac{1}{2}\left[\frac{10}{21}\right] \end{aligned}$$

So, the Gini coefficient is

$$1-2B = 1-\frac{10}{21} = \frac{11}{21} \approx 0.52$$

S-55:

(a) Using the Fundamental Theorem of Calculus,

$$TC = \int MC\mathrm{d}q + C$$

for some constant C.

$$\int MC\mathrm{d}q = \int\left(\frac{1}{q+1}+q+2\right)\mathrm{d}q = \ln|q+1| + \frac{1}{2}q^2+2q+C$$

Since $q \geqslant 0$,

$$TC = \ln(q+1)+\frac{1}{2}q^2+2q+C$$

Now we use the fact that TC(0) =FC = 1000

$$\begin{aligned} 1000 = TC(0) &= \ln(0+1)+\frac{1}{2}0^2+2(0)+C \\ 1000 &= C \\ TC &= \ln(q+1)+\frac{1}{2}q^2+2q+1000 \end{aligned}$$

In particular, if we want to make 2000 units, the total cost will be

$$\begin{aligned} TC(2000) &= \ln(2001)+\frac{2000^2}{2}+4000+1000 \\ &= \ln(2001)+2,005,000 \end{aligned}$$

(b) As in the above part,

$$TC = \int MC\ \mathrm{d}q + C$$

for some constant C.

$$\int MC\mathrm{d}q = \int \left(40 - 10q + \frac{e^q}{10}\right)\mathrm{d}q = 40q - 5q^2 + \frac{e^q}{10} + C$$

Now we use the fact that TC(0) =FC = 50,000

$$\begin{aligned}
50,000 = TC(0) &= 40(0) - 5(0^2) + \frac{e^0}{10} + C = \frac{1}{10} + C \\
49,999.90 &= C \\
TC &= 40q - 5q^2 + \frac{e^q}{10} + 49,999.90
\end{aligned}$$

In particular, if we want to make 10 units, the total cost will be

$$\begin{aligned}
TC(10) &= 40(10) - 5(10^2) + \frac{e^{10}}{10} + 49,999.90 \\
&= 50,379.90 + \frac{e^{10}}{10}
\end{aligned}$$

S-56:

(a) By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\mathrm{TR} &= \int \mathrm{MR}\ \mathrm{d}q + C \\
\int \mathrm{MR}\ \mathrm{d}q &= \int \left(\cos(q) + \frac{q}{5} + 2\right)\mathrm{d}x \\
&= \sin q + \frac{q^2}{10} + 2q + C
\end{aligned}$$

To find C, we use $\mathrm{TR}(0) = 0$

$$\begin{aligned}
0 = \mathrm{TR}(0) &= \sin 0 + \frac{0^2}{10} + C \\
0 &= C
\end{aligned}$$

All together,

$$\mathrm{TR} = \sin q + \frac{q^2}{10} + 2q$$

The unit price is

$$P = \frac{\mathrm{TR}}{q} = \frac{\sin q}{q} + \frac{q}{10} + 2$$

(b) As in the previous part,

$$\begin{aligned}
\text{TR} &= \int \text{MR}\ \mathrm{d}q + C \\
\int \text{MR}\ \mathrm{d}q &= \int \left(\frac{e^q}{1000} + \frac{1}{2\sqrt{q}} \right) \mathrm{d}x \\
&= \frac{e^q}{1000} + \sqrt{q} + C
\end{aligned}$$

To find C, we use $\text{TR}(0) = 0$

$$\begin{aligned}
0 = \text{TR}(0) &= \frac{e^0}{1000} + \sqrt{0} + C = \frac{1}{1000} + C \\
-\frac{1}{1000} &= C
\end{aligned}$$

All together,

$$\text{TR} = \frac{e^q}{1000} + \sqrt{q} - \frac{1}{1000} = \frac{e^q - 1}{1000} + \sqrt{q}$$

The unit price is

$$P = \frac{\text{TR}}{q} = \frac{e^q + 1}{1000q} + \frac{1}{\sqrt{q}}$$

Solutions to Exercises 3.4 — Jump to TABLE OF CONTENTS

S-1: (a) This is true: it is an application of Theorem 3.4.2 in the text with $f(x) = \sin x$ and $u(x) = e^x$.

(b) This is false: the upper limit of integration is incorrect. Using Theorem 3.4.8 in the text, the correct form is

$$\int_0^1 \sin(e^x) \cdot e^x\ \mathrm{d}x = \int_1^e \sin(u)\ \mathrm{d}u = -\cos(e) + \cos(1) = \cos(1) - \cos(e).$$

Alternately, we can use the Fundamental Theorem of Calculus Part 2, and our answer from (a):

$$\int_0^1 \sin(e^x) \cdot e^x\ \mathrm{d}x = [-\cos(e^x) + C]_0^1 = \cos(1) - \cos(e)\ .$$

S-2: The reasoning is not sound: when we do a substitution, we need to take care of the differential ($\mathrm{d}x$). Remember the method of substitution comes from the chain rule: there should be a function *and its derivative*. Here's the way to do it:

Problem: Evaluate $\int (2x+1)^2 \mathrm{d}x$.

Work: We use the substitution $u = 2x+1$. Then $\mathrm{d}u = 2\mathrm{d}x$, so $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$:

$$\begin{aligned}\int (2x+1)^2 \mathrm{d}x &= \int u^2 \cdot \frac{1}{2}\,\mathrm{d}u \\ &= \frac{1}{6}u^3 + C \\ &= \frac{1}{6}\,(2x+1)^3 + C\end{aligned}$$

S-3: The problem is with the limits of integration, as in Question 1. Here's how it ought to go:

Problem: Evaluate $\int_1^{\pi} \frac{\cos(\ln t)}{t} \mathrm{d}t$.

Work: We use the substitution $u = \ln t$, so $\mathrm{d}u = \frac{1}{t}\mathrm{d}t$. When $t = 1$, we have $u = \ln 1 = 0$ and when $t = \pi$, we have $u = \ln(\pi)$. Then:

$$\begin{aligned}\int_1^{\pi} \frac{\cos(\ln t)}{t} \mathrm{d}t &= \int_{\ln 1}^{\ln(\pi)} \cos(u)\mathrm{d}u \\ &= \int_0^{\ln(\pi)} \cos(u)\mathrm{d}u \\ &= \sin(\ln(\pi)) - \sin(0) = \sin(\ln(\pi)).\end{aligned}$$

S-4: Perhaps shorter ways exist, but the reasoning here is valid.

Problem: Evaluate $\int_0^{\pi/4} x\tan(x^2)\,\mathrm{d}x$.

Work: We begin with the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$:
If $u = x^2$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, so indeed $\mathrm{d}u = 2x\mathrm{d}x$.

$$\begin{aligned}\int_0^{\pi/4} x\tan(x^2)\,\mathrm{d}x &= \int_0^{\pi/4} \frac{1}{2}\tan(x^2)\cdot 2x\mathrm{d}x && \text{algebra} \\ &= \int_0^{\pi^2/16} \frac{1}{2}\tan u\,\mathrm{d}u\end{aligned}$$

Every piece is changed from x to u: integrand, differential, limits.

$$= \frac{1}{2}\int_0^{\pi^2/16} \frac{\sin u}{\cos u}\mathrm{d}u \qquad \tan u = \frac{\sin u}{\cos u}$$

Now we use the substitution $v = \cos u$, $\mathrm{d}v = -\sin u\,\mathrm{d}u$:

$$= \frac{1}{2}\int_{\cos 0}^{\cos(\pi^2/16)} -\frac{1}{v}\mathrm{d}v$$

Every piece is changed from u to v: integrand, differential, limits.

$$\begin{aligned}
&= -\frac{1}{2}\int_1^{\cos(\pi^2/16)} \frac{1}{v}\mathrm{d}v && \cos(0) = 1\\
&= -\frac{1}{2}\Big[\ln|v|\Big]_1^{\cos(\pi^2/16)} && \text{FTC Part 2}\\
&= -\frac{1}{2}\left(\ln\left(\cos(\pi^2/16)\right) - \ln(1)\right)\\
&= -\frac{1}{2}\ln\left(\cos(\pi^2/16)\right) && \ln(1) = 0
\end{aligned}$$

S-5: We substitute:

$$\begin{aligned}
u &= \sin x,\\
\mathrm{d}u &= \cos x\,\mathrm{d}x,\\
\cos x &= \sqrt{1-\sin^2 x} = \sqrt{1-u^2},\\
\mathrm{d}x &= \frac{\mathrm{d}u}{\cos x} = \frac{\mathrm{d}u}{\sqrt{1-u^2}}\\
u(0) &= \sin 0 = 0\\
u\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) = 1
\end{aligned}$$

So,

$$\int_{x=0}^{x=\pi/2} f(\sin x)\,\mathrm{d}x = \int_{u=0}^{u=1} f(u)\,\frac{\mathrm{d}u}{\sqrt{1-u^2}}$$

S-6: Using the chain rule, we see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))g'(x)$$

So, $f(g(x))$ is an antiderivative of $f'(g(x))g'(x)$. All antiderivatives of $f'(g(x))g'(x)$ differ by only a constant, so:

$$\begin{aligned}
\int f'(g(x))g'(x)\,\mathrm{d}x - f(g(x)) &= f(g(x)) + C - f(g(x))\\
&= C
\end{aligned}$$

That is, our expression simplifies to some constant C.

Remark: since

$$\int f'(g(x))g'(x)\,\mathrm{d}t - f(g(x)) = C$$

we conclude

$$\int f'(g(x))g'(x)\,\mathrm{d}t = f(g(x)) + C$$

which is precisely how we perform substitution on integrals.

S-7: We write $u(x) = e^{x^2}$ and find $\mathrm{d}u = u'(x)\,\mathrm{d}x = 2xe^{x^2}\mathrm{d}x$. Note that $u(1) = e^{1^2} = e$ when $x = 1$, and $u(0) = e^{0^2} = 1$ when $x = 0$. Therefore:

$$\begin{aligned}
\int_0^1 xe^{x^2}\cos(e^{x^2})\,\mathrm{d}x &= \frac{1}{2}\int_{x=0}^{x=1}\cos(u(x))u'(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{u=1}^{u=e}\cos(u)\,\mathrm{d}u \\
&= \frac{1}{2}\Big[\sin(u)\Big]_1^e = \frac{1}{2}\big(\sin(e) - \sin(1)\big).
\end{aligned}$$

S-8: Substituting $y = x^3$, $\mathrm{d}y = 3x^2\,\mathrm{d}x$:

$$\int_1^2 x^2 f(x^3)\,\mathrm{d}x = \frac{1}{3}\int_1^8 f(y)\,\mathrm{d}y = \frac{1}{3}$$

S-9: Setting $u = x^3 + 1$, we have $\mathrm{d}u = 3x^2\,\mathrm{d}x$ and so

$$\begin{aligned}
\int \frac{x^2\,\mathrm{d}x}{(x^3+1)^{101}} &= \int \frac{\mathrm{d}u/3}{u^{101}} \\
&= \frac{1}{3}\int u^{-101}\,\mathrm{d}u \\
&= \frac{1}{3}\cdot\frac{u^{-100}}{-100} \\
&= -\frac{1}{3\times 100u^{100}} + C \\
&= -\frac{1}{300(x^3+1)^{100}} + C
\end{aligned}$$

S-10: Setting $u = \ln x$, we have $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and so

$$\int_e^{e^4}\frac{\mathrm{d}x}{x\cdot\ln x} = \int_{x=e}^{x=e^4}\frac{1}{\ln x}\cdot\frac{1}{x}\,\mathrm{d}x = \int_{u=1}^{u=4}\frac{1}{u}\,\mathrm{d}u,$$

since $u = \ln(e) = 1$ when $x = e$ and $u = \ln(e^4) = 4$ when $x = e^4$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_1^4 \frac{1}{u}\,\mathrm{d}u = \Big[\ln|u|\Big]_1^4 = \ln 4 - \ln 1 = \ln 4.$$

S-11: Setting $u = 1 + \sin x$, we have $\mathrm{d}u = \cos x \,\mathrm{d}x$ and so

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\mathrm{d}x = \int_{x=0}^{x=\pi/2} \frac{1}{1+\sin x} \cos x \,\mathrm{d}x = \int_{u=1}^{u=2} \frac{\mathrm{d}u}{u}$$

since $u = 1 + \sin 0 = 1$ when $x = 0$ and $u = 1 + \sin(\pi/2) = 2$ when $x = \pi/2$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_{u=1}^{u=2} \frac{\mathrm{d}u}{u} = \Big[\ln|u|\Big]_1^2 = \ln 2$$

S-12: Setting $u = \sin x$, we have $\mathrm{d}u = \cos x \,\mathrm{d}x$ and so

$$\int_0^{\pi/2} \cos x \cdot (1+\sin^2 x)\mathrm{d}x = \int_{x=0}^{x=\pi/2} (1+\sin^2 x)\cdot \cos x \,\mathrm{d}x = \int_{u=0}^{u=1} (1+u^2)\,\mathrm{d}u,$$

since $u = \sin 0 = 0$ when $x = 0$ and $u = \sin(\pi/2) = 1$ when $x = \pi/2$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_0^1 (1+u^2)\,\mathrm{d}u = \left[u + \frac{u^3}{3}\right]_0^1 = \left(1+\frac{1}{3}\right) - 0 = \frac{4}{3}.$$

S-13: Substituting $t = x^2 - x$, $\mathrm{d}t = (2x-1)\,\mathrm{d}x$ and noting that $t = 0$ when $x = 1$ and $t = 6$ when $x = 3$,

$$\int_1^3 (2x-1)e^{x^2-x}\mathrm{d}x = \int_0^6 e^t \,\mathrm{d}t = \left[e^t\right]_0^6 = e^6 - 1$$

S-14: We use the substitution $u = 4 - x^2$, for which $\mathrm{d}u = -2x\,\mathrm{d}x$:

$$\begin{aligned}
\int \frac{x^2-4}{\sqrt{4-x^2}}\,x\,\mathrm{d}x &= \int \frac{1}{2}\cdot\frac{4-x^2}{\sqrt{4-x^2}}(-2x)\,\mathrm{d}x \\
&= \frac{1}{2}\int \frac{u}{\sqrt{u}}\,\mathrm{d}u \\
&= \frac{1}{2}\int \sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \\
&= \frac{1}{3}(4-x^2)^{3/2} + C
\end{aligned}$$

S-15:

Solution 1: If we let $u = \sqrt{\ln x}$, then $du = \dfrac{1}{2x\sqrt{\ln x}}\,dx$, and:

$$\int \frac{e^{\sqrt{\ln x}}}{2x\sqrt{\ln x}}\,dx = \int e^u\,du = e^u + C = e^{\sqrt{\ln x}} + C$$

Solution 2: In Solution 1, we made a pretty slick choice. We might have tried to work with something a little less convenient. For example, it's not unnatural to think that $u = \ln x$, $du = \dfrac{1}{x}\,dx$ would be a good choice. In that case:

$$\int \frac{e^{\sqrt{\ln x}}}{2x\sqrt{\ln x}}\,dx = \int \frac{e^{\sqrt{u}}}{2\sqrt{u}} du$$

Now, we should be able to see that $w = \sqrt{u}$, $dw = \dfrac{1}{2\sqrt{u}}\,du$ is a good choice:

$$\begin{aligned}\int \frac{e^{\sqrt{u}}}{2\sqrt{u}}\,du &= \int e^w\,dw\\ &= e^{\sqrt{u}} + C\\ &= e^{\sqrt{\ln x}} + C\end{aligned}$$

S-16:

The slightly sneaky method: We note that $\dfrac{d}{dx}\left\{e^{x^2}\right\} = 2x\,e^{x^2}$, so that $\dfrac{1}{2}e^{x^2}$ is a antiderivative for the integrand xe^{x^2}. So

$$\int_{-2}^{2} xe^{x^2}\,dx = \left[\frac{1}{2}e^{x^2}\right]_{-2}^{2} = \frac{1}{2}e^4 - \frac{1}{2}e^4 = 0$$

The really sneaky method: The integrand $f(x) = xe^{x^2}$ is an odd function (meaning that $f(-x) = -f(x)$). So by Theorem 3.2.11 in the text every integral of the form $\int_{-a}^{a} xe^{x^2}\,dx$ is zero.

S-17: The given sum is of the form

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sin\left(1+\frac{j^2}{n^2}\right) = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j^*)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x\sin(1+x^2)$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}
\int_0^1 f(x)\,\mathrm{d}x &= \int_0^1 x\sin(1+x^2)\,\mathrm{d}x \\
&= \frac{1}{2}\int_1^2 \sin(y)\,\mathrm{d}y \qquad \text{with } y = 1+x^2,\ \mathrm{d}y = 2x\,\mathrm{d}x \\
&= \frac{1}{2}\Big[-\cos(y)\Big]_{y=1}^{y=2} \\
&= \frac{1}{2}[\cos 1 - \cos 2]
\end{aligned}$$

Using a calculator, we see this is close to 0.478.

S-18: Often, the denominator of a function is a good guess for the substitution. So, let's try setting $w = u^2+1$. Then $\mathrm{d}w = 2u\,\mathrm{d}u$:

$$\int_0^1 \frac{u^3}{u^2+1}\,\mathrm{d}u = \frac{1}{2}\int_0^1 \frac{u^2}{u^2+1}\,2u\,\mathrm{d}u$$

The numerator now is u^2, and looking at our substitution, we see $u^2 = w-1$:

$$\begin{aligned}
&= \frac{1}{2}\int_1^2 \frac{w-1}{w}\,\mathrm{d}w \\
&= \frac{1}{2}\int_1^2 \left(1-\frac{1}{w}\right)\mathrm{d}w \\
&= \frac{1}{2}\left[w - \ln|w|\right]_{w=1}^{w=2} \\
&= \frac{1}{2}(2-\ln 2 - 1) = \frac{1}{2} - \frac{1}{2}\ln 2
\end{aligned}$$

S-19: The only thing we really have to work with is a tangent, so it's worth considering what would happen if we substituted $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. This doesn't show up in the integrand as it's written, but we can try and bring it out by using the identity $\tan^2 = \sec^2\theta - 1$:

$$\begin{aligned}
\int \tan^3\theta\,\mathrm{d}\theta &= \int \tan\theta\cdot\tan^2\,\theta\mathrm{d}\theta \\
&= \int \tan\theta\cdot\left(\sec^2\,\theta - 1\right)\mathrm{d}\theta \\
&= \int \tan\theta\cdot\sec^2\,\theta\mathrm{d}\theta - \int\tan\theta\,\mathrm{d}\theta
\end{aligned}$$

In Example 3.4.17 of the text, we learned $\int \tan\theta \, d\theta = \ln|\sec\theta| + C$

$$\begin{aligned}
&= \int u \, du - \ln|\sec\theta| + C \\
&= \frac{1}{2}u^2 - \ln|\sec\theta| + C \\
&= \frac{1}{2}\tan^2\theta - \ln|\sec\theta| + C
\end{aligned}$$

S-20: At first glance, it's not clear what substitution to use. If we try the denominator, $u = e^x + e^{-x}$, then $du = (e^x - e^{-x})\,dx$, but it's not clear how to make this work with our integral. So, we can try something else.

If we want to tidy things up, we might think to take $u = e^x$ as a substitution. Then $du = e^x\,dx$, so we need an e^x in the numerator. That can be arranged.

$$\begin{aligned}
\int \frac{1}{e^x + e^{-x}} \cdot \left(\frac{e^x}{e^x}\right) dx &= \int \frac{e^x}{(e^x)^2 + 1}\,dx \\
&= \int \frac{1}{u^2+1}\,du \\
&= \arctan(u) + C \\
&= \arctan(e^x) + C
\end{aligned}$$

S-21: We often like to take the "inside" function as our substitution, in this case $u = 1 - x^2$, so $du = -2x\,dx$. This takes care of *part* of the integral:

$$\int_0^1 (1-2x)\sqrt{1-x^2}\,dx = \int_0^1 \sqrt{1-x^2}\,dx + \int_0^1 (-2x)\sqrt{1-x^2}\,dx$$

The left integral is tough to solve with substitution, but luckily we don't have to–it's the area of a quarter of a circle of radius 1.

$$\begin{aligned}
&= \frac{\pi}{4} + \int_1^0 \sqrt{u}\,du \\
&= \frac{\pi}{4} + \left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=0} \\
&= \frac{\pi}{4} + 0 - \frac{2}{3} = \frac{\pi}{4} - \frac{2}{3}
\end{aligned}$$

S-22:

Solution 1: We often find it useful to take "inside" functions as our substitutions, so let's try $u = \cos x$, $\mathrm{d}u = -\sin x\, \mathrm{d}x$. In order to dig up a sine, we use the identity $\tan x = \dfrac{\sin x}{\cos x}$:

$$\begin{aligned}\int \tan x \cdot \ln(\cos x)\, \mathrm{d}x &= -\int \frac{-\sin x}{\cos x} \cdot \ln(\cos x)\, \mathrm{d}x \\ &= -\int \frac{1}{u} \ln(u)\, \mathrm{d}u\end{aligned}$$

Now, it is convenient to let $w = \ln u$, $\mathrm{d}w = \frac{1}{u}\, \mathrm{d}u$:

$$\begin{aligned}-\int \frac{1}{u}\ln(u)\, \mathrm{d}u &= -\int w\, \mathrm{d}w \\ &= -\frac{1}{2}w^2 + C \\ &= -\frac{1}{2}(\ln u)^2 + C \\ &= -\frac{1}{2}(\ln(\cos x))^2 + C\end{aligned}$$

Solution 2: We might guess that it's useful to have $u = \ln(\cos x)$, $\mathrm{d}u = \dfrac{-\sin x}{\cos x}\, \mathrm{d}x = -\tan x\, \mathrm{d}x$:

$$\begin{aligned}\int \tan x \cdot \ln(\cos x) \mathrm{d}x &= -\int -\tan x \cdot \ln(\cos x) \mathrm{d}x \\ &= -\int u\, \mathrm{d}u \\ &= -\frac{1}{2}u^2 + C \\ &= -\frac{1}{2}(\ln(\cos x))^2 + C\end{aligned}$$

S-23: The given sum is of the form

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{j}{n^2} \cos\left(\frac{j^2}{n^2}\right) = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x\cos(x^2)$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}\int_0^1 f(x)\, \mathrm{d}x &= \int_0^1 x\cos(x^2)\, \mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \cos(y)\, \mathrm{d}y \qquad \text{with } y = x^2,\ \mathrm{d}y = 2x\, \mathrm{d}x \\ &= \frac{1}{2}\Big[\sin(y)\Big]_0^1 \\ &= \frac{1}{2}\sin 1\end{aligned}$$

S-24: The given sum is of the form

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sqrt{1+\frac{j^2}{n^2}}=\lim_{n\to\infty}\sum_{j=1}^{n}f(x_j^*)\Delta x$$

with $\Delta x=\frac{1}{n}$, $x_j^*=\frac{j}{n}$ and $f(x)=x\sqrt{1+x^2}$. Since $x_0^*=0$ and $x_n^*=1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &= \int_0^1 x\sqrt{1+x^2}\,\mathrm{d}x\\ &= \frac{1}{2}\int_1^2\sqrt{y}\,\mathrm{d}y \qquad \text{with } y=1+x^2,\ \mathrm{d}y=2x\,\mathrm{d}x\\ &= \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_{y=1}^{y=2}\\ &= \frac{1}{3}[2\sqrt{2}-1]\end{aligned}$$

Using a calculator, we see this is approximately 0.609.

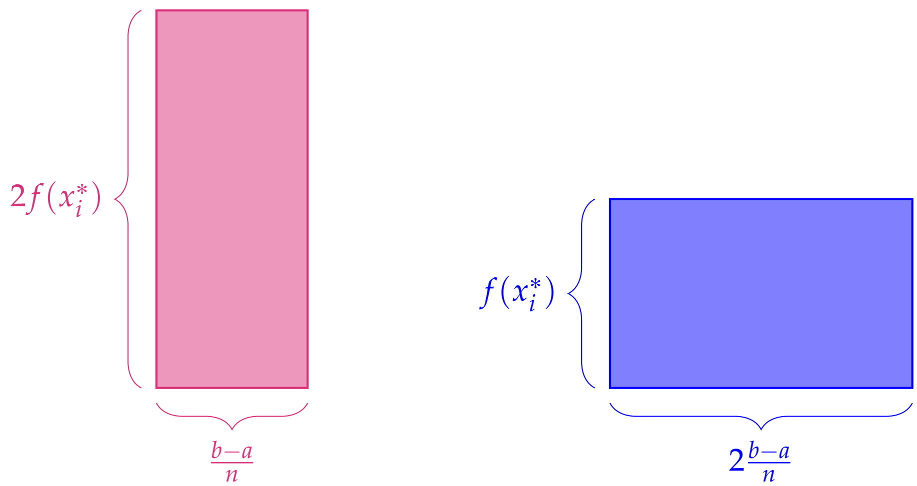
S-25: Using the definition of a definite integral with right Riemann sums:

$$\begin{aligned}\int_a^b 2f(2x)\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\cdot 2f(2(a+i\Delta x)) \qquad \Delta x=\frac{b-a}{n}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)\cdot 2f\left(2\left(a+i\left(\frac{b-a}{n}\right)\right)\right)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2b-2a}{n}\right)\cdot f\left(2a+i\left(\frac{2b-2a}{n}\right)\right)\\ \int_{2a}^{2b} f(x)\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\cdot f(2a+i\Delta x) \qquad \Delta x=\frac{2b-2a}{n}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2b-2a}{n}\right)\cdot f\left(2a+i\left(\frac{2b-2a}{n}\right)\right)\end{aligned}$$

Since the Riemann sums are exactly the same,

$$\int_a^b 2f(2x)\mathrm{d}x=\int_{2a}^{2b}f(x)\mathrm{d}x$$

Looking at the Riemann sum in this way is instructive, because it is very clear why the two integrals should be equal (without using substitution). The rectangles in the first Riemann sum are half as wide, but twice as tall, as the rectangles in the second Riemann sum. So, the two Riemann sums have rectangles of the same area.



(Not every substitution corresponds to such a simple picture.)

S-26:

(a) By the Fundamention Theorem of Calculus,

$$\begin{aligned} \mathrm{TC} &= \int \mathrm{MC}\,\mathrm{d}q + C \\ &= \int \frac{6q^2-80}{\sqrt{2q^3-80q}}\mathrm{d}q + C \end{aligned}$$

Let $u = 2q^3 - 80q$, so $\mathrm{d}u = (6q^2-80)\,\mathrm{d}q$

$$\begin{aligned} &= \int \frac{1}{\sqrt{u}}\mathrm{d}u + C = \int u^{-1/2}\mathrm{d}u \\ &= 2\sqrt{u} + C = 2\sqrt{2q^3-80q} + C \end{aligned}$$

Since $\mathrm{TC}(0) = FC= 2000$,

$$\begin{aligned} 2000 - \mathrm{TC}(0) &= 2\sqrt{0} + C \\ 2000 &= C \end{aligned}$$

All together,

$$\mathrm{TC} = 2\sqrt{2q^3-80q} + 2000$$

A company has $\mathrm{MC} = \dfrac{6q^2-80}{\sqrt{2q^3-80q}}$. Find the company's TC function $FC = 2,000$ dollars.

(b) By the Fundamental Theorem of Calculus,

$$\begin{aligned} \mathrm{TR} &= \int \mathrm{MR}\,\mathrm{d}q + C \\ &= \int \left(\frac{q^3}{\sqrt{q^2+1}}\right)\mathrm{d}q + C \end{aligned}$$

The most likely substitution seems to be $u = q^2 + 1$, with $\mathrm{d}u = 2q\mathrm{d}q$. Also $q^2 = u - 1$ and $q\ \mathrm{d}q = \frac{1}{2}\mathrm{d}u$.

$$\begin{aligned}
&= \int \left(\frac{q^2}{\sqrt{q^2+1}} \right) q\mathrm{d}q + C \\
&= \int \left(\frac{u-1}{\sqrt{u}} \right) \cdot \frac{1}{2}\mathrm{d}u + C \\
&= \frac{1}{2} \int \left(u^{1/2} - u^{-1/2} \right) \mathrm{d}u + C \\
&= \frac{1}{2} \left[\frac{2}{3}u^{3/2} - 2u^{1/2} \right] + C \\
&= \frac{1}{3}u^{3/2} - u^{1/2} + C \\
&= \frac{1}{3} \left(q^2 + 1 \right)^{3/2} - \left(q^2 + 1 \right)^{1/2} + C
\end{aligned}$$

To find C, we use $\mathrm{TR}(0) = 0$.

$$\begin{aligned}
0 = \mathrm{TR}(0) &= \frac{1}{3} \left(0^2 + 1 \right)^{3/2} - \left(0^2 + 1 \right)^{1/2} + C = -\frac{2}{3} + C \\
C &= \frac{2}{3}
\end{aligned}$$

All together,

$$\mathrm{TR} = \frac{1}{3} \left(q^2 + 1 \right)^{3/2} - \left(q^2 + 1 \right)^{1/2} + \frac{2}{3}$$

(c) Profit is revenue minus cost. In this case,

$$\left(\frac{1}{3} \left(q^2 + 1 \right)^{3/2} - \left(q^2 + 1 \right)^{1/2} + \frac{2}{3} \right) - \left(2\sqrt{2q^3 - 80q} + 2000 \right)$$

(d) If $2q^3 - 80q < 0$, then TC is not defined, as it involves the square root of a negative number. Note

$$2q^3 - 80q = 2q(q^2 - 40)$$

So for $0 \leqslant q \leqslant \sqrt{40}$, q is a number that makes sense as a quantity of production (i.e. q isn't negative), but it isn't in the domain of our cost functions. The company needs to make at least 7 units for q to be in the domain of the functions we're using.

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S-1: Integration by substitution is just using the chain rule, backwards:

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))g'(x) \\
\Leftrightarrow \quad & \int \frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\}\mathrm{d}x + C = \int f'(g(x))g'(x)\mathrm{d}x \\
\Leftrightarrow \quad & \underbrace{f(g(x))}_{f(u)} + C = \int \underbrace{f'(g(x))}_{f'(u)}\underbrace{g'(x)\mathrm{d}x}_{\mathrm{d}u}
\end{aligned}$$

Similarly, integration by parts comes from the product rule:

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x) \\
\Leftrightarrow & \int \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\}\mathrm{d}x + C = \int f'(x)g(x) + f(x)g'(x)\mathrm{d}x \\
\Leftrightarrow \quad & f(x)g(x) + C = \int f'(x)g(x)\mathrm{d}x + \int f(x)g'(x)\mathrm{d}x \\
\Leftrightarrow \quad & \int \underbrace{f(x)}_{u}\underbrace{g'(x)\mathrm{d}x}_{\mathrm{d}v} = \underbrace{f(x)}_{u}\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\underbrace{f'(x)\mathrm{d}x}_{\mathrm{d}u}
\end{aligned}$$

S-2: Remember our rule: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$. So, we take u and use it to make $\mathrm{d}u$–that is, we differentiate it. We take $\mathrm{d}v$ and use it to make v–that is, we antidifferentiate it.

S-3: We'll use the same ideas that lead to the methods of substitution and integration by parts. (You can review these in your text, or see the solution to Question 1 in this section.) According to the quotient rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Antidifferentiating both sides gives us:

$$\begin{aligned}
\int \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\}\mathrm{d}x + C &= \int \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}\mathrm{d}x \\
\frac{f(x)}{g(x)} + C &= \int \frac{f'(x)}{g(x)}\mathrm{d}x - \int \frac{f(x)g'(x)}{g^2(x)}\mathrm{d}x \\
\int \frac{f'(x)}{g(x)}\mathrm{d}x &= \frac{f(x)}{g(x)} + \int \frac{f(x)g'(x)}{g^2(x)}\mathrm{d}x + C
\end{aligned}$$

This isn't quite at catchy as integration by parts–which is probably why it hasn't caught on as a rule with its own name.

S-4: All the antiderivatives differ only by a constant, so we can write them all as $v(x) + C$ for some C. Then, using the formula for integration by parts,

$$\begin{aligned}\int u(x) \cdot v'(x)\mathrm{d}x &= \underbrace{u(x)}_{u}\underbrace{[v(x)+C]}_{v} - \int \underbrace{[v(x)+C]}_{v}\underbrace{u'(x)\mathrm{d}x}_{\mathrm{d}u} \\ &= u(x)v(x) + Cu(x) - \int v(x)u'(x)\mathrm{d}x - \int Cu'(x)\mathrm{d}x \\ &= u(x)v(x) + Cu(x) - \int v(x)u'(x)\mathrm{d}x - Cu(x) + D \\ &= u(x)v(x) - \int v(x)u'(x)\mathrm{d}x + D\end{aligned}$$

where D is any constant.

Since the terms with C cancel out, it didn't matter what we chose for C–all choices end up the same.

S-5: Suppose we choose $\mathrm{d}v = f(x)\mathrm{d}x$, $u = 1$. Then $v = \displaystyle\int f(x)\mathrm{d}x$, and $\mathrm{d}u = \mathrm{d}x$. So, our integral becomes:

$$\int \underbrace{(1)}_{u}\underbrace{f(x)\mathrm{d}x}_{\mathrm{d}v} = \underbrace{(1)}_{u}\underbrace{\int f(x)\mathrm{d}x}_{v} - \int \underbrace{\left(\int f(x)\mathrm{d}x\right)}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}$$

In order to figure out the first product (and the second integrand), you need to know the antiderivative of $f(x)$–but that's exactly what you're trying to figure out! So, using integration by parts has not eased your pain.

We note here that in certain cases, such as $\int \ln x \, \mathrm{d}x$ (Example 3.5.8 in the text), it is useful to choose $\mathrm{d}v = 1\mathrm{d}x$. This is similar to, but crucially different from, the do-nothing method in this problem.

S-6: For integration by parts, we want to break the integrand into two pieces, multiplied together. There is an obvious choice for how to do this: one piece is x, and the other is $\ln x$. Remember that one piece will be integrated, while the other is differentiated. The question is, which choice will be more helpful. After some practice, you'll get the hang of making the choice. For now, we'll present both choices–but when you're writing a solution, you don't have to write this part down. It's enough to present your choice, and then a successful computation is justification enough.

Option 1:	$u = x$ $\mathrm{d}v = \ln x \, \mathrm{d}x$	$\mathrm{d}u = 1 \, \mathrm{d}x$ $v = ??$
Option 2:	$u = \ln x$ $\mathrm{d}v = x \, \mathrm{d}x$	$\mathrm{d}u = \frac{1}{x} \, \mathrm{d}x$ $v = \frac{1}{2}x^2$

In Example 3.5.8 of the text, we found the antiderivative of logarithm, but it wasn't trivial. We might reasonably want to avoid using this complicated antiderivative. Indeed, Option 2 (differentiating logarithm, antidifferentiating x) looks promising–when we multiply the blue equations, we get something easily integrable– so let's not even bother going deeper into Option 1.

That is, we perform integration by parts with $u = \ln x$ and $\mathrm{d}v = x \, \mathrm{d}x$, so that $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = \frac{x^2}{2}$.

$$\begin{aligned} \int \underbrace{\ln x}_{u} \underbrace{x \, \mathrm{d}x}_{\mathrm{d}v} &= \underbrace{\frac{x^2 \ln x}{2}}_{uv} - \int \underbrace{\frac{x^2}{2}}_{v} \underbrace{\frac{\mathrm{d}x}{x}}_{\mathrm{d}u} = \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, \mathrm{d}x \\ &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C \end{aligned}$$

S-7: Our integrand is the product of two functions, and there's no clear substitution. So, we might reasonably want to try integration by parts. Again, we have two obvious pieces: $\ln x$, and x^{-7}. We'll consider our options for assigning these to u and $\mathrm{d}v$:

Option 1:	$u = \ln x$ $\mathrm{d}v = x^{-7} \, \mathrm{d}x$	$\mathrm{d}u = \frac{1}{x} \, \mathrm{d}x$ $v = \frac{1}{-6}x^{-6}$
Option 2:	$u = x^{-7}$ $\mathrm{d}v = \ln x \, \mathrm{d}x$	$\mathrm{d}u = -7x^{-8} \, \mathrm{d}x$ $v = ??$

Again, we remember that logarithm has some antiderivative we found in Example 3.5.8

of the text, but it was something complicated. Luckily, we don't need to bother with it: when we multiply the red equations in Option 1, we get a perfectly workable integral.

We perform integration by parts with $u = \ln x$ and $\mathrm{d}v = x^{-7}\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = -\frac{x^{-6}}{6}$.

$$\int \frac{\ln x}{x^7}\,\mathrm{d}x = \underbrace{-\ln x\,\frac{x^{-6}}{6}}_{uv} + \int \underbrace{\frac{x^{-6}}{6}}_{-v}\,\underbrace{\frac{\mathrm{d}x}{x}}_{du} = -\frac{\ln x}{6x^6} + \frac{1}{6}\int x^{-7}\,\mathrm{d}x$$
$$= -\frac{\ln x}{6x^6} - \frac{1}{36x^6} + C$$

S-8: To integrate by parts, we need to decide what to use as u, and what to use as $\mathrm{d}v$. The salient parts of this integrand are x and $\sin x$, so we only need to decide which is u and which $\mathrm{d}v$. Again, this process will soon become familiar, but to help you along we show you both options below.

Option 1:	$u = x$ $\mathrm{d}v = \sin x\,\mathrm{d}x$	$\mathrm{d}u = 1\,\mathrm{d}x$ $v = -\cos x$
Option 2:	$u = \sin x$ $\mathrm{d}v = x\,\mathrm{d}x$	$\mathrm{d}u = \cos x\,\mathrm{d}x$ $v = \frac{1}{2}x^2$

The derivative and antiderivative of sine are almost the same, but x turns into something simpler when we differentiate it. So, we choose Option 1.

We integrate by parts, using $u = x$, $\mathrm{d}v = \sin x\,\mathrm{d}x$ so that $v = -\cos x$ and $\mathrm{d}u = \mathrm{d}x$:

$$\int_0^{\pi} x\sin x\,\mathrm{d}x = \underbrace{-x\cos x}_{uv}\Big|_0^{\pi} - \int_0^{\pi} \underbrace{(-\cos x)}_{v}\,\underbrace{\mathrm{d}x}_{\mathrm{d}u} = \Big[-x\cos x + \sin x\Big]_0^{\pi} = -\pi(-1) = \pi$$

S-9: When we have two functions multiplied like this, and there's no obvious substitution, our minds turn to integration by parts. We hope that our integral will be improved by differentiating one part and antidifferentiating the other. Let's see what our choices are:

Option 1:	$u = x$	$\mathrm{d}u = 1\,\mathrm{d}x$
	$\mathrm{d}v = \cos x\,\mathrm{d}x$	$v = \sin x$
Option 2:	$u = \cos x$	$\mathrm{d}u = -\sin x\,\mathrm{d}x$
	$\mathrm{d}v = x\,\mathrm{d}x$	$v = \dfrac{1}{2}x^2$

Option 1 seems preferable. We integrate by parts, using $u = x$, $\mathrm{d}v = \cos x\,\mathrm{d}x$ so that $v = \sin x$ and $\mathrm{d}u = \mathrm{d}x$:

$$\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = \underbrace{x}_{u}\underbrace{\sin x}_{v}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \underbrace{\sin x}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u} = \Big[x\sin x + \cos x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

S-10: This integrand is the product of two functions, with no obvious substitution. So, let's try integration by parts, with one part e^x and one part x^3.

Option 1:	$u = e^x$	$\mathrm{d}u = e^x\,\mathrm{d}x$
	$\mathrm{d}v = x^3\,\mathrm{d}x$	$v = \frac{1}{4}x^4$
Option 2:	$u = x^3$	$\mathrm{d}u = 3x^2\,\mathrm{d}x$
	$\mathrm{d}v = e^x\,\mathrm{d}x$	$v = e^x$

At first glance, multiplying the red functions and multiplying the blue functions give largely equivalent integrands to what we started with–none of them with obvious antiderivatives. In previous questions, we were able to choose $u = x$, and then $\mathrm{d}u = \mathrm{d}x$, so the "$x$" in the integrand effectively went away. Here, we see that choosing $u = x^3$ will lead to $\mathrm{d}u = 3x^2\mathrm{d}x$, which has a lower power. If we repeatedly perform integration by parts, choosing u to be the power of x each time, then after a few iterations it should go away, because the third derivative of x^3 is a constant.

So, we start with Option 2: $u = x^3$, $\mathrm{d}v = e^x\mathrm{d}x$, $\mathrm{d}u = 3x^2\mathrm{d}x$, and $v = e^x$.

$$\begin{aligned}\int x^3e^x\mathrm{d}x &= \underbrace{x^3}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \cdot \underbrace{3x^2\mathrm{d}x}_{\mathrm{d}u}\\ &= x^3e^x - 3\int e^x \cdot x^2\mathrm{d}x\end{aligned}$$

Now, we take $u = x^2$ and $\mathrm{d}v = e^x\mathrm{d}x$, so $\mathrm{d}u = 2x\mathrm{d}x$ and $v = e^x$. We're only using integration by parts on the actual integral–the rest of the function stays the way it is.

$$\begin{aligned}&= x^3e^x - 3\left[\underbrace{x^2e^x}_{uv} - \int \underbrace{e^x}_{v} \cdot \underbrace{2x\mathrm{d}x}_{\mathrm{d}u}\right]\\ &= x^3e^x - 3x^2e^x + 6\int xe^x\mathrm{d}x\end{aligned}$$

Continuing, we take $u = x$ and $\mathrm{d}v = e^x\mathrm{d}x$, so $\mathrm{d}u = \mathrm{d}x$ and $v = e^x$. This is the step where the polynomial part of the integrand finally disappears.

$$\begin{aligned}&= x^3e^x - 3x^2e^x + 6\left[\underbrace{xe^x}_{uv} - \int \underbrace{e^x}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}\right]\\ &= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C\\ &= e^x\left(x^3 - 3x^2 + 6x - 6\right) + C\end{aligned}$$

Let's check that this makes sense: the derivative of $e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$ should be x^3e^x. We differentiate using the product rule.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^x\left(x^3 - 3x^2 + 6x - 6\right) + C\right\} &= e^x\left(x^3 - 3x^2 + 6x - 6\right) + e^x\left(3x^2 - 6x + 6\right)\\ &= e^x\left(x^3 - 3x^2 + 3x^2 + 6x - 6x - 6 + 6\right) = x^3e^x\end{aligned}$$

Remark: In order to be technically correct in our antidifferentiation, we should add the $+C$ as soon as we do the first integration by parts. However, when we are using integration by parts, we usually end up evaluating an integral at the end, and we add the $+C$ at that point. Since the $+C$ comes up eventually, it is common practice to not clutter our calculations with it until the end.

S-11: Since our integrand is two functions multiplied together, and there isn't an obvious substitution, let's try integration by parts. Here are our salient options.

Option 1:	$u = x$ $\mathrm{d}v = \ln^3 x\ \mathrm{d}x$	$\mathrm{d}u = 1\ \mathrm{d}x$ $v = ??$
Option 2:	$u = \ln^3 x$ $\mathrm{d}v = x\ \mathrm{d}x$	$\mathrm{d}u = 3\ln^2 x \cdot \frac{1}{x}\ \mathrm{d}x$ $v = \dfrac{1}{2}x^2$

This calls for some strategizing. Using the template of Example 3.5.8 in the text, we could probably figure out the antiderivative of $\ln^3 x$. Option 1 is tempting, because our x-term goes away. So, there might be a benefit there, but on the other hand, the antiderivative of $\ln^3 x$ is probably pretty complicated.

Now let's consider Option 2. When we multiply the blue functions together, we get something similar to our original integrand, but the power of logarithm is smaller. If we were to iterate this method (using integration by parts a few times, always choosing u to be the part with a logarithm) then eventually we would end up differentiating logarithm. This seems like a safer plan: let's do Option 2.

We use integration by parts with $u = \ln^3 x$, $\mathrm{d}v = x\mathrm{d}x$, $\mathrm{d}u = \frac{3}{x}\ln^2 x\mathrm{d}x$, and $v = \frac{1}{2}x^2$.

$$\begin{aligned}\int x\ln^3 x\ \mathrm{d}x &= \underbrace{\frac{1}{2}x^2\ln^3 x}_{uv} - \int \underbrace{\frac{3}{2}x\ln^2 x\mathrm{d}x}_{v\mathrm{d}u} \\ &= \frac{1}{2}x^2\ln^3 x - \frac{3}{2}\int x\ln^2 x\mathrm{d}x\end{aligned}$$

Continuing our quest to differentiate away the logarithm, we use integration by parts with $u = \ln^2 x$, $\mathrm{d}v = x\mathrm{d}x$, $\mathrm{d}u = \dfrac{2}{x}\ln x\mathrm{d}x$, and $v = \dfrac{1}{2}x^2$.

$$\begin{aligned} &= \frac{1}{2}x^2\ln^3 x - \frac{3}{2}\left[\underbrace{\frac{1}{2}x^2\ln^2 x}_{uv} - \int \underbrace{x\ln x\mathrm{d}x}_{v\mathrm{d}u}\right] \\ &= \frac{1}{2}x^2\ln^3 x - \frac{3}{4}x^2\ln^2 x + \frac{3}{2}\int x\ln x\mathrm{d}x\end{aligned}$$

One last integration by parts: $u = \ln x$, $\mathrm{d}v = x\mathrm{d}x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, and $v = \frac{1}{2}x^2$.

$$\begin{aligned}
&= \frac{1}{2}x^2\ln^3 x - \frac{3}{4}x^2\ln^2 x + \frac{3}{2}\left[\underbrace{\frac{1}{2}x^2\ln x}_{uv} - \int \underbrace{\frac{1}{2}x\mathrm{d}x}_{v\mathrm{d}u}\right] \\
&= \frac{1}{2}x^2\ln^3 x - \frac{3}{4}x^2\ln^2 x + \frac{3}{4}x^2\ln x - \frac{3}{4}\int x\mathrm{d}x \\
&= \frac{1}{2}x^2\ln^3 x - \frac{3}{4}x^2\ln^2 x + \frac{3}{4}x^2\ln x - \frac{3}{8}x^2 + C
\end{aligned}$$

Once again, technically there is a $+C$ in the work after the first integration by parts, but we follow convention by conveniently suppressing it until the final integration.

S-12: The integrand is the product of two functions, without an obvious substitution, so let's see what integration by parts can do for us.

Option 1:	$u = x^2$	$\mathrm{d}u = 2x\ \mathrm{d}x$
	$\mathrm{d}v = \sin x\ \mathrm{d}x$	$v = -\cos x$
Option 2:	$u = \sin x$	$\mathrm{d}u = \cos x\ \mathrm{d}x$
	$\mathrm{d}v = x^2\ \mathrm{d}x$	$v = \frac{1}{3}x^3$

Neither option gives us something immediately integrable, but Option 1 replaces our x^2 term with a *lower* power of x. If we repeatedly apply integration by parts, we can reduce this power to zero. So, we start by choosing $u = x^2$ and $\mathrm{d}v = \sin x\mathrm{d}x$, so $\mathrm{d}u = 2x\mathrm{d}x$ and $v = -\cos x$.

$$\begin{aligned}
\int x^2\sin x\ \mathrm{d}x &= \underbrace{-x^2\cos x}_{uv} + \underbrace{\int 2x\cos x\mathrm{d}x}_{-v\mathrm{d}u} \\
&= -x^2\cos x + 2\int x\cos x\mathrm{d}x
\end{aligned}$$

Using integration by parts again, we want to be differentiating (not antidifferentiating) x, so we choose $u = x$, $\mathrm{d}v = \cos x \mathrm{d}x$, and then $\mathrm{d}u = \mathrm{d}x$ (x went away!), $v = \sin x$.

$$\begin{aligned}
&= -x^2 \cos x + 2 \left[\underbrace{x \sin x}_{uv} - \int \underbrace{\sin x \mathrm{d}x}_{v\mathrm{d}u} \right] \\
&= -x^2 \cos x + 2x \sin x + 2 \cos x + C \\
&= (2 - x^2) \cos x + 2x \sin x + C
\end{aligned}$$

S-13: This problem is similar to Questions 6 and 7: integrating a polynomial multiplied by a logarithm. Just as in these questions, if we use integration by parts with $u = \ln t$, then $\mathrm{d}u = \dfrac{1}{t}\mathrm{d}t$, and our new integrand will consist of powers of t–which are easy to antidifferentiate.

So, we use $u = \ln t$, $\mathrm{d}v = 3t^2 - 5t + 6$, $\mathrm{d}u = \frac{1}{t}\mathrm{d}t$, and $v = t^3 - \frac{5}{2}t^2 + 6t$.

$$\begin{aligned}
\int (3t^2 - 5t + 6) \ln t \, \mathrm{d}t &= \underbrace{\ln t}_{u} \left(\underbrace{t^3 - \frac{5}{2}t^2 + 6t}_{v} \right) - \int \underbrace{\frac{1}{t} \left(t^3 - \frac{5}{2}t^2 + 6t \right) \mathrm{d}t}_{v\mathrm{d}u} \\
&= \left(t^3 - \frac{5}{2}t^2 + 6t \right) \ln t - \int \left(t^2 - \frac{5}{2}t + 6 \right) \mathrm{d}t \\
&= \left(t^3 - \frac{5}{2}t^2 + 6t \right) \ln t - \frac{1}{3}t^3 + \frac{5}{4}t^2 - 6t + C
\end{aligned}$$

S-14: Before we jump to integration by parts, we notice that the square roots lend themselves to substitution. Let's take $w = \sqrt{s}$. Then $\mathrm{d}w = \dfrac{1}{2\sqrt{s}}\,\mathrm{d}s$, so $2w \, \mathrm{d}w = \mathrm{d}s$.

$$\int \sqrt{s} e^{\sqrt{s}} \mathrm{d}s = \int w \cdot e^w \cdot 2w \mathrm{d}w = 2 \int w^2 e^w \mathrm{d}w$$

Now we have nearly the situation of Question 10. We can repeatedly use integration by parts, with u as the power of w, to get rid of the polynomial part. We'll start with $u = w^2$, $\mathrm{d}v = e^w \mathrm{d}w$, $\mathrm{d}u = 2w\mathrm{d}w$, and $v = e^w$.

$$\begin{aligned}
&= 2 \left[\underbrace{w^2 e^w}_{uv} - \int \underbrace{2we^w \mathrm{d}w}_{v\mathrm{d}u} \right] \\
&= 2w^2 e^w - 4 \int we^w \mathrm{d}w
\end{aligned}$$

We use integration by parts again, this time with $u = w$, $\mathrm{d}v = e^w \mathrm{d}w$, $\mathrm{d}u = \mathrm{d}w$, and $v = e^w$.

$$\begin{aligned}
&= 2w^2e^w - 4\left[\underbrace{we^w}_{uv} - \int \underbrace{e^w \mathrm{d}w}_{v\mathrm{d}u}\right] \\
&= 2w^2e^w - 4we^w + 4e^w + C \\
&= e^w\left(2w^2 - 4w + 4\right) + C \\
&= e^{\sqrt{s}}\left(2s - 4\sqrt{s} + 4\right) + C
\end{aligned}$$

S-15: Let's use integration by parts. What are our parts? We have a few options.

Solution 1: Following Example 3.5.8 in the text, we choose $u = \ln^2 x$ and $\mathrm{d}v = \mathrm{d}x$, so that $\mathrm{d}u = \frac{2}{x}\ln x\ \mathrm{d}x$ and $v = x$.

$$\int \ln^2 x\mathrm{d}x = \underbrace{x\ln^2 x}_{uv} - \int \underbrace{2\ln x\mathrm{d}x}_{v\mathrm{d}u}$$

Here we can either use the antiderivative of logarithm from memory, or re-derive it. We do the latter, using integration by parts with $u = \ln x$, $\mathrm{d}v = 2\mathrm{d}x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, and $v = 2x$.

$$\begin{aligned}
&= x\ln^2 x - \left[\underbrace{2x\ln x}_{uv} - \int \underbrace{2\mathrm{d}x}_{v\mathrm{d}u}\right] \\
&= x\ln^2 x - 2x\ln x + 2x + C
\end{aligned}$$

Solution 2: Our integrand is two functions multiplied together: $\ln x$ and $\ln x$. So, we will use integration by parts with $u = \ln x$, $\mathrm{d}v = \ln x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, and (using the antiderivative of logarithm, found in Example 3.5.8 in the text) $v = x\ln x - x$.

$$\begin{aligned}
\int \ln^2 x\ \mathrm{d}x &= (\underbrace{\ln x}_{u})(\underbrace{x\ln x - x}_{v}) - \int (\underbrace{x\ln x - x}_{v})\ \underbrace{\frac{1}{x}\mathrm{d}x}_{\mathrm{d}u} \\
&= x\ln^2 x - x\ln x - \int (\ln x - 1)\ \mathrm{d}x \\
&= x\ln^2 x - x\ln x - [(x\ln x - x) - x] + C \\
&= x\ln^2 x - 2x\ln x + 2x + C
\end{aligned}$$

S-16: This is your friendly reminder that to a person with a hammer, everything looks like a nail. The integral in the problem is a classic example of an integral to solve using substitution. We have an "inside function," x^2+1, whose derivative shows up multiplied to the rest of the integrand. We take $u = x^2+1$, then $\mathrm{d}u = 2x\mathrm{d}x$, so

$$\int 2xe^{x^2+1}\mathrm{d}x = \int e^u \mathrm{d}u = e^u + C = e^{x^2+1} + C$$

S-17: In Example 3.5.9 of the text, we saw that integration by parts was useful when the integrand has a derivative that works nicely when multiplied by x. We use the same idea here. Let $u = \arccos y$ and $\mathrm{d}v = \mathrm{d}y$, so that $v = y$ and $du = -\frac{\mathrm{d}y}{\sqrt{1-y^2}}$.

$$\int \arccos y\,\mathrm{d}y = \underbrace{y\arccos y}_{uv} + \underbrace{\int \frac{y}{\sqrt{1-y^2}}\mathrm{d}y}_{-v\mathrm{d}u}$$

Using the substitution $u = 1-y^2$, $\mathrm{d}u = -2y\mathrm{d}y$,

$$\begin{aligned}
&= y\arccos y - \frac{1}{2}\int u^{-1/2}\mathrm{d}u \\
&= y\arccos y - u^{1/2} + C \\
&= y\arccos y - \sqrt{1-y^2} + C
\end{aligned}$$

S-18: We integrate by parts, using $u = \arctan(2y)$, $\mathrm{d}v = 4y\,\mathrm{d}y$, so that $v = 2y^2$ and $du = \frac{2\,\mathrm{d}y}{1+(2y)^2}$:

$$\int 4y\arctan(2y)\,\mathrm{d}y = \underbrace{2y^2\arctan(2y)}_{uv} - \underbrace{\int \frac{4y^2}{(2y)^2+1}\,\mathrm{d}y}_{v\mathrm{d}u}$$

The integrand $\frac{4y^2}{(2y)^2+1}$ is a rational function. So the remaining integral can be evaluated using the method of partial fractions, starting with long division. But it is easier to just notice that $\frac{4y^2}{4y^2+1} = \frac{4y^2+1}{4y^2+1} - \frac{1}{4y^2+1}$. We therefore have:

$$\int \frac{4y^2}{4y^2+1}\,\mathrm{d}y = \int \left(1 - \frac{1}{4y^2+1}\right)\,\mathrm{d}y = y - \frac{1}{2}\arctan(2y) + C$$

The final answer is then

$$\int 4y\arctan(2y)\,\mathrm{d}y = 2y^2\arctan(2y) - y + \frac{1}{2}\arctan(2y) + C$$

S-19: We've got an integrand that consists of two functions multiplied together, and no obvious substitution. So, we think about integration by parts. Let's consider our options. Note in Example 3.5.9 of the text, we found that the antiderivative of arctangent is $x \arctan x - \frac{1}{2}\ln(1+x^2) + C$.

Option 1:	$u = \arctan x$	$\mathrm{d}u = \dfrac{1}{1+x^2}\,\mathrm{d}x$
	$\mathrm{d}v = x^2\,\mathrm{d}x$	$v = \dfrac{1}{3}x^3$
Option 2:	$u = x^2$	$\mathrm{d}u = 2x\,\mathrm{d}x$
	$\mathrm{d}v = \arctan x\,\mathrm{d}x$	$v = x\arctan x - \dfrac{1}{2}\ln(1+x^2)$

Option 1 Option 1 seems likelier. Let's see how it plays out. We use integration by parts with $u = \arctan x$, $\mathrm{d}v = x^2\mathrm{d}x$, $\mathrm{d}u = \frac{\mathrm{d}x}{1+x^2}$, and $v = \frac{1}{3}x^3$.

$$\begin{aligned}\int x^2 \arctan x\,\mathrm{d}x &= \underbrace{\frac{x^3}{3}\arctan x}_{uv} - \underbrace{\int \frac{x^3}{3(1+x^2)}\mathrm{d}x}_{v\mathrm{d}u}\\ &= \frac{x^3}{3}\arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}\mathrm{d}x\end{aligned}$$

This is starting to look like a candidate for a substitution! Let's try the denominator, $s = 1+x^2$. Then $\mathrm{d}s = 2x\mathrm{d}x$, and $x^2 = s-1$.

$$\begin{aligned} &= \frac{x^3}{3}\arctan x - \frac{1}{6}\int \frac{x^2}{1+x^2}\cdot 2x\mathrm{d}x\\ &= \frac{x^3}{3}\arctan x - \frac{1}{6}\int \frac{s-1}{s}\mathrm{d}s\\ &= \frac{x^3}{3}\arctan x - \frac{1}{6}\int 1 - \frac{1}{s}\,\mathrm{d}s\\ &= \frac{x^3}{3}\arctan x - \frac{1}{6}s + \frac{1}{6}\ln|s| + C\\ &= \frac{x^3}{3}\arctan x - \frac{1}{6}(1+x^2) + \frac{1}{6}\ln(1+x^2) + C\end{aligned}$$

Option 2: What if we had tried the other option? That is, $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$, $\mathrm{d}v = \arctan x$, and $v = x\arctan x - \frac{1}{2}\ln(1+x^2)$. It's not always the case that both options work, but sometimes they do. (They are almost never of equal difficulty.) This solution takes advantage of two previously hard-won results: the antiderivatives of logarithm and arctangent.

$$\int x^2 \arctan x\,\mathrm{d}x = \underbrace{x^2}_{u}\underbrace{\left(x\arctan x - \frac{1}{2}\ln(1+x^2)\right)}_{v} - \int \underbrace{\left(x\arctan x - \frac{1}{2}\ln(1+x^2)\right)}_{v} \cdot \underbrace{2x\mathrm{d}x}_{\mathrm{d}u}$$
$$= x^3\arctan x - \frac{x^2}{2}\ln(1+x^2) - 2\int x^2\arctan x\,\mathrm{d}x + \int x\ln(1+x^2)\mathrm{d}x$$

Adding $\quad 2\int x^2\arctan x\,\mathrm{d}x \quad$ to both sides:

$$3\int x^2\arctan x\,\mathrm{d}x = x^3\arctan x - \frac{x^2}{2}\ln(1+x^2) + \int x\ln(1+x^2)\mathrm{d}x$$
$$\int x^2\arctan x\,\mathrm{d}x = \frac{x^3}{3}\arctan x - \frac{x^2}{6}\ln(1+x^2) + \frac{1}{3}\int x\ln(1+x^2)\mathrm{d}x$$

Using the substitution $s = 1+x^2$, $\mathrm{d}s = 2x\mathrm{d}x$:

$$= \frac{x^3}{3}\arctan x - \frac{x^2}{6}\ln(1+x^2) + \frac{1}{6}\int \ln s\,\mathrm{d}s$$

Using the antiderivative of logarithm found in Example 3.5.8 of the text,

$$= \frac{x^3}{3}\arctan x - \frac{x^2}{6}\ln(1+x^2) + \frac{1}{6}\left(s\ln s - s\right) + C$$
$$= \frac{x^3}{3}\arctan x - \frac{x^2}{6}\ln(1+x^2) + \frac{1}{6}\Big((1+x^2)\ln(1+x^2) - (1+x^2)\Big) + C$$
$$= \frac{x^3}{3}\arctan x + \left[-\frac{x^2}{6} + \frac{1+x^2}{6}\right]\ln(1+x^2) - \frac{1}{6}(1+x^2) + C$$
$$= \frac{x^3}{3}\arctan x + \frac{1}{6}\ln(1+x^2) - \frac{1}{6}(1+x^2) + C$$

S-20: This example is similar to Example 3.5.10 in the text. The functions $e^{x/2}$ and $\cos(2x)$ both do not substantially alter when we differentiate or antidifferentiate them. If we use integration by parts twice, we'll end up with an expression that includes our original integral. Then we can just solve for the original integral in the equation, without actually antidifferentiating.

Let's use $u = \cos(2x)$ and $\mathrm{d}v = e^{x/2}\mathrm{d}x$, so $\mathrm{d}u = -2\sin(2x)\mathrm{d}x$ and $v = 2e^{x/2}$.

$$\int e^{x/2}\cos(2x)\mathrm{d}x = \underbrace{2e^{x/2}\cos(2x)}_{uv} - \int \underbrace{-4e^{x/2}\sin(2x)\mathrm{d}x}_{v\mathrm{d}u}$$
$$= 2e^{x/2}\cos(2x) + 4\int e^{x/2}\sin(2x)\mathrm{d}x$$

Similarly to our first integration by parts, we use $u = \sin(2x)$, $\mathrm{d}v = e^{x/2}\mathrm{d}x$, $\mathrm{d}u = 2\cos(2x)\mathrm{d}x$, and $v = 2e^{x/2}$.

$$= 2e^{x/2}\cos(2x) + 4\left[\underbrace{2e^{x/2}\sin(2x)}_{uv} - \int \underbrace{4e^{x/2}\cos(2x)\mathrm{d}x}_{v\mathrm{d}u}\right]$$

So, we've found the equation

$$\int e^{x/2}\cos(2x)\mathrm{d}x = 2e^{x/2}\cos(2x) + 8e^{x/2}\sin(2x) - 16\int e^{x/2}\cos(2x)\mathrm{d}x + C$$

We add $\quad 16\displaystyle\int e^{x/2}\cos(2x)\mathrm{d}x \quad$ to both sides.

$$\begin{aligned} 17\int e^{x/2}\cos(2x)\mathrm{d}x &= 2e^{x/2}\cos(2x) + 8e^{x/2}\sin(2x) + C \\ \int e^{x/2}\cos(2x)\mathrm{d}x &= \frac{2}{17}e^{x/2}\cos(2x) + \frac{8}{17}e^{x/2}\sin(2x) + C \end{aligned}$$

Remark: remember that C is a stand-in for "we can add any real constant". Since C can be any number in $(-\infty, \infty)$, also $\frac{C}{17}$ can be any number in $(-\infty, \infty)$. So, rather than write $\frac{C}{17}$ in the last line, we re-named $\frac{C}{17}$ to C.

S-21:

Solution 1: This question looks like a substitution, since we have an "inside function." So, let's see where that leads: let $u = \ln x$. Then $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. We don't see this right away in our function, but we can bring it into the function by multiplying and dividing by x, and noting from our substitution that $e^u = x$, $e^u\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned} \int \sin(\ln x)\mathrm{d}x &= \int \frac{x\sin(\ln x)}{x}\mathrm{d}x \\ &= \int \frac{e^u \sin\left(\ln\left(e^u\right)\right)}{e^u}e^u\mathrm{d}u \\ &= \int e^u \sin u\ \mathrm{d}u \end{aligned}$$

Using the result of Example 3.5.11 in the text:

$$\begin{aligned} &= \frac{1}{2}e^u\left(\sin u - \cos u\right) + C \\ &= \frac{1}{2}e^{\ln x}\left(\sin(\ln x) - \cos(\ln x)\right) + C \\ &= \frac{1}{2}x\left(\sin(\ln x) - \cos(\ln x)\right) + C \end{aligned}$$

Solution 2: It's not clear how to antidifferentiate the integrand, but we can certainly differentiate it. So, keeping in mind the method of Example 3.5.11 in the text, we take $u = \sin(\ln x)$ and $dv = dx$, so $du = \frac{1}{x}\cos(\ln x)dx$ and $v = x$.

$$\int \sin(\ln x)dx = \underbrace{x\sin(\ln x)}_{uv} - \int \underbrace{\cos(\ln x)dx}_{vdu}$$

Continuing on, we again use integration by parts, with $u = \cos(\ln x)$, $dv = dx$, $du = -\frac{1}{x}\sin(\ln x)dx$, and $v = x$.

$$= x\sin(\ln x) - \left[\underbrace{x\cos(\ln x)}_{uv} + \int \underbrace{\sin(\ln x)}_{-vdu} dx\right]$$

That is, we have

$$\int \sin(\ln x)dx = x\left[\sin(\ln x) - \cos(\ln x)\right] - \int \sin(\ln x)dx + C$$

Adding $\int \sin(\ln x)dx$ to both sides,

$$2\int \sin(\ln x)dx = x\left[\sin(\ln x) - \cos(\ln x)\right] + C$$

$$\int \sin(\ln x)dx = \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C$$

Remark: remember that C is a stand-in for "we can add any real constant". Since C can be any number in $(-\infty, \infty)$, also $\frac{C}{2}$ can be any number in $(-\infty, \infty)$. So, rather than write $\frac{C}{2}$ in the last line, we re-named $\frac{C}{2}$ to C.

S-22: We begin by simplifying the integrand.

$$\int 2^{x+\log_2 x}dx = \int 2^x \cdot 2^{\log_2 x}dx = \int 2^x \cdot x\, dx$$

This is similar to the integral $\int xe^x dx$, which we saw in Example 3.5.1 of the text. Let's write $2 = e^{\ln 2}$ to take advantage of the easy integrability of e^x.

$$= \int x \cdot e^{x\ln 2}dx$$

We use integration by parts with $u = x$, $dv = e^{x\ln 2}dx$; $du = dx$, $v = \frac{1}{\ln 2}e^{x\ln 2}$. (Remember $\ln 2$ is a constant. If you'd prefer, you can do a substitution with $s = x\ln 2$ first, to have a simpler exponent of e.)

$$\begin{aligned}
&= \underbrace{\frac{x}{\ln 2}e^{x\ln 2}}_{uv} - \int \underbrace{\frac{1}{\ln 2}e^{x\ln 2}dx}_{vdu} \\
&= \frac{x}{\ln 2}e^{x\ln 2} - \frac{1}{(\ln 2)^2}e^{x\ln 2} + C \\
&= \frac{x}{\ln 2}2^x - \frac{1}{(\ln 2)^2}2^x + C
\end{aligned}$$

S-23: It's not obvious where to start, but in general it's nice to have the arguments of our trig functions the same. So, we use the identity $\sin(2x) = 2\sin x\cos x$.

$$\int e^{\cos x}\sin(2x)\mathrm{d}x = 2\int e^{\cos x}\cos x\sin x\mathrm{d}x$$

Now we can use the substitution $w = \cos x$, $\mathrm{d}w = -\sin x\mathrm{d}x$.

$$= -2\int we^{w}\mathrm{d}w$$

From here the integral should look more familiar. We can use integration by parts with $u = w$, $\mathrm{d}v = e^{w}\mathrm{d}w$, $\mathrm{d}u = \mathrm{d}w$, and $v = e^{w}$.

$$\begin{aligned}
&= -2\left[\underbrace{we^{w}}_{uv} - \int\underbrace{e^{w}\mathrm{d}w}_{v\mathrm{d}u}\right]\\
&= 2e^{w}\left[1-w\right] + C\\
&= 2e^{\cos x}[1-\cos x] + C
\end{aligned}$$

S-24: We've got an integrand that consists of several functions multiplied together, and no obvious substitution. So, we think about integration by parts. We know an antiderivative for $\frac{1}{(1-x)^2}$, because we know $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{1}{(1-x)^2}$. So let's try $\mathrm{d}v = \frac{\mathrm{d}x}{(1-x)^2}$ and $u = xe^{-x}$. Then $v = \frac{1}{1-x}$ and $\mathrm{d}u = (1-x)e^{-x}\,\mathrm{d}x$. So, by integration by parts,

$$\begin{aligned}
\int\underbrace{xe^{-x}}_{u}\underbrace{\frac{\mathrm{d}x}{(1-x)^2}}_{\mathrm{d}v} &= \underbrace{\frac{xe^{-x}}{1-x}}_{uv} - \int\underbrace{\frac{1}{1-x}}_{v}\underbrace{(1-x)e^{-x}\,\mathrm{d}x}_{\mathrm{d}u}\\
&= \frac{xe^{-x}}{1-x} - \int e^{-x}\,\mathrm{d}x\\
&= \frac{xe^{-x}}{1-x} + e^{-x} + C = \frac{e^{-x}}{1-x} + C
\end{aligned}$$

S-25: (a) The "parts" in the integrand are powers of sine. Looking at the right hand side of the reduction formula, we see that it looks a little like the derivative of $\sin^{n-1}x$, although not exactly. So, let's integrate by parts with $u = \sin^{n-1}x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$, so that $\mathrm{d}u = (n-1)\sin^{n-2}x\cos x$ and $v = -\cos x$.

$$\int\sin^n x\,\mathrm{d}x = \underbrace{-\sin^{n-1}x\ \cos x}_{uv} + \underbrace{(n-1)\int\cos^2 x\ \sin^{n-2}x\,\mathrm{d}x}_{-\int v\mathrm{d}u}$$

Using the identity $\sin^2 x + \cos^2 x = 1$,

$$= -\sin^{n-1} x \ \cos x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x \, dx$$
$$= -\sin^{n-1} x \ \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

Moving the last term on the right hand side to the left hand side gives

$$n \int \sin^n x \, dx = -\sin^{n-1} x \ \cos x + (n-1) \int \sin^{n-2} x \, dx$$

Dividing across by n gives the desired reduction formula.

(b) By the reduction formula of part (a), if $n \geqslant 2$,

$$\int_0^{\pi/2} \sin^n(x) \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2}(x) \, dx$$

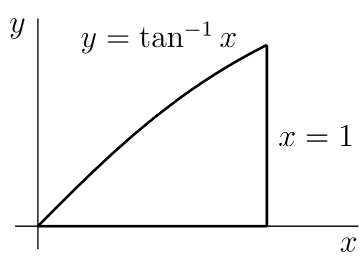
since $\sin 0 = \cos \frac{\pi}{2} = 0$. Applying this reduction formula, with $n = 8, 6, 4, 2$:

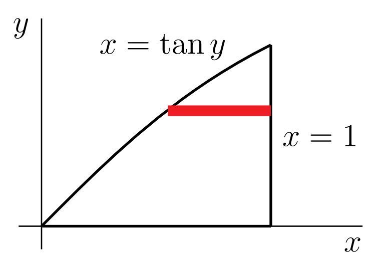
$$\begin{aligned} \int_0^{\pi/2} \sin^8(x) \, dx &= \frac{7}{8} \int_0^{\pi/2} \sin^6(x) \, dx = \frac{7}{8} \cdot \frac{5}{6} \int_0^{\pi/2} \sin^4(x) \, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \int_0^{\pi/2} \sin^2(x) \, dx \\ &= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35}{256}\pi \end{aligned}$$

Using a calculator, we see this is approximately 0.4295.

S-26: The sketch is the figure on the left below. By integration by parts with $u = \arctan x$, $dv = dx$, $v = x$ and $du = \frac{1}{1+x^2} dx$, and then the substitution $s = 1 + x^2$,

$$A = \int_0^1 \arctan x \, dx = \underbrace{x \arctan x}_{uv} \Big|_0^1 - \int_0^1 \underbrace{\frac{x}{1+x^2} dx}_{v du} = \arctan 1 - \tfrac{1}{2} \ln(1+x^2) \Big|_0^1$$
$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$





S-27: For a fixed value of x, if we rotate about the x-axis, we form a washer of inner radius $B(x)$ and outer radius $T(x)$ and hence of area $\pi[T(x)^2 - B(x)^2]$. We integrate this function from $x = 0$ to $x = 3$ to find the total volume V:

$$\begin{aligned}
V &= \int_0^3 \pi[T(x)^2 - B(x)^2]\,\mathrm{d}x \\
&= \pi \int_0^3 (\sqrt{x}e^{3x})^2 - (\sqrt{x}(1+2x))^2\,\mathrm{d}x \\
&= \pi \int_0^3 \big(xe^{6x} - (x + 4x^2 + 4x^3)\big)\,\mathrm{d}x \\
&= \pi \int_0^3 xe^{6x}\,\mathrm{d}x - \pi\Big[\frac{x^2}{2} + \frac{4x^3}{3} + x^4\Big]_0^3 \\
&= \pi \int_0^3 xe^{6x}\,\mathrm{d}x - \pi\Big[\frac{3^2}{2} + \frac{4\cdot 3^3}{3} + 3^4\Big]
\end{aligned}$$

For the first integral, we use integration by parts with $u(x) = x$, $\mathrm{d}v = e^{6x}\mathrm{d}x$, so that $\mathrm{d}u = \mathrm{d}x$ and $v(x) = \frac{1}{6}e^{6x}$:

$$\begin{aligned}
\int_0^3 xe^{6x}\,\mathrm{d}x &= \underbrace{\frac{xe^{6x}}{6}}_{uv}\Bigg|_0^3 - \int_0^3 \underbrace{\frac{1}{6}e^{6x}\,\mathrm{d}x}_{v\mathrm{d}u} \\
&= \frac{3e^{18}}{6} - 0 - \frac{1}{36}e^{6x}\Bigg|_0^3 = \frac{e^{18}}{2} - \left(\frac{e^{18}}{36} - \frac{1}{36}\right).
\end{aligned}$$

Therefore, the total volume is

$$V = \pi\left[\frac{e^{18}}{2} - \left(\frac{e^{18}}{36} - \frac{1}{36}\right)\right] - \pi\left[\frac{3^2}{2} + \frac{4\cdot 3^3}{3} + 3^4\right] = \pi\left(\frac{17e^{18} - 4373}{36}\right).$$

S-28: To get rid of the square root in the argument of f'', we make the change of variables (also called "substitution") $x = t^2,\ \mathrm{d}x = 2t\,\mathrm{d}t$.

$$\int_0^4 f''\big(\sqrt{x}\big)\,\mathrm{d}x = 2\int_0^2 tf''(t)\,\mathrm{d}t$$

Then, to convert f'' into f', we integrate by parts with $u = t$, $\mathrm{d}v = f''(t)\,\mathrm{d}t$, $v = f'(t)$.

$$\begin{aligned}
\int_0^4 f''\big(\sqrt{x}\big)\,\mathrm{d}x &= 2\left\{\Big[\underbrace{tf'(t)}_{uv}\Big]_0^2 - \int_0^2 \underbrace{f'(t)\,\mathrm{d}t}_{v\mathrm{d}u}\right\} \\
&= 2\Big[tf'(t) - f(t)\Big]_0^2 \\
&= 2\big[2f'(2) - f(2) + f(0)\big] = 2\big[2\times 4 - 3 + 1\big] \\
&= 12
\end{aligned}$$

S-29: As we saw in Section 3.1 of the text, there are many different ways to interpret a limit as a Riemann sum. In the absence of instructions that restrain our choices, we go with the most convenient interpretations.

With that in mind, we choose:

- that our Riemann sum is a right Riemann sum (because we see i, not $i-1$ or $i-\frac{1}{2}$)
- $\Delta x = \frac{2}{n}$ (because it is multiplied by the rest of the integrand, and also shows up multiplied by i),
- then $x_i = a + i\Delta x = \frac{2}{n}i - 1$, which leads us to $a = -1$ and
- $f(x) = xe^x$.
- Finally, since $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and $a = -1$, we have $b = 1$.

So, the limit is equal to the definite integral

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\frac{2}{n}i-1\right)e^{\frac{2}{n}i-1} = \int_{-1}^{1} xe^x\,\mathrm{d}x$$

which we evaluate using integration by parts with $u = x$, $\mathrm{d}v = e^x\mathrm{d}x$, $\mathrm{d}u = \mathrm{d}x$, and $v = e^x$.

$$\begin{aligned} &= \Big[\underbrace{xe^x}_{uv}\Big]_{-1}^{1} - \int_{-1}^{1}\underbrace{e^x\mathrm{d}x}_{v\mathrm{d}u} \\ &= \left(e+\frac{1}{e}\right) - \left(e-\frac{1}{e}\right) = \frac{2}{e} \end{aligned}$$

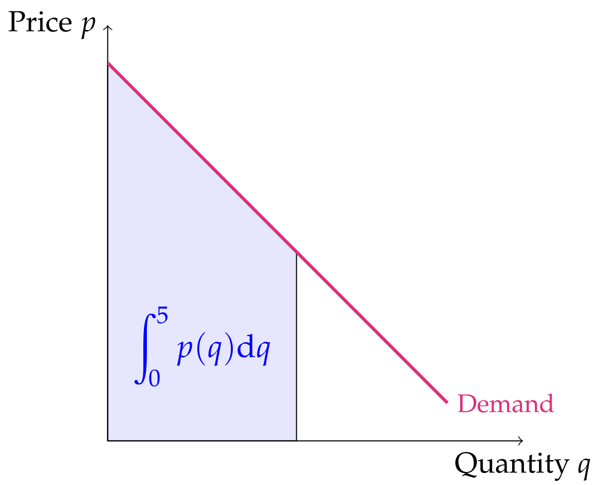
S-30: The first thing to do is find p_e, which is going to be the price at which demand is 5. That is:

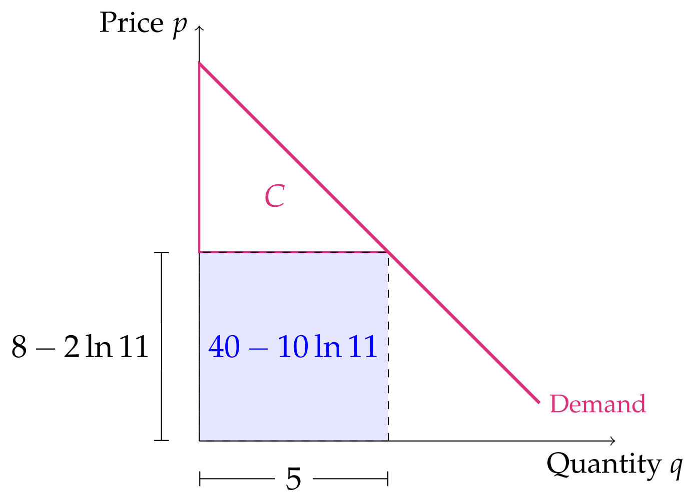
$$p_e = p(5) = 8 - 2\ln(2\cdot 5+1) = 8 - 2\ln 11$$

What we want to find is the area underneath the curve $p(q) = 8 - 2\ln(2q+1)$ and above the line $p = 8 - 2\ln 11$ on the interval from $q = 0$ to $q = 5$. We'll do this in two steps. The integral

$$\int_0^5 8 - 2\ln(2q+1)\mathrm{d}q$$

gives the area under the demand curve and above the horizontal axis. The area underneath the line $p = p_e$ and the horizontal axis is a rectangle. That rectangle has width 5 and height $8 - 2\ln 11$, so its area is $40 - 10\ln 11$. To find C, we subtract this number from our integral.





So, all together, the consumer surplus is $\int_0^5 (8-2\ln(2q+1))\mathrm{d}q - (40-10\ln 11)$.

$$\begin{aligned}
\mathrm{CS} &= \left(\int_0^5 (8-2\ln(2q+1))\mathrm{d}q\right) - (40-10\ln 11)\\
&= \left(\int_0^5 8\mathrm{d}q\right) - 40 + 10\ln 11 - \int_0^5 2\ln(2q+1)\mathrm{d}q\\
&= 10\ln 11 - \int_0^5 2\ln(2q+1)\mathrm{d}q
\end{aligned}$$

Let $s = 2q+1$, so $2\mathrm{d}q = \mathrm{d}s$

$$= 10\ln 11 - \int_{2\cdot 0+1}^{2\cdot 5+1} \ln s\mathrm{d}s = 10\ln 11 - \int_1^{11} \ln s\mathrm{d}s$$

We've already found the antiderivative of natural log, but we'll find it again for practice. Let $u = \ln s$, $\mathrm{d}v = \mathrm{d}s$; $\mathrm{d}u = \frac{1}{s}\mathrm{d}s$, $v = s$

$$\begin{aligned}
&= 10\ln 11 - \left[s\ln s\Big|_1^{11} - \int_1^{11} s\cdot\frac{1}{s}\mathrm{d}s\right]\\
&= 10\ln 11 - \left[(11\ln 11 - 1\ln 1) - \int_1^{11} 1\mathrm{d}s\right]\\
&= 10\ln 11 - 11\ln 11 + \int_1^{11} 1\mathrm{d}s\\
&= 10\ln 11 - 11\ln 11 + 10 = 10 - \ln 11
\end{aligned}$$

S-31:

(a) Using the Fundamental Theorem of Calculus,

$$TC = \int \mathrm{MC}\mathrm{d}q + C$$

for some constant C.

$$\int \mathrm{MC}\mathrm{d}q = \int \left(\frac{qe^{\frac{q}{10}}}{100} - q + 30 \right) \mathrm{d}q = \left(\int \frac{1}{100} qe^{q/10} \mathrm{d}q \right) - \frac{1}{2}q^2 + 30q$$

To make things a little easier to see, we'll use the substitution $\frac{q}{10} = s$, $\frac{1}{10}\mathrm{d}q = \mathrm{d}s$:

$$= \left(\int \frac{q}{10} \cdot e^{q/10} \cdot \frac{1}{10} \mathrm{d}q \right) - \frac{1}{2}q^2 + 30q$$
$$= \left(\int se^s \mathrm{d}s \right) - \frac{1}{2}q^2 + 30q$$

Let $u = s$, $\mathrm{d}v = e^s \mathrm{d}s$; $\mathrm{d}u = \mathrm{d}s$, $v = e^s$

$$= \left(se^s - \int e^s \mathrm{d}s \right) - \frac{1}{2}q^2 + 30q$$
$$= (se^s - e^s) - \frac{1}{2}q^2 + 30q + C$$

Now we can comfortably eliminate our invention s, as its usefulness has expired.

$$= \left(\frac{q}{10} e^{q/10} - e^{q/10} \right) - \frac{1}{2}q^2 + 30q + C$$
$$= \left(\frac{q}{10} - 1 \right) e^{q/10} - \frac{1}{2}q^2 + 30q + C$$

Now, we use the observation at the start of this solution:

$$\mathrm{TC} = \left(\frac{q}{10} - 1 \right) e^{q/10} - \frac{1}{2}q^2 + 30q + C$$

The problem mentions $\mathrm{TC}(0) = $FC, and FC$= 1000$, so:

$$1000 = \mathrm{TC}(0) = \left(\frac{0}{10} - 1 \right) e^0 - \frac{1}{2}(0^2) + 30(0) + C = -1 + C$$
$$C = 1000 + 1 = 1001$$
$$\mathrm{TC} = \left(\frac{q}{10} - 1 \right) e^{q/10} - \frac{1}{2}q^2 + 30q + 1001$$

(b) The cost of making 10 items is

$$\mathrm{TC}(10) = \left(\frac{10}{10} - 1 \right) e^{10/10} - \frac{1}{2}(10^2) + 30(10) + 1001$$
$$= 1251$$

So, the average cost-per-item is

$$\frac{1251}{10} = 125.10$$

Solutions to Exercises 3.6 — Jump to TABLE OF CONTENTS

S-1: If $u = \cos x$, then $-\mathrm{d}u = \mathrm{d}x$. If $n \neq -1$, then

$$\int_0^{\pi/4} \sin x \cos^n x \, \mathrm{d}x = -\int_1^{1/\sqrt{2}} u^n \mathrm{d}u = \left[-\frac{1}{n+1}u^{n+1}\right]_1^{1/\sqrt{2}} = \frac{1}{n+1}\left(1 - \frac{1}{\sqrt{2}^{n+1}}\right)$$

If $n = -1$, then

$$\begin{aligned}\int_0^{\pi/4} \sin x \cos^n x \, \mathrm{d}x &= -\int_1^{1/\sqrt{2}} u^n \mathrm{d}u = -\int_1^{1/\sqrt{2}} \frac{1}{u} \mathrm{d}u = \Big[-\ln|u|\Big]_1^{1/\sqrt{2}} \\ &= -\ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\ln 2\end{aligned}$$

So, (e) n can be any real number.

S-2: We use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x \, \mathrm{d}x$.

$$\int \sec^n x \tan x \mathrm{d}x = \int \sec^{n-1} x \cdot \sec x \tan x \, \mathrm{d}x = \int u^{n-1} \mathrm{d}u$$

Since n is positive, $n - 1 \neq -1$, so we antidifferentiate using the power rule.

$$= \frac{u^n}{n} + C = \frac{1}{n}\sec^n x + C$$

S-3: We divide both sides by $\cos^2 x$, and simplify.

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \frac{\sin^2 x + \cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\ \frac{\sin^2 x}{\cos^2 x} + 1 &= \sec^2 x \\ \tan^2 x + 1 &= \sec^2 x\end{aligned}$$

S-4: The power of cosine is odd, and the power of sine is even (zero). Following the strategy in the text, we make the substitution $u = \sin x$, so that $\mathrm{d}u = \cos x\,\mathrm{d}x$ and $\cos^2 x = 1 - \sin^2 x = 1 - u^2$:

$$\begin{aligned}\int \cos^3 x\,\mathrm{d}x &= \int (1-\sin^2 x)\cos x\,\mathrm{d}x = \int (1-u^2)\,\mathrm{d}u \\ &= u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C\end{aligned}$$

S-5: Using the trig identity $\cos^2 x = \dfrac{1+\cos(2x)}{2}$, we have

$$\int \cos^2 x\mathrm{d}x = \frac{1}{2}\int_0^\pi \left[1+\cos(2x)\right]\mathrm{d}x = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right]_0^\pi = \frac{\pi}{2}$$

S-6: Since the power of cosine is odd, following the strategies in the text, we make the substitution $u = \sin t$, so that $\mathrm{d}u = \cos t\,\mathrm{d}t$ and $\cos^2 t = 1 - \sin^2 t = 1 - u^2$.

$$\begin{aligned}\int \sin^{36} t\cos^3 t\,\mathrm{d}t &= \int \sin^{36} t\,(1-\sin^2 t)\cos t\,\mathrm{d}t = \int u^{36}(1-u^2)\,\mathrm{d}u \\ &= \frac{u^{37}}{37} - \frac{u^{39}}{39} + C = \frac{\sin^{37} t}{37} - \frac{\sin^{39} t}{39} + C\end{aligned}$$

S-7: Since the power of sine is odd (and positive), we can reserve one sine for $\mathrm{d}u$, and turn the rest into cosines using the identity $\sin^2 + \cos^2 x = 1$. This allows us to use the substitution $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$, and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^4 x}\,\mathrm{d}x &= \int \frac{\sin^2 x}{\cos^4 x}\sin x\,\mathrm{d}x = \int -\frac{1-u^2}{u^4}\mathrm{d}u \\ &= \int\left(-\frac{1}{u^4} + \frac{1}{u^2}\right)\,\mathrm{d}u = \frac{1}{3u^3} - \frac{1}{u} + C \\ &= \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C\end{aligned}$$

S-8: Both sine and cosine have even powers (four and zero, respectively), so we don't have the option of using a substitution like $u = \sin x$ or $u = \cos x$. Instead, we use the identity $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$.

$$\begin{aligned}
\int_0^{\pi/3} \sin^4 x \, dx &= \int_0^{\pi/3} \left(\sin^2 x\right)^2 \, dx = \int_0^{\pi/3} \left(\frac{1-\cos(2x)}{2}\right)^2 \, dx \\
&= \frac{1}{4}\int_0^{\pi/3} \left(1 - 2\cos(2x) + \cos^2(2x)\right) \, dx \\
&= \frac{1}{4}\int_0^{\pi/3} \left(1 - 2\cos(2x)\right) \, dx + \frac{1}{4}\int_0^{\pi/3} \cos^2(2x) \, dx
\end{aligned}$$

We can antidifferentiate the first integral right away. For the second integral, we use the identity $\quad \cos^2\theta = \dfrac{1+\cos(2\theta)}{2}, \quad$ with $\theta = 2x$.

$$\begin{aligned}
&= \frac{1}{4}\Big[x - \sin(2x)\Big]_0^{\pi/3} + \frac{1}{8}\int_0^{\pi/3} (1+\cos(4x)) \, dx \\
&= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right] + \frac{1}{8}\Big[x + \frac{1}{4}\sin(4x)\Big]_0^{\pi/3} \\
&= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right] + \frac{1}{8}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{8}\right] \\
&= \frac{\pi}{8} - \frac{9\sqrt{3}}{64}
\end{aligned}$$

S-9: Since the power of sine is odd, we can reserve one sine for du, and change the remaining four into cosines. This sets us up to use the substitution $u = \cos x$, $du = -\sin x \, dx$.

$$\begin{aligned}
\int \sin^5 x \, dx &= \int \sin^4 x \cdot \sin x \, dx = \int (1-\cos^2 x)^2 \sin x \, dx \\
&= -\int (1-u^2)^2 \, du = -\int (1 - 2u^2 + u^4) du \\
&= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\
&= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C
\end{aligned}$$

S-10: If we use the substitution $u = \sin x$, then $du = \cos x \, dx$, which very conveniently shows up in the integrand.

$$\int \sin^{1.2} x \cos x \, dx = \int u^{1.2} du = \frac{u^{2.2}}{2.2} + C = \frac{1}{2.2}\sin^{2.2} x + C$$

Note this is exactly the strategy described in the text when the power of cosine is odd. The non-integer power of sine doesn't cause a problem.

S-11:

Solution 1: Let's use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$:

$$\int \tan x \sec^2 x \mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$$

Solution 2: We can also use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$:

$$\int \tan x \sec^2 x \mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sec^2 x + C$$

We note that because $\tan^2 x$ and $\sec^2 x$ only differ by a constant, the two answers are equivalent.

S-12:

Solution 1: Substituting $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$, $\sin^2 x = 1 - \cos^2 x = 1 - u^2$, gives

$$\begin{aligned}\int \tan^3 x \sec^5 x\,\mathrm{d}x &= \int \frac{\sin^3 x}{\cos^8 x}\,\mathrm{d}x = \int \frac{(1-\cos^2 x)\sin x}{\cos^8 x}\,\mathrm{d}x = -\int \frac{1-u^2}{u^8}\,\mathrm{d}u \\ &= -\left[\frac{u^{-7}}{-7} - \frac{u^{-5}}{-5}\right] + C = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C\end{aligned}$$

Solution 2: Alternatively, substituting $u = \sec x$, $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$, $\tan^2 x = \sec^2 x - 1 = u^2 - 1$, gives

$$\begin{aligned}\int \tan^3 x \sec^5 x\,\mathrm{d}x &= \int \tan^2 x \sec^4 x\,(\tan x \sec x)\,\mathrm{d}x = \int (u^2-1)u^4\,\mathrm{d}u \\ &= \left[\frac{u^7}{7} - \frac{u^5}{5}\right] + C = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C\end{aligned}$$

S-13: Use the substitution $u = \tan x$, so that $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$:

$$\begin{aligned}\int \sec^4 x\ \tan^{46} x\,\mathrm{d}x &= \int (\tan^2 x + 1)\tan^{46} x\ \sec^2 x\,\mathrm{d}x = \int (u^2+1)u^{46}\,\mathrm{d}u \\ &= \frac{u^{49}}{49} + \frac{u^{47}}{47} + C = \frac{\tan^{49} x}{49} + \frac{\tan^{47} x}{47} + C\end{aligned}$$

S-14: We use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$. Then $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.

$$\begin{aligned}\int \tan^3 x \sec^{1.5} x\,\mathrm{d}x &= \int \tan^2 x \cdot \sec^{0.5} x \cdot \sec x \tan x \mathrm{d}x \\ &= \int (u^2-1)u^{0.5}\,\mathrm{d}u = \int \left(u^{2.5} - u^{0.5}\right)\,\mathrm{d}u \\ &= \frac{u^{3.5}}{3.5} - \frac{u^{1.5}}{1.5} + C \\ &= \frac{1}{3.5}\sec^{3.5} x - \frac{1}{1.5}\sec^{1.5} x + C\end{aligned}$$

Note this solution used the same method as Example 3.6.13 in the text for the case that the power of tangent is odd and there is at least one secant.

S-15: As in Question 14, we have an odd power of tangent and at least one secant. So, we can use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x\, \mathrm{d}x$, and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.

$$\begin{aligned}
\int \tan^3 x \sec^2 x\, \mathrm{d}x &= \int \tan^2 x \sec x \cdot \sec x \tan x\, \mathrm{d}x \\
&= \int (u^2-1)u\, \mathrm{d}u = \int \left(u^3 - u\right)\, \mathrm{d}u \\
&= \frac{1}{4}u^4 - \frac{1}{2}u^2 + C \\
&= \frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + C
\end{aligned}$$

S-16: In contrast to Questions 14 and 15, we do not have an odd power of tangent, so we should consider a different substitution. Luckily, if we choose $u = \tan x$, then $\mathrm{d}u = \sec^2 x\, \mathrm{d}x$, and this fits our integrand nicely.

$$\int \tan^4 x \sec^2 x\, \mathrm{d}x = \int u^4\, \mathrm{d}u = \frac{1}{5}u^5 + C = \frac{1}{5}\tan^5 x + C$$

S-17:

Solution 1: Since the power of tangent is odd, let's try to use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x\, \mathrm{d}x$, and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$, as in Questions 14 and 15. In order to make this work, we need to see $\sec x \tan x\, \mathrm{d}x$ in the integrand, so we do a little algebraic manipulation.

$$\begin{aligned}
\int \tan^3 x \sec^{-0.7} x\, \mathrm{d}x &= \int \frac{\tan^3 x}{\sec^{0.7 x}}\, \mathrm{d}x = \int \frac{\tan^3 x}{\sec^{1.7 x}} \sec x\, \mathrm{d}x \\
&= \int \frac{\tan^2 x}{\sec^{1.7} x} \cdot \sec x \tan x\, \mathrm{d}x \\
&= \int \frac{u^2 - 1}{u^{1.7}}\, \mathrm{d}u = \int \left(u^{0.3} - u^{-1.7}\right)\, \mathrm{d}u \\
&= \frac{u^{1.3}}{1.3} + \frac{1}{0.7u^{0.7}} + C \\
&= \frac{1}{1.3}\sec^{1.3} x + \frac{1}{0.7\sec^{0.7} x} + C \\
&= \frac{1}{1.3}\sec^{1.3} x + \frac{1}{0.7}\cos^{0.7} x + C
\end{aligned}$$

Solution 2: Let's convert the secants and tangents to sines and cosines.

$$\begin{aligned}\int \tan^3 x \sec^{-0.7} x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} \cdot \cos^{0.7} x \, dx \\ &= \int \frac{\sin^3 x}{\cos^{2.3} x} \, dx = \int \frac{\sin^2 x}{\cos^{2.3} x} \cdot \sin x \, dx\end{aligned}$$

Using the substitution $u = \cos x$, $du = -\sin\ dx$, and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$:

$$\begin{aligned}&= -\int \frac{1-u^2}{u^{2.3}} \, du = \int \left(-u^{-2.3} + u^{-0.3}\right) \, du \\ &= \frac{1}{1.3}u^{-1.3} + \frac{1}{0.7}u^{0.7} + C \\ &= \frac{1}{1.3}\sec^{1.3} x + \frac{1}{0.7}\cos^{0.7} x + C\end{aligned}$$

S-18: We replace $\tan x$ with $\dfrac{\sin x}{\cos x}$.

$$\int \tan^5 x \, dx = \int \left(\frac{\sin x}{\cos x}\right)^5 \, dx = \int \frac{\sin^4 x}{\cos^5 x} \cdot \sin x \, dx$$

Now we use the substitution $u = \cos x$, $du = -\sin x \, dx$, and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned}&= -\int \frac{(1-u^2)^2}{u^5} \, du = \int \left(-u^{-5} + 2u^{-3} - u^{-1}\right) \, du \\ &= \frac{1}{4}u^{-4} - u^{-2} - \ln|u| + C \\ &= \frac{1}{4}\sec^4 x - \sec^2 x - \ln|\cos x| + C \\ &= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\end{aligned}$$

where in the last line, we used the logarithm rule $\log(b^a) = a \log b$, with $b^a = \cos x = (\sec x)^{-1}$.

S-19: Integrating even powers of tangent is surprisingly different from integrating odd powers of tangent. For even powers, we use the identity $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x\, \mathrm{d}x$ on (parhaps only a part of) the resulting integral.

$$\begin{aligned}
\int_0^{\pi/6} \tan^6 x\, \mathrm{d}x &= \int_0^{\pi/6} \tan^4 x(\sec^2 x - 1)\, \mathrm{d}x \\
&= \int_0^{\pi/6} \left(\underbrace{\tan^4 x \sec^2 x}_{u^4\, \mathrm{d}u} - \tan^4 x \right) \mathrm{d}x \\
&= \int_0^{\pi/6} \left(\tan^4 x \sec^2 x - \tan^2 x(\sec^2 x - 1) \right) \mathrm{d}x \\
&= \int_0^{\pi/6} \left(\tan^4 x \sec^2 x - \underbrace{\tan^2 x \sec^2 x}_{u^2\, \mathrm{d}u} + \tan^2 x \right) \mathrm{d}x \\
&= \int_0^{\pi/6} \left(\tan^4 x \sec^2 x - \tan^2 x \sec^2 x + (\underbrace{\sec^2 x}_{\mathrm{d}u} - 1) \right) \mathrm{d}x \\
&= \int_0^{\pi/6} \left(\tan^4 x - \tan^2 x + 1 \right) \sec^2 x\, \mathrm{d}x - \int_0^{\pi/6} 1 \mathrm{d}x
\end{aligned}$$

Note $\tan(0) = 0$, and $\tan(\pi/6) = 1/\sqrt{3}$.

$$\begin{aligned}
&= \int_0^{1/\sqrt{3}} (u^4 - u^2 + 1)\, \mathrm{d}u - \big[x\big]_0^{\pi/6} \\
&= \left[\frac{1}{5}u^5 - \frac{1}{3}u^3 + u \right]_0^{1/\sqrt{3}} - \frac{\pi}{6} \\
&= \frac{1}{5\sqrt{3}^5} - \frac{1}{3\sqrt{3}^3} + \frac{1}{\sqrt{3}} - \frac{\pi}{6} \\
&= \frac{41}{45\sqrt{3}} - \frac{\pi}{6}
\end{aligned}$$

S-20: Since there is an even power of secant in the integrand, we can reserve two secants for $\mathrm{d}u$ and change the rest to tangents. That sets us up nicely to use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x\, \mathrm{d}x$. Note $\tan(0) = 0$ and $\tan(\pi/4) = 1$.

$$\begin{aligned}
\int_0^{\pi/4} \tan^8 x \sec^4 x\, \mathrm{d}x &= \int_0^{\pi/4} \tan^8 x\, (\tan^2 x + 1) \sec^2 x\, \mathrm{d}x \\
&= \int_0^1 u^8\, (u^2 + 1)\, \mathrm{d}u \\
&= \int_0^1 u^{10} + u^8\, \mathrm{d}u \\
&= \frac{1}{11} + \frac{1}{9}
\end{aligned}$$

S-21:

Solution 1: Let's use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$. In order to make this work, we need to see $\sec x \tan x$ in the integrand, so we start with some algebraic manipulation.

$$\begin{aligned}\int \tan x \sqrt{\sec x}\left(\frac{\sqrt{\sec x}}{\sqrt{\sec x}}\right)\,\mathrm{d}x &= \int \frac{1}{\sqrt{\sec x}} \sec x \tan x\,\mathrm{d}x\\ &= \int \frac{1}{\sqrt{u}}\,\mathrm{d}u = 2\sqrt{u} + C\\ &= 2\sqrt{\sec x} + C\end{aligned}$$

Solution 2: Let's turn our secants and tangents into sines and cosines.

$$\int \tan x \sqrt{\sec x}\,\mathrm{d}x = \int \frac{\sin x}{\cos x \cdot \sqrt{\cos x}}\,\mathrm{d}x = \int \frac{\sin x}{\cos^{1.5} x}\,\mathrm{d}x$$

We use the substitution $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$.

$$\begin{aligned}&= \int -u^{-1.5}\,\mathrm{d}u = \frac{2}{\sqrt{u}} + C\\ &= 2\sqrt{\sec x} + C\end{aligned}$$

S-22: Since the power of secant is even and positive, we can reserve two secants for $\mathrm{d}u$, and change the rest into tangents, setting the stage for the substitution $u = \tan\theta$, $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$.

$$\begin{aligned}\int \sec^8\theta \tan^e\theta\,\mathrm{d}\theta &= \int \sec^6\theta \tan^e\theta \sec^2\theta\,\mathrm{d}\theta\\ &= \int (\tan^2\theta + 1)^3 \tan^e\theta \sec^2\theta\,\mathrm{d}\theta\\ &= \int (u^2+1)^3 \cdot u^e\,\mathrm{d}u\\ &= \int (u^6 + 3u^4 + 3u^2 + 1)\cdot u^e\,\mathrm{d}u\\ &= \int (u^{6+e} + 3u^{4+e} + 3u^{2+e} + u^e)\,\mathrm{d}u\\ &= \frac{1}{7+e}u^{7+e} + \frac{3}{5+e}u^{5+e} + \frac{3}{3+e}u^{3+e} + \frac{1}{1+e}u^{1+e} + C\\ &= \frac{1}{7+e}\tan^{7+e}\theta + \frac{3}{5+e}\tan^{5+e}\theta + \frac{3}{3+e}\tan^{3+e}\theta + \frac{1}{1+e}\tan^{1+e}\theta + C\\ &= \tan^{1+e}\theta\left(\frac{\tan^6\theta}{7+e} + \frac{3\tan^4\theta}{5+e} + \frac{3\tan^2\theta}{3+e} + \frac{1}{1+e}\right) + C\end{aligned}$$

S-23: (a) Using the trig identity $\tan^2 x = \sec^2 x - 1$ and the substitution $y = \tan x$, $\mathrm{d}y = \sec^2 x\,\mathrm{d}x$,

$$\begin{aligned}
\int \tan^n x\,\mathrm{d}x = \int \tan^{n-2} x\ \tan^2 x\,\mathrm{d}x \qquad &= \int \tan^{n-2} x\ \sec^2 x\,\mathrm{d}x - \int \tan^{n-2} x\,\mathrm{d}x \\
&= \int y^{n-2}\,\mathrm{d}y - \int \tan^{n-2} x\,\mathrm{d}x = \frac{y^{n-1}}{n-1} - \int \tan^{n-2} x\,\mathrm{d}x \\
&= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,\mathrm{d}x
\end{aligned}$$

(b) By the reduction formula of part (a),

$$\begin{aligned}
\int_0^{\pi/4} \tan^n(x)\,\mathrm{d}x &= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2}(x)\,\mathrm{d}x \\
&= \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2}(x)\,\mathrm{d}x
\end{aligned}$$

for all integers $n \geqslant 2$, since $\tan 0 = 0$ and $\tan \frac{\pi}{4} = 1$. We apply this reduction formula, with $n = 6, 4, 2$.

$$\begin{aligned}
\int_0^{\pi/4} \tan^6(x)\,\mathrm{d}x &= \frac{1}{5} - \int_0^{\pi/4} \tan^4(x)\,\mathrm{d}x = \frac{1}{5} - \frac{1}{3} + \int_0^{\pi/4} \tan^2(x)\,\mathrm{d}x = \frac{1}{5} - \frac{1}{3} + 1 - \int_0^{\pi/4} \mathrm{d}x \\
&= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}
\end{aligned}$$

Using a calculator, we see this is approximately 0.0813.

Notice how much faster this was than the method of Question 19.

S-24: Recall $\tan x = \dfrac{\sin x}{\cos x}$.

$$\int \tan^5 x \cos^2 x\,\mathrm{d}x = \int \frac{\sin^5 x}{\cos^5 x} \cos^2 x\,\mathrm{d}x = \int \frac{\sin^5 x}{\cos^3 x}\,\mathrm{d}x$$

Substitute $u = \cos x$, so $\mathrm{d}u = -\sin x\,\mathrm{d}x$ and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned}
&= \int \frac{\sin^4 x}{\cos^3 x}\ \sin x\,\mathrm{d}x = -\int \frac{(1-u^2)^2}{u^3}\,\mathrm{d}u \\
&= -\int \frac{1 - 2u^2 + u^4}{u^3}\,\mathrm{d}u = \int \left(-\frac{1}{u^3} + \frac{2}{u} - u\right)\mathrm{d}u \\
&= \frac{1}{2u^2} + 2|u| - \frac{1}{2}u^2 + C \\
&= \frac{1}{2\cos^2 x} + 2|\cos x| - \frac{1}{2}\cos^2 x + C
\end{aligned}$$

S-25: We can use the definition of secant to make this integral look more familiar.

$$\int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$$

S-26: We re-write $\cot x = \dfrac{\cos x}{\sin x}$, and use the substitution $u = \sin x$, $du = \cos x\,dx$.

$$\begin{aligned}\int \cot x\,dx &= \int \frac{\cos x}{\sin x}\,dx = \int \frac{1}{u}\,du \\ &= |u| + C = \ln|\sin x| + C\end{aligned}$$

S-27:

Solution 1: We begin with the obvious substitution, $w = e^x$, $dw = e^x dw$.

$$\int e^x \sin(e^x)\cos(e^x)\,dx = \int \sin w \cos w\,dw$$

Now we see another substitution, $u = \sin w$, $du = \cos w\,dw$.

$$\begin{aligned}&= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 w + C \\ &= \frac{1}{2}\sin^2(e^x) + C\end{aligned}$$

Solution 2: Notice that $\frac{d}{dx}\{\sin(e^x)\} = e^x\cos(e^x)$. This suggests to us the substitution $u = \sin(e^x)$, $du = e^x\cos(e^x)\,dx$.

$$\int e^x \sin(e^x)\cos(e^x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(e^x) + C$$

S-28: Since we have an "inside function," we start with the substitution $s = \cos x$, so $-ds = \sin x\,dx$ and $\sin^2 x = 1 - \cos^2 x = 1 - s^2$.

$$\begin{aligned}\int \sin(\cos x)\sin^3 x\,dx &= \int \sin(\cos x)\cdot\sin^2 x\cdot\sin x dx \\ &= -\int \sin(s)\cdot(1-s^2)\,ds\end{aligned}$$

We use integration by parts with $u = (1-s^2)$, $dv = \sin s\,ds$; $du = -2s\,ds$, and $v = -\cos s$.

$$\begin{aligned}&= -\left[-(1-s^2)\cos s - \int 2s\cos s\,ds\right] \\ &= (1-s^2)\cos s + \int 2s\cos s\,ds\end{aligned}$$

We integrate by parts again, with $u = 2s$, $dv = \cos s \, ds$; $du = 2 \, ds$, and $v = \sin s$.

$$\begin{aligned}
&= (1-s^2)\cos s + 2s\sin s - \int 2\sin s \, ds \\
&= (1-s^2)\cos s + 2s\sin s + 2\cos s + C \\
&= \sin^2 x \cdot \cos(\cos x) + 2\cos x \cdot \sin(\cos x) + 2\cos(\cos x) + C \\
&= (\sin^2 x + 2)\cos(\cos x) + 2\cos x \cdot \sin(\cos x) + C
\end{aligned}$$

S-29:

Since the integrand is the product of polynomial and trigonometric functions, we suspect it might yield to integration by parts. There are a number of ways this can be accomplished.

Solution 1: Before we choose parts, let's use the identity $\sin(2x) = 2\sin x \cos x$.

$$\int x \sin x \cos x dx = \frac{1}{2}\int x\sin(2x)dx$$

Now let $u = x$, $dv = \sin(2x)dx$; $du = dx$, and $v = -\frac{1}{2}\cos(2x)$. Using integration by parts:

$$\begin{aligned}
&= \frac{1}{2}\left[-\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)dx\right] \\
&= -\frac{x}{4}\cos(2x) + \frac{1}{8}\sin(2x) + C \\
&= -\frac{x}{4}(1-2\sin^2 x) + \frac{1}{4}\sin x\cos x + C \\
&= -\frac{x}{4} + \frac{x}{2}\sin^2 x + \frac{1}{4}\sin x \cos x + C
\end{aligned}$$

Solution 2: If we let $u = x$, then $du = dx$, and this seems desirable for integration by parts. If $u = x$, then $dv = \sin x \cos x dx$. To find v we can use the substitution $u = \sin x$, $du = \cos x dx$.

$$v = \int \sin x \cos x dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

So, we take $v = \frac{1}{2}\sin^2 x$. Now we can apply integration by parts to our original integral.

$$\int x \sin x \cos x \, dx = \frac{x}{2}\sin^2 x - \int \frac{1}{2}\sin^2 x dx$$

Apply the identity $\sin^2 x = \dfrac{1-\cos(2x)}{2}$.

$$= \frac{x}{2}\sin^2 x - \frac{1}{4}\int 1-\cos(2x)\mathrm{d}x$$
$$= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{8}\sin(2x) + C$$
$$= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{4}\sin x\cos x + C$$

Solution 3: Let $u = x\sin x$ and $\mathrm{d}v = \cos x\mathrm{d}x$; then $\mathrm{d}u = (x\cos x + \sin x)\mathrm{d}x$ and $v = \sin x$.

$$\begin{aligned}\int x\sin x\cos x\mathrm{d}x &= x\sin^2 x - \int \sin x(x\cos x + \sin x)\mathrm{d}x\\ &= x\sin^2 x - \int x\sin x\cos x\mathrm{d}x - \int \sin^2 x\mathrm{d}x\end{aligned}$$

Apply the identity $\sin^2 x = \dfrac{1-\cos(2x)}{2}$ to the second integral.

$$\begin{aligned}&= x\sin^2 x - \int x\sin x\cos x\mathrm{d}x - \int \frac{1-\cos(2x)}{2}\mathrm{d}x\\ &= x\sin^2 x - \int x\sin x\cos x\mathrm{d}x - \frac{x}{2} + \frac{1}{4}\sin(2x) + C\end{aligned}$$

So, we have the equation

$$\begin{aligned}\int x\sin x\cos x\mathrm{d}x &= x\sin^2 x - \int x\sin x\cos x\mathrm{d}x - \frac{x}{2} + \frac{1}{4}\sin(2x) + C\\ 2\int x\sin x\cos x\mathrm{d}x &= x\sin^2 x - \frac{x}{2} + \frac{1}{4}\sin(2x) + C\\ \int x\sin x\cos x\mathrm{d}x &= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{8}\sin(2x) + \frac{C}{2}\\ &= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{4}\sin x\cos x + \frac{C}{2}\end{aligned}$$

Since C is an arbitrary constant that can take any number in $(-\infty, \infty)$, also $\frac{C}{2}$ is an arbitrary constant that can take any number in $(-\infty, \infty)$, so we're free to rename $\frac{C}{2}$ to C.

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S-1: In the text, there is a template for choosing an appropriate substitution, but for this problem we will explain the logic of the choices.

The trig identities that we can use are:

$$1-\sin^2\theta = \cos^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

They have the following forms:

$$\text{constant} - \text{function} \qquad \text{function} + \text{constant} \qquad \text{function} - \text{constant}$$

In order to cancel out the square root, we should choose a substitution that will match the argument under the square root with the trig identity of the corresponding form.

(a) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the square root in the denominator. Under the square root is the function $9x^2 - 16$, which has the form (function) − (constant). This form matches the trig identity $\sec^2\theta - 1 = \tan^2\theta$. We can set x to be whatever we need it to be, but we don't have the same control over the constant, 16. So, to make the substitution work, we use a different form of the trig identity: multiplying both sides by 16, we get

$$16\sec^2\theta - 16 = 16\tan^2\theta$$

What we want is a substitution that gives us

$$\begin{aligned} 9x^2 - 16 &= 16\sec^2\theta - 16 \\ \text{So,} \qquad 9x^2 &= 16\sec^2\theta \\ x &= \frac{4}{3}\sec\theta \end{aligned}$$

Using this substitution,

$$\begin{aligned} \sqrt{9x^2 - 16} &= \sqrt{16\sec^2\theta - 16} \\ &= \sqrt{16\tan^2\theta} \\ &= 4|\tan\theta| \end{aligned}$$

So, we eliminated the square root.

(b) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the square root in the denominator. Under the square root is the function $1 - 4x^2$, which has the form (constant) − (function). This form matches the trig identity $1 - \sin^2\theta = \cos^2\theta$. What we want is a substitution that gives us

$$\begin{aligned} 1 - 4x^2 &= 1 - \sin^2\theta \\ \text{So,} \qquad 4x^2 &= \sin^2\theta \\ x &= \frac{1}{2}\sin\theta \end{aligned}$$

Using this substitution,

$$\begin{aligned} \sqrt{1 - 4x^2} &= \sqrt{1 - \sin^2\theta} \\ &= \sqrt{\cos^2\theta} \\ &= |\cos\theta| \end{aligned}$$

So, we eliminated the square root.

(c) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the fractional power. (That is, we want to eliminate the square root.) The function under the power is $25 + x^2$, which has the form (constant) + (function). This form matches the trig identity $\tan^2\theta + 1 = \sec^2\theta$. We can set x to be whatever we need it to be, but we don't have the same control over the constant, 25. So, to make the substitution work, we use a different form of the trig identity: multiplying both sides by 25, we get

$$25\tan^2\theta + 25 = 25\sec^2\theta$$

What we want is a substitution that gives us

$$\begin{aligned} 25 + x^2 &= 25\tan^2\theta + 25 \\ \text{So,} \qquad x^2 &= 25\tan^2\theta \\ x &= 5\tan\theta \end{aligned}$$

Using this substitution,

$$\begin{aligned} (25 + x^2)^{-5/2} &= (25 + 25\tan^2\theta)^{-5/2} \\ &= (25\sec^2\theta)^{-5/2} \\ &= (5|\sec\theta|)^{-5} \end{aligned}$$

So, we eliminated the square root.

S-2: Just as in Question 1, we want to choose a trigonometric substitution that will allow us to eliminate the square roots. Before we can make that choice, though, we need to complete the square. In subsequent problems, we won't show the algebra behind completing the square, but for this problem we'll work it out explicitly. After some practice, you'll be able to do this step in your head for many cases.

After the squares are completed, the choice of trig substitution follows the logic outlined in the solutions to Question 1, or (equivalently) the template in the text.

(a) The quadratic function under the square root is $x^2 - 4x + 1$. To complete the square, we match the non-constant terms to those of a perfect square.

$$\begin{aligned} (ax + b)^2 &= a^2x^2 + 2abx + b^2 \\ x^2 - 4x + 1 &= a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c \end{aligned}$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-4 = 2ab = 2b$, so $b = -2$.
- Finally, the constant terms give us $1 = b^2 + c = 4 + c$, so $c = -3$.

$$\int \frac{1}{\sqrt{x^2 - 4x + 1}}\,dx = \int \frac{1}{\sqrt{(x-2)^2 - 3}}\,dx = \int \frac{1}{\sqrt{(x-2)^2 - \sqrt{3}^2}}\,dx$$

So we use the substitution $(x - 2) = \sqrt{3}\sec u$, which eliminates the square root:

$$\sqrt{(x-2)^2 - 3} = \sqrt{3\sec^2 u - 3} = \sqrt{3\tan^2 u} = \sqrt{3}|\tan u|$$

(b) The quadratic function under the square root is $-x^2+2x+4 = -[x^2-2x-4]$. To complete the square, we match the non-constant terms to those of a perfect square. We factored out the negative to make things a little easier–don't forget to put it back in before choosing a substitution!

$$\begin{aligned}(ax+b)^2 &= a^2x^2+2abx+b^2\\ x^2-2x-4 &= a^2x^2+2abx+b^2+c \quad \text{for some constant } c\end{aligned}$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-2 = 2ab = 2b$, so $b = -1$.
- Finally, the constant terms give us $-4 = b^2+c = 1+c$, so $c = -5$.
- Then $-x^2+2x+4 = -[x^2-2x-4] = -[(x-1)^2-5] = 5-(x-1)^2$.

$$\int \frac{(x-1)^6}{(-x^2+2x+4)^{3/2}}\,dx = \int \frac{(x-1)^6}{(5-(x-1)^2)^{3/2}}\,dx = \int \frac{(x-1)^6}{\left(\sqrt{5}^2-(x-1)^2\right)^{3/2}}\,dx$$

So we use the substitution $(x-1) = \sqrt{5}\sin u$, which eliminates the square root (fractional power):

$$(5-(x-1)^2)^{3/2} = \left(5-5\sin^2 u\right)^{3/2} = \left(5\cos^2 u\right)^{3/2} = 5\sqrt{5}|\cos^3 u|$$

(c) The quadratic function under the square root is $4x^2+6x+10$. To complete the square, we match the non-constant terms to those of a perfect square.

$$\begin{aligned}(ax+b)^2 &= a^2x^2+2abx+b^2\\ 4x^2+6x+10 &= a^2x^2+2abx+b^2+c \quad \text{for some constant } c\end{aligned}$$

- Looking at the leading term tells us $a = 2$.
- Then the second term tells us $6 = 2ab = 4b$, so $b = \frac{3}{2}$.
- Finally, the constant terms give us $10 = b^2+c = \frac{9}{4}+c$, so $c = \frac{31}{4}$.

$$\int \frac{1}{\sqrt{4x^2+6x+10}}\,dx = \int \frac{1}{\sqrt{\left(2x+\frac{3}{2}\right)^2+\frac{31}{4}}}\,dx = \int \frac{1}{\sqrt{\left(2x+\frac{3}{2}\right)^2+\left(\frac{\sqrt{31}}{2}\right)^2}}\,dx$$

So we use the substitution $\left(2x+\frac{3}{2}\right) = \frac{\sqrt{31}}{2}\tan u$, which eliminates the square root:

$$\sqrt{\left(2x+\frac{3}{2}\right)^2+\frac{31}{4}} = \sqrt{\frac{31}{4}\tan^2 u+\frac{31}{4}} = \sqrt{\frac{31}{4}\sec^2 u} = \frac{\sqrt{31}}{2}|\sec u|$$

(d) The quadratic function under the square root is x^2-x. To complete the square, we match the non-constant terms to those of a perfect square.

$$\begin{aligned}(ax+b)^2 &= a^2x^2+2abx+b^2\\ x^2-x &= a^2x^2+2abx+b^2+c \quad \text{for some constant } c\end{aligned}$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-1 = 2ab = 2b$, so $b = -\frac{1}{2}$.
- Finally, the constant terms give us $0 = b^2 + c = \frac{1}{4} + c$, so $c = -\frac{1}{4}$.

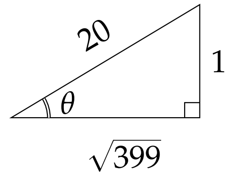
$$\int \sqrt{x^2 - x}\, dx = \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}}\, dx = \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\, dx$$

So we use the substitution $(x - 1/2) = \frac{1}{2}\sec u$, which eliminates the square root:

$$\sqrt{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}} = \sqrt{\frac{1}{4}\sec^2 u - \frac{1}{4}} = \sqrt{\frac{1}{4}\tan^2 u} = \frac{1}{2}|\tan u|$$

S-3:

(a) If $\sin\theta = \dfrac{1}{20}$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has opposite side length 1, and hypotenuse length 20. By the Pythagorean Theorem, the adjacent side has length $\sqrt{20^2 - 1^2} = \sqrt{399}$. So, $\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{399}}{20}$.



We can do a quick "reasonableness" check here: $\frac{1}{20}$ is pretty close to 0, so we might expect θ to be pretty close to 0, and so $\cos\theta$ should be pretty close to 1. Indeed it is: $\dfrac{\sqrt{399}}{20} \approx \dfrac{\sqrt{400}}{20} = \dfrac{20}{20} = 1$.

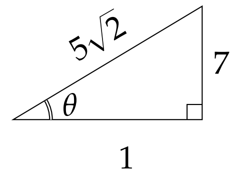
Alternately, we can solve this problem using identities.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{1}{20}\right)^2 + \cos^2\theta = 1$$
$$\cos\theta = \pm\sqrt{1 - \frac{1}{400}} = \pm\frac{\sqrt{399}}{20}$$

Since $0 \leqslant \theta \leqslant \frac{\pi}{2}$, $\cos\theta \geqslant 0$, so

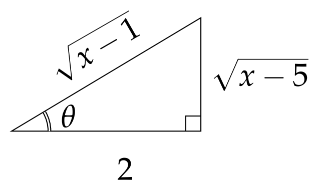
$$\cos\theta = \frac{\sqrt{399}}{20}$$

(b) If $\tan\theta = 7$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has opposite side length 7 and adjacent side length 1. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$. So, $\csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{5\sqrt{2}}{7}$.



Again, we can do a quick reasonableness check. Since 7 is much larger than 1, the triangle we're thinking of doesn't look much like the triangle in our standardized picture above: it's really quite tall, with a small base. So, the opposite side and hypotenuse are pretty close in length. Indeed, $\frac{5\sqrt{2}}{7} \approx 7.071$, so this dimension seems reasonable.

(c) If $\sec\theta = \frac{\sqrt{x-1}}{2}$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has hypotenuse length $\sqrt{x-1}$ and adjacent side length 2. By the Pythagorean Theorem, the opposite side has length $\sqrt{\sqrt{x-1}^2 - 2^2} = \sqrt{x-1-4} = \sqrt{x-5}$. So, $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x-5}}{2}$.



We can also solve this using identities. Note that since $\sec\theta$ exists, $\theta \neq \frac{\pi}{2}$.

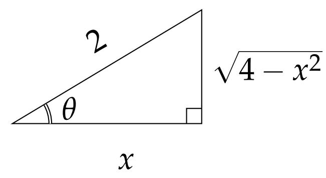
$$\begin{aligned}
\tan^2\theta + 1 &= \sec^2\theta \\
\tan^2\theta + 1 &= \left(\frac{\sqrt{x-1}}{2}\right)^2 = \frac{x-1}{4} \\
\tan\theta &= \pm\sqrt{\frac{x-1}{4} - 1} = \pm\frac{\sqrt{x-5}}{2}
\end{aligned}$$

Since $0 \leqslant \theta < \frac{\pi}{2}$, $\tan\theta \geqslant 0$, so

$$\tan\theta = \frac{\sqrt{x-5}}{2}$$

S-4:

(a) Let $\theta = \arccos\left(\frac{x}{2}\right)$. That is, $\cos(\theta) = \frac{x}{2}$, and $0 \leqslant \theta \leqslant \pi$. Then we can draw the corresponding right triangle with angle θ with adjacent side of signed length x (we note that if $\theta > \frac{\pi}{2}$, then x is negative) and hypotenuse of length 2. By the Pythagorean Theorem, the opposite side of the triangle has length $\sqrt{4-x^2}$.



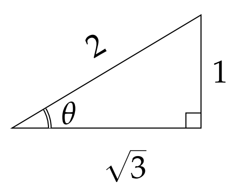
So,

$$\sin\left(\arccos\left(\frac{x}{2}\right)\right) = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$$

(b) Let $\theta = \arctan\left(\frac{1}{\sqrt{3}}\right)$. That is, $\tan(\theta) = \frac{1}{\sqrt{3}}$, and $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

Solution 1: Then $\theta = \dfrac{\pi}{6}$, so $\sin\theta = \dfrac{1}{2}$.

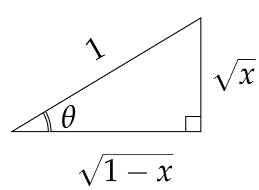
Solution 2: Then we can draw the corresponding right triangle with angle θ with opposite side of length 1 and adjacent side of length $\sqrt{3}$. By the Pythagorean Theorem, the hypotenuse of the triangle has length $\sqrt{\sqrt{3}^2 + 1^2} = 2$.



So,

$$\sin\left(\arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$$

(c) Let $\theta = \arcsin\left(\sqrt{x}\right)$. That is, $\sin(\theta) = \sqrt{x}$, and $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$. Then we can draw the corresponding right triangle with angle θ with opposite side of length $\sqrt{x}$ and hypotenuse of length 1. By the Pythagorean Theorem, the adjacent side of the triangle has length $\sqrt{1-x}$.

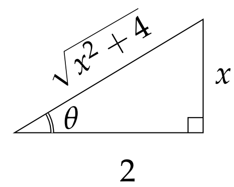


So,

$$\sec\left(\arcsin\left(\sqrt{x}\right)\right) = \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\sqrt{1-x}}$$

S-5: Let $x = 2\tan\theta$, so that $x^2+4 = 4\tan^2\theta+4 = 4\sec^2\theta$ and $\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$. Then

$$\begin{aligned}\int \frac{1}{(x^2+4)^{3/2}}\,\mathrm{d}x &= \int \frac{1}{(4\sec^2\theta)^{3/2}}\cdot 2\sec^2\theta\,\mathrm{d}\theta\\ &= \int \frac{2\sec^2\theta}{8\sec^3\theta}\,\mathrm{d}\theta\\ &= \frac{1}{4}\int \cos\theta\,\mathrm{d}\theta\\ &= \frac{1}{4}\sin\theta + C = \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C\end{aligned}$$



To find $\sin\theta$ in terms of x, we construct the right triangle above. Since $\tan\theta = \frac{x}{2} = \frac{\text{opp}}{\text{adj}}$, we label the opposite side x and the adjacent side 2. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{x^2+4}$. Then $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+4}}$.

To see why we could write $(\sec^2\theta)^{3/2} = \sec^3\theta$, as opposed to $(\sec^2\theta)^{3/2} = \left|\sec^3\theta\right|$, in the second line above, see Example 3.7.5 in the text.

As a check, we observe that the derivative of the answer

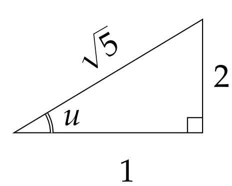
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C\right) &= \frac{1}{4}\frac{1}{\sqrt{x^2+4}} - \frac{1}{2\times 4}\frac{x(2x)}{\left(x^2+4\right)^{3/2}} = \frac{\frac{x^2}{4}+1-\frac{x^2}{4}}{\left(x^2+4\right)^{3/2}}\\ &= \frac{1}{\left(x^2+4\right)^{3/2}}\end{aligned}$$

is exactly the integrand.

S-6:

Solution 1: As in Question 5, substitute $x = 2\tan u$, $\mathrm{d}x = 2\sec^2 u\,\mathrm{d}u$. Note that when $x = 4$ we have $4 = 2\tan u$, so that $\tan u = 2$.

$$\begin{aligned}\int_0^4 \frac{1}{\left(4+x^2\right)^{3/2}}\,\mathrm{d}x &= \int_0^{\arctan 2} \frac{1}{\left(4+4\tan^2 u\right)^{3/2}}\,2\sec^2 u\,\mathrm{d}u\\ &= \int_0^{\arctan 2} \frac{2\sec^2 u}{\left(2\sec u\right)^3}\,\mathrm{d}u\\ &= \frac{1}{4}\int_0^{\arctan 2} \frac{\sec^2 u}{\sec^3 u}\,\mathrm{d}u\\ &= \frac{1}{4}\int_0^{\arctan 2} \cos u\,\mathrm{d}u\\ &= \left[\frac{1}{4}\sin u\right]_0^{\arctan 2}\\ &= \frac{1}{4}\big(\sin(\arctan 2) - 0\big) = \frac{1}{2\sqrt{5}}\end{aligned}$$



To find $\sin(\arctan 2)$, we use the right triangle above, with angle $u = \arctan 2$. Since $\tan u = 2 = \frac{\text{opp}}{\text{adj}}$, we label the opposite side as 2, and the adjacent side as 1. The Pythagorean Theorem tells us the hypotenuse has length $\sqrt{5}$, so $\sin u = \frac{\text{opp}}{\text{hyp}} = \frac{2}{\sqrt{5}}$.

Solution 2: Using our result from Question 5,

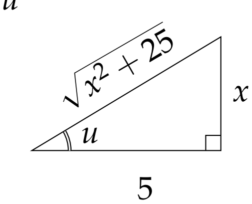
$$\begin{aligned}\int_0^4 \frac{1}{(4+x^2)^{3/2}}\,\mathrm{d}x &= \frac{1}{4}\left[\frac{x}{\sqrt{x^2+4}}\right]_0^4 \\ &= \frac{1}{4}\cdot\frac{4}{\sqrt{4^2+4}} = \frac{1}{2\sqrt{5}}\end{aligned}$$

S-7: Make the change of variables $x = 5\sin\theta$, $\mathrm{d}x = 5\cos\theta\,\mathrm{d}\theta$. Since $x = 0$ corresponds to $\theta = 0$ and $x = \frac{5}{2}$ correponds to $\sin\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$,

$$\int_0^{5/2} \frac{\mathrm{d}x}{\sqrt{25-x^2}} = \int_0^{\pi/6} \frac{5\cos\theta\,\mathrm{d}\theta}{\sqrt{25-25\sin^2\theta}} = \int_0^{\pi/6} \mathrm{d}\theta = \frac{\pi}{6}$$

S-8: Substitute $x = 5\tan u$, so that $\mathrm{d}x = 5\sec^2 u\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+25}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{25\tan^2 u+25}}\,5\sec^2 u\,\mathrm{d}u \\ &= \int \frac{5\sec^2 u}{5\sec u}\,\mathrm{d}u = \int \sec u\,\mathrm{d}u \\ &= \ln\left|\sec u + \tan u\right| + C \\ &= \ln\left|\sqrt{1+\frac{x^2}{25}} + \frac{x}{5}\right| + C\end{aligned}$$



To find $\sec u$ and $\tan u$, we have two options. One is to set up a right triangle with angle u and $\tan u = \frac{x}{5}$. Then we can label the opposite side x and the adjacent side 5, and use Pythagorus to find that the hypotenuse is $\sqrt{x^2+25}$.

Another option is to look back at our work a little more closely–in fact, we've already found what we're looking for. Since we used the substitution $x = 5\tan u$, this gives us $\tan u = \frac{x}{5}$. In the denominator of the integrand, we simplified $\sqrt{x^2+25} = 5\sec u$, so $\sec u = \frac{1}{5}\sqrt{x^2+25} = \sqrt{1+\frac{x^2}{25}}$.

To see why we could write $\sqrt{x^2+25} = 5\sec u$, as opposed to $\sqrt{x^2+25} = 5|\sec u|$, see Example 3.7.5 in the text.

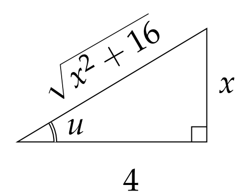
S-9: The quadratic formula underneath the square root makes us think of a trig substitution, but in the interest of developing good habits, let's check for an easier way first. If we let $u = 2x^2+4x$, then $\mathrm{d}u = (4x+4)\,\mathrm{d}x$, so $\frac{1}{4}\,\mathrm{d}u = (x+1)\,\mathrm{d}x$. This substitution looks easier than a trig substitution (which would start with completing the square).

$$\int \frac{x+1}{\sqrt{2x^2+4x}}\,dx = \frac{1}{4}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\sqrt{u}+C = \frac{1}{2}\sqrt{2x^2+4x}+C$$

S-10: Substitute $x = 4\tan u$, $dx = 4\sec^2 u\,du$.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+16}}\,dx &= \int \frac{1}{16\tan^2 u\sqrt{16\tan^2 u+16}}\,4\sec^2 u\,du \\ &= \int \frac{\sec^2 u}{16\tan^2 u\sec u}\,du = \frac{1}{16}\int \frac{\sec u}{\tan^2 u}\,du \\ &= \frac{1}{16}\int \frac{\cos u}{\sin^2 u}\,du\end{aligned}$$

To finish off the integral, we'll substitute $v = \sin u$, $dv = \cos u\,du$.



$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+16}}\,dx = \frac{1}{16}\int \frac{\cos u}{\sin^2 u}\,du = \frac{1}{16}\int \frac{dv}{v^2} = -\frac{1}{16v}+C \\ = -\frac{1}{16\sin u}+C = -\frac{1}{16}\frac{\sqrt{x^2+16}}{x}+C\end{aligned}$$

To find $\sin u$, we draw a right triangle with angle u and $\tan u = \frac{x}{4}$. We label the opposite side x and the adjacent side 4, and then from Pythagorus we find that the hypotenuse has length $\sqrt{x^2+16}$. So, $\sin u = \dfrac{\sqrt{x^2+16}}{x}$.

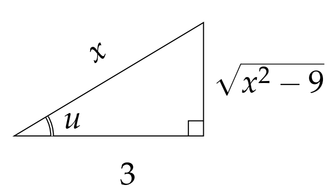
As a check, we observe that the derivative of the answer

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{16}\frac{\sqrt{x^2+16}}{x}+C\right) &= \frac{1}{16}\frac{\sqrt{x^2+16}}{x^2} - \frac{1}{16}\frac{x}{x\sqrt{x^2+16}} = \frac{1}{16}\frac{(x^2+16)-x^2}{x^2\sqrt{x^2+16}} \\ &= \frac{1}{x^2\sqrt{x^2+16}}\end{aligned}$$

is exactly the integrand.

S-11: Substitute $x = 3\sec u$ with $0 \leqslant u < \frac{\pi}{2}$. Then $dx = 3\sec u\tan u\,du$ and $\sqrt{x^2-9} = \sqrt{9\sec^2 u - 9} = \sqrt{9\tan^2 u} = 3\tan u$, so that

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-9}} &= \int \frac{3\sec u\tan u\,du}{9\sec^2 u\sqrt{9\tan^2 u}} \\ &= \frac{1}{9}\int \frac{du}{\sec u} \\ &= \frac{1}{9}\int \cos u\,du = \frac{1}{9}\sin u + C.\end{aligned}$$



To evaluate $\sin u$, we make a right triangle with angle u. Since $\sec u = \frac{x}{3} = \frac{\text{hyp}}{\text{adj}}$, we label the hypotenuse x and the adjacent side 3. Using the Pythagorean Theorem, the opposite side has length $\sqrt{x^2-9}$. So, $\sin u = \frac{\sqrt{x^2-9}}{x}$ and

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}} = \frac{\sqrt{x^2-9}}{9x} + C.$$

As a check, we observe that the derivative of the answer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{x^2-9}}{9x} + C\right) &= -\frac{\sqrt{x^2-9}}{9x^2} + \frac{x}{9x\sqrt{x^2-9}} = \frac{1}{9}\,\frac{-(x^2-9)+x^2}{x^2\sqrt{x^2-9}} \\ &= \frac{1}{x^2\sqrt{x^2-9}}\end{aligned}$$

is exactly the integrand. (We remark that this is the case even for $x \leqslant -3$.)

S-12: (a) We'll use the trig identity $\cos 2\theta = 2\cos^2\theta - 1$. It implies that

$$\begin{aligned}\cos^2\theta = \frac{\cos 2\theta + 1}{2} \implies \cos^4\theta &= \frac{1}{4}\big[\cos^2 2\theta + 2\cos 2\theta + 1\big] = \frac{1}{4}\left[\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right] \\ &= \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\end{aligned}$$

So,

$$\begin{aligned}\int_0^{\pi/4} \cos^4\theta \,\mathrm{d}\theta &= \int_0^{\pi/4} \left(\frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\right) \mathrm{d}\theta \\ &= \left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3}{8}\theta\right]_0^{\pi/4} \\ &= \frac{1}{4} + \frac{3}{8}\cdot\frac{\pi}{4} \\ &= \frac{8+3\pi}{32}\end{aligned}$$

as required.

(b) We'll use the trig substitution $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta \,\mathrm{d}\theta$. Note that when $\theta = \pm\frac{\pi}{4}$, we have $x = \pm 1$. Also note that dividing the trig identity $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ gives the trig identity $\tan^2\theta + 1 = \sec^2\theta$. So

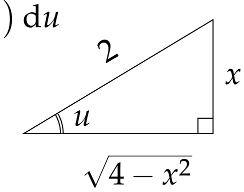
$$\begin{aligned}
\int_{-1}^{1} \frac{\mathrm{d}x}{\left(x^2+1\right)^3} &= 2\int_0^1 \frac{\mathrm{d}x}{\left(x^2+1\right)^3} && \text{(even integrand)}\\
&= 2\int_0^{\pi/4} \frac{\sec^2\theta \,\mathrm{d}\theta}{\left(\tan^2\theta+1\right)^3}\\
&= 2\int_0^{\pi/4} \frac{\sec^2\theta \,\mathrm{d}\theta}{\left(\sec^2\theta\right)^3}\\
&= 2\int_0^{\pi/4} \cos^4\theta \,\mathrm{d}\theta\\
&= \frac{8+3\pi}{16}
\end{aligned}$$

by part (a).

S-13: The integrand is an odd function, and the limits of integration are symmetric, so $\displaystyle\int_{-\pi/12}^{\pi/12} \frac{15x^3}{(x^2+1)\sqrt{9-x^2}^5}\,\mathrm{d}x = 0$.

S-14: Substitute $x = 2\sin u$, so that $\mathrm{d}x = 2\cos u\,\mathrm{d}u$.

$$\begin{aligned}
\int \sqrt{4-x^2}\,\mathrm{d}x &= \int \sqrt{4-4\sin^2 u}\, 2\cos u\,\mathrm{d}u\\
&= \int \sqrt{4\cos^2 u}\, 2\cos u\,\mathrm{d}u\\
&= \int 4\cos^2 u\,\mathrm{d}u = 2\int \big(1+\cos(2u)\big)\,\mathrm{d}u\\
&= 2u + \sin(2u) + C\\
&= 2u + 2\sin u\cos u + C\\
&= 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C
\end{aligned}$$



To see why we could write $\sqrt{4\cos^2 u} = 2\cos u$, as opposed to $\sqrt{4\cos^2 u} = 2|\cos u|$, in the third line above, see Example 3.7.2 in the text.

We used the substitution $x = 2\sin u$, so we know $\sin u = \frac{x}{2}$ and $u = \arcsin(\frac{x}{2})$. We have three options for finding $\cos u$.

First, we can draw a right triangle with angle u. Since $\sin u = \frac{x}{2}$, we label the opposite side x and the hypotenuse 2, then by the Pythagorean Theorem the adjacent side has length $\sqrt{4-x^2}$. So, $\cos u = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{4-x^2}}{2}$.

Second, we can look back carefully at our work. We simplified $\sqrt{4-x^2} = 2\cos u$, so $\cos u = \dfrac{\sqrt{4-x^2}}{2}$.

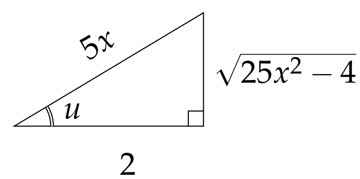
Third, we could use the identity $\sin^2 u + \cos^2 u = 1$. Then $\cos u = \pm\sqrt{1-\sin^2 u} = \pm\sqrt{1-\frac{x^2}{4}}$. Since $u = \arcsin(x/2)$, u is in the range of arcsine, which means $-\frac{\pi}{2} \leqslant u \leqslant \frac{\pi}{2}$. Therefore, $\cos u \geqslant 0$, so $\cos u = \sqrt{1-\frac{x^2}{4}} = \frac{\sqrt{4-x^2}}{2}$.

So,

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2u + 2\sin u\cos u + C = 2\arcsin\frac{x}{2} + x\cdot\frac{\sqrt{4-x^2}}{2} + C$$

S-15: Substitute $x = \frac{2}{5}\sec u$ with $0 < u < \frac{\pi}{2}$, so that $\mathrm{d}x = \frac{2}{5}\sec u\ \tan u\,\mathrm{d}u$ and $\sqrt{25x^2-4} = \sqrt{4(\sec^2 u - 1)} = \sqrt{4\tan^2 u} = 2\tan u$. Then

$$\begin{aligned}
\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x &= \int \frac{2\tan u}{\frac{2}{5}\sec u}\cdot\frac{2}{5}\sec u\tan u\,\mathrm{d}u \\
&= 2\int \tan^2 u\,\mathrm{d}u = 2\int \left(\sec^2 u - 1\right)\mathrm{d}u \\
&= 2\tan u - 2u + C \\
&= \sqrt{25x^2-4} - 2\,\mathrm{arcsec}\,\tfrac{5x}{2} + C
\end{aligned}$$



To find $\tan u$, we draw a right triangle with angle u. Since $\sec u = \dfrac{5x}{2}$, we label the hypotenuse $5x$ and the adjacent side 2. Then the Pythagorean Theorem gives us the opposite side as length $\sqrt{25x^2-4}$. Then $\tan u = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sqrt{25x^2-4}}{2}$.

Alternately, we can notice that in our work, we already showed $2\tan u = \sqrt{25x^2-4}$, so $\tan u = \frac{1}{2}\sqrt{25x^2-4}$.

As a check, we observe that the derivative of the answer

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{25x^2-4} - 2\,\mathrm{arcsec}\,\frac{5x}{2} + C\right) &= \frac{25x}{\sqrt{25x^2-4}} - 2\frac{\frac{5}{2}}{\left|\frac{5x}{2}\right|\sqrt{\frac{25x^2}{4}-1}} \\
&= \frac{25x}{\sqrt{25x^2-4}} - \frac{4}{x\sqrt{25x^2-4}} \qquad \text{since } x > 0 \\
&= \frac{25x^2-4}{x\sqrt{25x^2-4}} \\
&= \frac{\sqrt{25x^2-4}}{x}
\end{aligned}$$

is exactly the integrand (provided $x > \frac{2}{5}$).

S-16: The integrand has a quadratic polynomial under a square root, which makes us think of trig substitutions. However, it's good practice to look for simpler methods before we jump into more complicated ones, and in this case we find something nicer than a trig substitution: the substitution $u = x^2 - 1$, $\mathrm{d}u = 2x\,\mathrm{d}x$. Then $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$, and $x^2 = u + 1$. When $x = \sqrt{10}$, $u = 9$, and when $x = \sqrt{17}$, $u = 16$.

$$\begin{aligned}
\int_{\sqrt{10}}^{\sqrt{17}} \frac{x^3}{\sqrt{x^2-1}}\,\mathrm{d}x &= \int_{\sqrt{10}}^{\sqrt{17}} \frac{x^2}{\sqrt{x^2-1}} \cdot x\mathrm{d}x \\
&= \frac{1}{2}\int_9^{16} \frac{u+1}{\sqrt{u}}\,\mathrm{d}u \\
&= \frac{1}{2}\int_9^{16} \left(u^{1/2} + u^{-1/2}\right)\,\mathrm{d}u \\
&= \frac{1}{2}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_9^{16} \\
&= \frac{1}{2}\left[\frac{2}{3}\cdot 4^3 + 2\cdot 4 - \frac{2}{3}\cdot 3^3 - 2\cdot 3\right] \\
&= \frac{40}{3}
\end{aligned}$$

S-17: This integrand looks very different from those above. But it is only slightly disguised. If we complete the square

$$\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} = \int \frac{\mathrm{d}x}{\sqrt{4-(x+1)^2}}$$

and make the substitution $y = x + 1$, $\mathrm{d}y = \mathrm{d}x$

$$\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} = \int \frac{\mathrm{d}x}{\sqrt{4-(x+1)^2}} = \int \frac{\mathrm{d}y}{\sqrt{4-y^2}}$$

we get a typical trig substitution integral. So, we substitute $y = 2\sin\theta$, $\mathrm{d}y = 2\cos\theta\,\mathrm{d}\theta$ to get

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} &= \int \frac{\mathrm{d}y}{\sqrt{4-y^2}} = \int \frac{2\cos\theta\,\mathrm{d}\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta\,\mathrm{d}\theta}{\sqrt{4\cos^2\theta}} \\
&= \int \mathrm{d}\theta = \theta + C = \arcsin\frac{y}{2} + C \\
&= \arcsin\frac{x+1}{2} + C
\end{aligned}$$

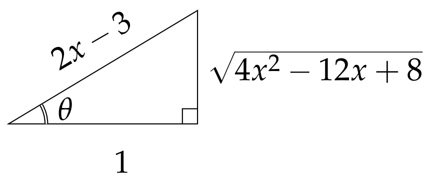
An experienced integrator would probably substitute $x + 1 = 2\sin\theta$ directly, without going through y.

S-18: Completing the square, we see $4x^2 - 12x + 8 = (2x-3)^2 - 1$.

$$\int \frac{1}{(2x-3)^3\sqrt{4x^2-12x+8}}\,\mathrm{d}x = \int \frac{1}{(2x-3)^3\sqrt{(2x-3)^2-1}}\,\mathrm{d}x$$

As $x > 2$, we have $2x - 3 > 1$. We use the substitution $2x - 3 = \sec\theta$ with $0 \leqslant \theta < \frac{\pi}{2}$. So $2\,\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$ and $\sqrt{(2x-3)^2 - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$.

$$\begin{aligned}
&= \frac{1}{2}\int \frac{1}{\sec^3\theta\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\int \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\int \frac{1}{\sec^2\theta}\,\mathrm{d}\theta \\
&= \frac{1}{2}\int \cos^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{4}\int (1 + \cos(2\theta))\,\mathrm{d}\theta \\
&= \frac{1}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{1}{4}(\theta + \sin\theta\cos\theta) + C \\
&= \frac{1}{4}\left(\arccos\left(\frac{1}{2x-3}\right) + \frac{\sqrt{4x^2 - 12x + 8}}{(2x-3)^2}\right) + C
\end{aligned}$$

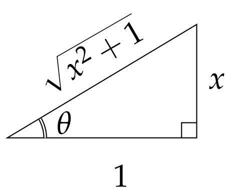


Since $2x - 3 = \sec\theta$, we know $\cos\theta = \frac{1}{2x-3}$ and $\theta = \arccos\left(\frac{1}{2x-3}\right)$. (Equivalently, $\theta = \operatorname{arcsec}(2x-3)$.) To find $\sin\theta$, we draw a right triangle with adjacent side of length 1, and hypotenuse of length $2x - 3$. By the Pythagorean Theorem, the opposite side has length $\sqrt{4x^2 - 12x + 8}$.

S-19: We use the substitution $x = \tan u$, $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$. Note $\tan 0 = 0$ and $\tan\frac{\pi}{4} = 1$.

$$\begin{aligned}
\int_0^1 \frac{x^2}{\sqrt{x^2+1}^3}\mathrm{d}x &= \int_0^{\pi/4} \frac{\tan^2 u}{\sqrt{\tan^2 u + 1}^3}\sec^2 u\,\mathrm{d}u \\
&= \int_0^{\pi/4} \frac{\tan^2 u}{\sqrt{\sec^2 u}^3}\sec^2 u\,\mathrm{d}u \\
&= \int_0^{\pi/4} \frac{\tan^2 u}{\sec u}\,\mathrm{d}u \\
&= \int_0^{\pi/4} \frac{\sec^2 u - 1}{\sec u}\,\mathrm{d}u \\
&= \int_0^{\pi/4} \left(\sec u - \cos u\right)\mathrm{d}u \\
&= \Big[\ln|\sec u + \tan u| - \sin u\Big]_0^{\pi/4} \\
&= \left(\ln\left|\sqrt{2} + 1\right| - \frac{1}{\sqrt{2}}\right) - (\ln|1 + 0| - 0) \\
&= \ln(1 + \sqrt{2}) - \frac{1}{\sqrt{2}}
\end{aligned}$$

S-20: There's no square root, but we can still make use of the substitution $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$.



$$\begin{aligned}
\int \frac{1}{(x^2+1)^2}\, dx &= \int \frac{1}{(\tan^2\theta+1)^2} \sec^2\theta\, d\theta \\
&= \int \frac{1}{\sec^4\theta} \sec^2\theta\, d\theta = \int \cos^2\theta\, d\theta \\
&= \frac{1}{2}\int \big(1+\cos(2\theta)\big)\, d\theta \\
&= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{1}{2}\left(\theta + \sin\theta\cos\theta\right) + C \\
&= \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + C
\end{aligned}$$

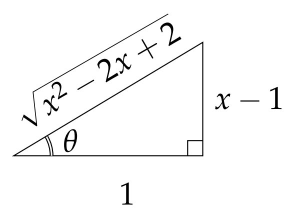
Since $x = \tan\theta$, we can draw a right triangle with angle θ, opposite side x, and adjacent side 1. Then by the Pythagorean Theorem, its hypotenuse has length $\sqrt{x^2+1}$, which allows us to find $\sin\theta$ and $\cos\theta$.

S-21: We complete the square to find $x^2 - 2x + 2 = (x-1)^2 + 1$.

$$\int \frac{x^2}{\sqrt{x^2-2x+2}}\, dx = \int \frac{x^2}{\sqrt{(x-1)^2+1}}\, dx$$

We use the substitution $x - 1 = \tan\theta$, which implies $dx = \sec^2\theta\, d\theta$ and $x = \tan\theta + 1$

$$\begin{aligned}
&= \int \frac{(\tan\theta+1)^2}{\sqrt{(\tan\theta)^2+1}} \sec^2\theta\, d\theta \\
&= \int \frac{\tan^2\theta + 2\tan\theta + 1}{\sec\theta} \sec^2\theta\, d\theta \\
&= \int (\sec^2\theta + 2\tan\theta)\sec\theta\, d\theta \\
&= \int \big(\sec^3\theta + 2\tan\theta\sec\theta\big)\, d\theta \\
&= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + 2\sec\theta + C \\
&= \frac{1}{2}\sqrt{x^2-2x+2}(x-1) + \frac{1}{2}\ln\left|\sqrt{x^2-2x+2} + x - 1\right| \\
&\qquad + 2\sqrt{x^2-2x+2} + C \\
&= \frac{3+x}{2}\sqrt{x^2-2x+2} + \frac{1}{2}\ln\left|\sqrt{x^2-2x+2} + x - 1\right| + C
\end{aligned}$$



To see why we could write $\sqrt{(\tan\theta)^2+25} = \sec\theta$, as opposed to $\sqrt{(\tan\theta)^2+25} = |\sec\theta|$, see Example 3.7.5 in the text.

From our substitution, we know $\tan\theta = x-1$. To find $\sec\theta$, we can notice that in our work we already simplified $\sqrt{x^2-2x+1} = \sec\theta$. Alternately, we can draw a right triangle with angle θ, opposite side $x-1$, adjacent side 1, and use the Pythagorean Theorem to find the hypotenuse.

S-22: First, we complete the square. The constants aren't integers, but we can still use the same method as in Question 2. The quadratic function under the square root is $3x^2+5x$. We match the non-constant terms to those of a perfect square.

$$(ax+b)^2 = a^2x^2 + 2abx + b^2$$
$$3x^2 + 5x = a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c$$

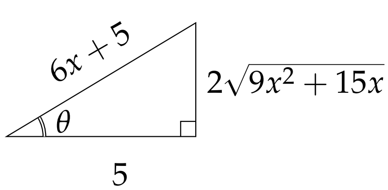
- Looking at the leading term tells us $a = \sqrt{3}$.
- Then the second term tells us $5 = 2ab = 2\sqrt{3}b$, so $b = \frac{5}{2\sqrt{3}}$.
- Finally, the constant terms give us $0 = b^2 + c = \frac{25}{12} + c$, so $c = -\frac{25}{12}$.

So, $3x^2+5x = \left(\sqrt{3}x + \frac{5}{2\sqrt{3}}\right)^2 - \frac{25}{12}$.

$$\int \frac{1}{\sqrt{3x^2+5x}}\,dx = \int \frac{1}{\sqrt{\left(\sqrt{3}x+\frac{5}{2\sqrt{3}}\right)^2 - \frac{25}{12}}}\,dx$$

We use the substitution $\sqrt{3}x + \frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}}\sec\theta$, which leads to $\sqrt{3}dx = \frac{5}{2\sqrt{3}}\sec\theta\tan\theta\,d\theta$, i.e. $dx = \frac{5}{6}\sec\theta\tan\theta\,d\theta$.

$$\begin{aligned}
&= \int \frac{1}{\sqrt{\left(\frac{5}{2\sqrt{3}}\sec\theta\right)^2 - \frac{25}{12}}} \cdot \frac{5}{6}\sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{\sqrt{\frac{25}{12}\sec^2\theta - \frac{25}{12}}} \cdot \frac{5}{6}\sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{\sqrt{\frac{25}{12}\tan^2\theta}} \cdot \frac{5}{6}\sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{\frac{5}{2\sqrt{3}}\tan\theta} \cdot \frac{5}{6}\sec\theta\tan\theta\,d\theta \\
&= \frac{1}{\sqrt{3}}\int \sec\theta\,d\theta \\
&= \frac{1}{\sqrt{3}}\ln|\sec\theta + \tan\theta| + C \\
&= \frac{1}{\sqrt{3}}\ln\left|\left(\frac{6}{5}x+1\right) + \frac{2}{5}\sqrt{9x^2+15x}\right| + C
\end{aligned}$$



Since we used the substitution $\sqrt{3}x + \frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}}\sec\theta$, we have $\sec\theta = \frac{6}{5}x + 1 = \frac{6x+5}{5}$. To find $\tan\theta$ in terms of x, we have two options. We can make a right triangle with angle θ, hypotenuse $6x+5$, and adjacent side 5, then use the Pythagorean Theorem to find the opposite side. Or, we can look through our work and see that $\sqrt{3x^2+5} = \frac{5}{2\sqrt{3}}\tan\theta$, so $\tan\theta = \frac{2\sqrt{3}}{5}\sqrt{3x^2+5} = \frac{2}{5}\sqrt{9x^2+15}$.

As a check, we observe that the derivative of the answer

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{3}}\ln\left|\left(\frac{6}{5}x+1\right)+\frac{2}{5}\sqrt{9x^2+15x}\right|+C\right) &= \frac{1}{\sqrt{3}}\,\frac{\frac{6}{5}+\frac{1}{5}\frac{18x+15}{\sqrt{9x^2+15x}}}{\left(\frac{6}{5}x+1\right)+\frac{2}{5}\sqrt{9x^2+15x}} \\
&= \frac{1}{\sqrt{3}}\,\frac{6+3\frac{6x+5}{\sqrt{9x^2+15x}}}{(6x+5)+2\sqrt{9x^2+15x}} = \sqrt{3}\,\frac{2+\frac{6x+5}{\sqrt{9x^2+15x}}}{(6x+5)+2\sqrt{9x^2+15x}} = \frac{\sqrt{3}}{\sqrt{9x^2+15x}} \\
&= \frac{1}{\sqrt{3x^2+5x}}
\end{aligned}$$

is exactly the integrand.

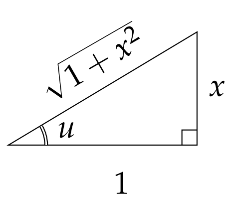
Remark: in applications, often the numbers involved are messier than they are in textbooks. The ideas of this problem are similar to other problems in this section, but it's good practice to apply them in a slightly messy context.

S-23: We use the substitution $x=\tan u$, $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$.

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}^3}{x}\mathrm{d}x &= \int \frac{\sqrt{1+\tan^2 u}^3}{\tan u}\sec^2 u\,\mathrm{d}u \\
&= \int \frac{\sec^3 u}{\tan u}\sec^2 u\,\mathrm{d}u \\
&= \int \frac{(\sec^2 u)^2}{\tan u}\sec u\,\mathrm{d}u \\
&= \int \frac{(\tan^2 u+1)^2}{\tan u}\sec u\,\mathrm{d}u \\
&= \int \frac{\tan^4 u + 2\tan^2 u + 1}{\tan u}\sec u\,\mathrm{d}u \\
&= \int \tan^3 u\sec u\,\mathrm{d}u + \int 2\sec u\tan u\,\mathrm{d}u + \int \frac{\sec u}{\tan u}\,\mathrm{d}u
\end{aligned}$$

For the first integral, we use the substitution $w=\sec u$. The second is the antiderivative of $2\sec u$. The third we simplify as $\frac{\sec u}{\tan u} = \frac{1}{\cos u}\cdot\frac{\cos u}{\sin u} = \csc u$.

$$\begin{aligned}
&= \int\left((\sec^2 u - 1)\sec u\tan u\right)\mathrm{d}u + 2\sec u + \ln|\cot u - \csc u| + C \\
&= \int (w^2-1)\,\mathrm{d}w + 2\sec u + \ln|\cot u - \csc u| + C \\
&= \frac{1}{3}w^3 - w + 2\sec u + \ln|\cot u - \csc u| + C \\
&= \frac{1}{3}\sec^3 u - \sec u + 2\sec u + \ln|\cot u - \csc u| + C \\
&= \frac{1}{3}\sec^3 u + \sec u + \ln|\cot u - \csc u| + C
\end{aligned}$$



We began with the substitution $x = \tan u$. Then $\cot u = \frac{1}{x}$. To find $\csc u$ and $\sec u$, we draw a right triangle with angle u, opposite side x, and adjacent side 1. The Pythagorean Theorem gives us the hypotenuse.

$$= \frac{1}{3}\sqrt{1+x^2}^3 + \sqrt{1+x^2} + \ln\left|\frac{1}{x} - \frac{\sqrt{1+x^2}}{x}\right| + C$$
$$= \frac{1}{3}\sqrt{1+x^2}(4+x^2) + \ln\left|\frac{1-\sqrt{1+x^2}}{x}\right| + C$$

S-24: The half of the ellipse to the right of the y-axis is given by the equation

$$x = f(y) = 4\sqrt{1-\left(\frac{y}{2}\right)^2}$$

The area we want is the twice the area between the right-hand side of the curve and the y-axis, from $y = -1$ to $y = 1$. In other words,

$$\text{Area} = 2\int_{-1}^{1} 4\sqrt{1-\left(\frac{y}{2}\right)^2}\,\mathrm{d}y$$

Making use of symmetry,

$$= 16\int_0^1 \sqrt{1-\left(\frac{y}{2}\right)^2}\,\mathrm{d}y$$

We use the substitution $\frac{y}{2} = \sin\theta$, $\frac{1}{2}\mathrm{d}y = \cos\theta\,\mathrm{d}\theta$. Notice $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin 0 = 0$.

$$\begin{aligned}
&= 16\int_0^{\pi/6} \sqrt{1-(\sin\theta)^2}\;2\cos\theta\,\mathrm{d}\theta \\
&= 32\int_0^{\pi/6} \sqrt{\cos^2\theta}\;\cos\theta\,\mathrm{d}\theta \\
&= 32\int_0^{\pi/6} \cos^2\theta\,\mathrm{d}\theta \\
&= 16\int_0^{\pi/6} (1+\cos(2\theta))\,\mathrm{d}\theta \\
&= 16\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6} \\
&= 16\left(\frac{\pi}{6} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) \\
&= \frac{8\pi}{3} + 4\sqrt{3}
\end{aligned}$$

Remark: we also investigated areas of ellipses in Question 19, Section 1.2.

S-25: Note that $f(x)$ is an even function, nonnegative over its entire domain.
(a) To find the area of R, we evaluate

$$\text{Area} = \int_{-1/2}^{1/2} \frac{|x|}{\sqrt[4]{1-x^2}}\,\mathrm{d}x = 2\int_0^{1/2} \frac{x}{\sqrt[4]{1-x^2}}\,\mathrm{d}x$$

We use the substitution $u = 1 - x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$.

$$\begin{aligned}
&= -\int_1^{3/4} \frac{1}{u^{1/4}}\,\mathrm{d}u \\
&= -\left[\frac{4}{3}u^{3/4}\right]_1^{3/4} = -\frac{4}{3}\left(\left(\frac{3}{4}\right)^{3/4} - 1\right) \\
&= \frac{4}{3} - \sqrt[4]{\frac{4}{3}}
\end{aligned}$$

(b) We slice the solid of rotation into circular disks of width $\mathrm{d}x$ and radius $\dfrac{|x|}{\sqrt[4]{1-x^2}}$.

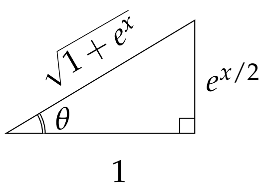
$$\begin{aligned}
\text{Volume} &= \int_{-1/2}^{1/2} \pi\left(\frac{|x|}{\sqrt[4]{1-x^2}}\right)^2\,\mathrm{d}x \\
&= 2\pi\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x
\end{aligned}$$

We use the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$, so $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. Note $\sin 0 = 0$ and $\sin\frac{\pi}{6} = \frac{1}{2}$.

$$\begin{aligned}
&= 2\pi\int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,\mathrm{d}\theta \\
&= 2\pi\int_0^{\pi/6} \sin^2\theta\,\mathrm{d}\theta \\
&= \pi\int_0^{\pi/6} \big(1 - \cos(2\theta)\big)\,\mathrm{d}\theta \\
&= \pi\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6} \\
&= \pi\left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) \\
&= \frac{\pi^2}{6} - \frac{\sqrt{3}\pi}{4}
\end{aligned}$$

S-26: If we think of e^x as $\left(e^{x/2}\right)^2$, the function under the square root suggests the substitution $e^{x/2} = \tan\theta$. Then $\frac{1}{2}e^{x/2}\,\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$, so $\mathrm{d}x = \frac{2}{e^{x/2}}\sec^2\theta\,\mathrm{d}\theta = \frac{2}{\tan\theta}\sec\theta\,\mathrm{d}\theta$.

$$\begin{aligned}
\int \sqrt{1+e^x}\,\mathrm{d}x &= \int \frac{2\sqrt{1+\tan^2\theta}}{\tan\theta}\sec^2\theta\,\mathrm{d}\theta \\
&= 2\int \frac{\sec^3\theta}{\tan\theta}\,\mathrm{d}\theta \\
&= 2\int \frac{\sec\theta(\tan^2\theta+1)}{\tan\theta}\,\mathrm{d}\theta \\
&= 2\int \left(\sec\theta\tan\theta + \frac{\sec\theta}{\tan\theta}\right)\,\mathrm{d}\theta \\
&= 2\int \big(\sec\theta\tan\theta + \csc\theta\big)\,\mathrm{d}\theta \\
&= 2\sec\theta + 2\ln|\cot\theta - \csc\theta| + C \\
&= 2\sqrt{1+e^x} + 2\ln\left|\frac{1}{e^{x/2}} - \frac{\sqrt{1+e^x}}{e^{x/2}}\right| + C \\
&= 2\sqrt{1+e^x} + 2\ln\left|1-\sqrt{1+e^x}\right| - 2\ln(e^{x/2}) + C \\
&= 2\sqrt{1+e^x} + 2\ln\left|1-\sqrt{1+e^x}\right| - x + C
\end{aligned}$$



We used the substitution $e^{x/2} = \tan\theta$, so $\cot\theta = \frac{1}{e^{x/2}}$. To find $\sec\theta$ and $\csc\theta$, we draw a right triangle with opposite side $e^{x/2}$ and adjacent side 1. They by the Pythagorean Theorem, the hypotenuse has length $\sqrt{1+e^x}$.

Remark: if we use the substitution $u = \sqrt{1+e^x}$, then we can change the integral to $\displaystyle\int \frac{2u^2}{u^2-1}\,\mathrm{d}u$. We can integrate this using the method of partial fractions, which we'll learn in the next section. You can explore this option in Question 23, Section 1.10.

S-27:

(a) We can save ourselves some trouble by applying logarithm rules before we differentiate.

$$\begin{aligned}
\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right| &= \ln|1+x| - \ln|\sqrt{1-x^2}| \\
&= \ln|1+x| - \frac{1}{2}\ln|1-x^2| \\
&= \ln|1+x| - \frac{1}{2}\ln|(1+x)(1-x)| \\
&= \ln|1+x| - \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| \\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right|\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\ln|1+x| - \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|\right\} \\
&= \frac{1}{1+x} - \frac{1/2}{1+x} + \frac{1/2}{1-x} \\
&= \frac{1/2}{1+x} + \frac{1/2}{1-x} \\
&= \frac{1}{1-x^2}
\end{aligned}$$

Notice this is the integrand from our work in blue.

(b) False: $\displaystyle\int_2^3 \frac{1}{1-x^2}\,\mathrm{d}x$ is a number, because it is the area under a finite portion of a continuous curve. (We note that the integrand is continuous over the interval $[2,3]$, although it is not continuous everywhere.) However, $\displaystyle\left[\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right|\right]_{x=2}^{x=3}$ is not defined, since the denominator takes the square root of a negative number. So, these two expressions are not the same.

(c) The work in the question is not correct. The most salient problem is that when we make the substitution $x = \sin\theta$, we restrict the possible values of x to $[-1,1]$, since this is the range of the sine function. However, the original integral had no such restriction.

How can we be sure we avoid this problem in the future? In the introductory text to Section 3.7 (before Example 3.7.1), the text tells us that we are allowed to write our old variable as a function of a new variable (say $x = s(u)$) *as long as that function is invertible to recover our original variable x*. There is one very obvious reason why invertibility is necessary: after we antidifferentiate using our new variable u, we need to get it back in terms of our original variable, so we need to be able to recover x. Moreover, invertibility reconciles potential problems with domains: if an inverse function $u = s^{-1}(x)$ exists, then for any x, there exists a u with $s(u) = x$. (This was not the case in the work for the question, because we chose $x = \sin\theta$, but if $x = 2$, there is no corresponding θ. Note, however, that $x = \sin\theta$ is invertible over $[-1,1]$, so the work is correct if we restrict x to those values.)

Remark: in the next section, you will learn to use partial fractions to find $\displaystyle\int \frac{1}{1-x^2}\,\mathrm{d}x = \ln|1+x| - \frac{1}{2}\ln|1-x|$. When $-1 < x < 1$, this is equivalent to

$$\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right|.$$

S-28: Remember that for any value X,

$$|X| = \begin{cases} X & \text{if } X \geqslant 0 \\ -X & \text{if } X \leqslant 0 \end{cases}$$

So, $|X| \neq X$ precisely when $X < 0$.

(a) The range of arcsine is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, since $u = \arcsin(x/a)$, u is in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore $\cos u \geqslant 0$. Since a is positive, $a\cos u \geqslant 0$, so $a\cos u = |a\cos u|$. That is,

$$\sqrt{a^2 - x^2} = |a\cos u| = a\cos u$$

all the time.

(b) The range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, since $u = \arctan(x/a)$, u is in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore $\sec u = \frac{1}{\cos u} > 0$. Since a is positive, $a\sec u > 0$, so $a\sec u = |a\sec u|$.That is,

$$\sqrt{a^2 + x^2} = |a\sec u| = a\sec u$$

all the time.

(c) The range of arccosine is $\left[0, \pi\right]$. So, since $u = \operatorname{arcsec}(x/a) = \arccos(a/x)$, u is in the range $\left[0, \pi\right]$. (Actually, it's in the range $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, since secant is undefined at $\pi/2$.) If $|a\tan u| \neq a\tan u$, then $\tan u < 0$, which happens when u is in the range $\left(\frac{\pi}{2}, \pi\right)$. This is the same range over which $-1 < \cos u < 0$, and so $-1 < \frac{a}{x} < 0$. Since $\frac{a}{x} < 0$, a and x have different signs, so $x < 0$. Then since $-1 < \frac{a}{x}$, also $x < -a$.

So,

$$\sqrt{x^2 - a^2} = |a\tan u| = -a\tan u \neq a\tan u$$

happens precisely when when $x < -a$.

Solutions to Exercises 3.8 — Jump to TABLE OF CONTENTS

S-1: If a quadratic function has roots a and b, then it can be written as $c(x-a)(x-b)$ for some constant c. In the case that is has two *different* roots ($a \neq b$), then this factorization has two distinct linear factors. In the case of a quadratic equation with exactly one root, we have $a = b$, so the equation has a repeated linear factor, $(x-a)^2$.

In graphs (a) and (d), the equations have only one root, so when they're factored they'll have a repeated linear factor, (ii). In graphs (b) and (c), the equations have two different roots, so they factor into distinct linear factors, (i).

S-2: Our first step is to fully factor the denominator:

$$(x^2-1)^2(x+1) = (x-1)^2(x+1)^2(x+1) = (x-1)^2(x+1)^3$$

So we have two repeated linear terms.

$$\begin{aligned}\frac{x^3+3}{(x^2-1)^2(x+1)} &= \frac{x^3+3}{(x-1)^2(x+1)^3}\\ &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}\end{aligned}$$

S-3: The partial fraction decomposition has the form

$$\frac{3x^3-2x^2+11}{x^2(x-1)(x+3)} = \frac{A}{x-1} + \text{various terms}$$

When we multiply through by the original denominator, this becomes

$$3x^3-2x^2+11 = x^2(x+3)A + (x-1)(\text{other terms}).$$

Evaluating both sides at $x=1$ yields $3\cdot 1^3 - 2\cdot 1^2 + 11 = 1^2(1+3)A + 0$, or $A=3$.

S-4:

(a) We start by dividing. The leading term of the numerator is x times the leading term of the denominator. The remainder is $x+2$.

$$\begin{array}{r|l} & \quad\;\; x \\ \hline x^2+1 & x^3+2x+2 \\ & -x^3\;\; -x \\ \hline & \qquad\;\; x+2 \end{array}$$

That is, $x^3+2x+2 = x(x^2+1)+(x+2)$. So,

$$\frac{x^3+2x+2}{x^2+1} = x + \frac{x+2}{x^2+1}$$

(b) We start by dividing. The leading term of the numerator is $3x^2$ times the leading term of the denominator.

$$\begin{array}{r|l} & \qquad\qquad 3x^2 \\ \hline 5x^2+2x+8 & 15x^4+6x^3+34x^2+4x+20 \\ & -15x^4-6x^3-24x^2 \\ \hline & \qquad\qquad\quad 10x^2+4x+20 \end{array}$$

Then $5x^2$ goes into $10x^2$ twice, so

$$\begin{array}{r|l} & \qquad\qquad 3x^2 \qquad\quad +2 \\ \hline 5x^2+2x+8 & 15x^4+6x^3+34x^2+4x+20 \\ & -15x^4-6x^3-24x^2 \\ \hline & \qquad\qquad\quad 10x^2+4x+20 \\ & \qquad\qquad\; -10x^2-4x-16 \\ \hline & \qquad\qquad\qquad\qquad\qquad\quad 4 \end{array}$$

Our remainder is 4. That is,

$$\frac{15x^4 + 6x^3 + 34x^2 + 4x + 20}{5x^2 + 2x + 8} = 3x^2 + 2 + \frac{4}{5x^2 + 2x + 8}.$$

(c) We start by dividing. The leading term of the numerator is x^3 times the leading term of the denominator.

$$\begin{array}{r|l} & \quad\; x^3 \\ \cline{2-2} 2x^2+5 & \;\; 2x^5 + 9x^3 + 12x^2 + 10x + 30 \\ & -2x^5 - 5x^3 \\ \cline{2-2} & \qquad\quad 4x^3 + 12x^2 + 10x \end{array}$$

Then $2x^2(2x)$ gives us $4x^3$.

$$\begin{array}{r|l} & \quad\; x^3 \qquad\qquad + 2x \\ \cline{2-2} 2x^2+5 & \;\; 2x^5 + 9x^3 + 12x^2 + 10x + 30 \\ & -2x^5 - 5x^3 \\ \cline{2-2} & \qquad\quad 4x^3 + 12x^2 + 10x \\ & \qquad -4x^3 \qquad\quad - 10x \\ \cline{2-2} & \qquad\qquad\quad 12x^2 \qquad\quad + 30 \end{array}$$

Finally, $2x^2$ goes into $12x^2$ six times.

$$\begin{array}{r|l} & \quad\; x^3 \qquad\qquad + 2x \;\; + 6 \\ \cline{2-2} 2x^2+5 & \;\; 2x^5 + 9x^3 + 12x^2 + 10x + 30 \\ & -2x^5 - 5x^3 \\ \cline{2-2} & \qquad\quad 4x^3 + 12x^2 + 10x \\ & \qquad -4x^3 \qquad\quad - 10x \\ \cline{2-2} & \qquad\qquad\quad 12x^2 \qquad\quad + 30 \\ & \qquad\qquad -12x^2 \qquad\quad - 30 \\ \cline{2-2} & \qquad\qquad\qquad\qquad\qquad\quad 0 \end{array}$$

Since there is no remainder,

$$\frac{2x^5 + 9x^3 + 12x^2 + 10x + 30}{2x^2 + 5} = x^3 + 2x + 6$$

Remark: if we wanted to be pedantic about the question statement, we could write our final answer as $x^3 + 2x + 6 + \frac{0}{x}$, so that we are indeed adding a polynomial to a rational function whose numerator has degree strictly smaller than its denominator.

S-5:

(a) The polynomial $5x^3 - 3x^2 - 10x + 6$ has a repeated pattern: the ratio of the first two coefficients is the same as the ratio of the last two coefficients. We can use this to factor.

$$\begin{aligned} 5x^3 - 3x^2 - 10x + 6 &= x^2(5x-3) - 2(5x-3) = (x^2-2)(5x-3) \\ &= (x+\sqrt{2})(x-\sqrt{2})(5x-3) \end{aligned}$$

(b) The polynomial $x^4 - 11x^2 + 30$ has only even powers of x, so we can (temporarily) replace them with $x^2 = y$ to turn our quartic polynomial into a quadratic.

$$x^4 - 11x^2 + 30 = y^2 - 11y + 30$$

This factors! (If it didn't, we could use the quadratic equation to find its roots and then its factors.)

$$\begin{aligned} &= (y-5)(y-6) \\ &= (x^2-5)(x^2-6) \\ &= \left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)\left(x+\sqrt{6}\right)\left(x-\sqrt{6}\right) \end{aligned}$$

S-6:

(a) Without seeing any obvious patterns, we start hunting for roots. Since we have all integer coefficients, if there are any integer roots, they will divide our constant term, 50. So, our candidates for integer roots are ± 1, ± 2, ± 5, ± 10, and ± 50. To save time, we don't need to know exactly the value of our polynomial at these points: only whether or not it is 0. Let $f(x) = x^4 - 3x^3 - 15x^2 + 15x + 50$.

$$\begin{array}{rrrrr} f(-1) \neq 0 & f(-2) = 0 & f(-5) \neq 0 & f(-10) \neq 0 & f(-50) \neq 0 \\ f(1) \neq 0 & f(2) \neq 0 & f(5) = 0 & f(10) \neq 0 & f(50) \neq 0 \end{array}$$

Since $x = -2$ and $x = 5$ are roots of our polynomial, it has factors $(x+2)$ and $(x-5)$. Note $(x+2)(x-5) = x^2 - 3x - 10$. We use long division to figure out what else is lurking in our polynomial.

$$\begin{array}{r|l} & x^2 \qquad\qquad -5 \\ \hline x^2 - 3x - 10 & x^4 - 3x^3 - 15x^2 + 15x + 50 \\ & -x^4 + 3x^3 + 10x^2 \\ \hline & \qquad\quad -5x^2 + 15x + 50 \\ & \qquad\quad 5x^2 - 15x - 50 \\ \hline & \qquad\qquad\qquad\qquad 0 \end{array}$$

So,

$$\begin{aligned} x^4 - 3x^3 - 15x^2 + 15x + 50 &= (x^2-3x-10)(x^2-5) \\ &= (x+2)(x-5)(x^2-5) \\ &= (x+2)(x-5)(x+\sqrt{5})(x-\sqrt{5}) \end{aligned}$$

(b) Without seeing any obvious patterns, we start hunting for roots. Since we have all integer coefficients, if there are any integer roots, they will divide our constant term, -15. So, our candidates for roots are ± 1, ± 3, ± 5, and ± 15. Write $f(x) = 2x^4 + 12x^3 - x^2 - 52x + 15$.

$$\begin{array}{cccc} f(-1) \neq 0 & f(-3) = 0 & f(-5) = 0 & f(-5) \neq 0 \\ f(1) \neq 0 & f(3) \neq 0 & f(5) \neq 0 & f(15) \neq 0 \end{array}$$

Since $x = -3$ and $x = -5$ are roots of our polynomial, it has factors $(x+3)$ and $(x+5)$. Note $(x+3)(x+5) = x^2 + 8x + 15$. We use long division to move forward.

$$\begin{array}{r|rrrrr} & & & 2x^2 & -4x & +1 \\ \hline x^2+8x+15 & 2x^4 & +12x^3 & -x^2 & -52x & +15 \\ & -2x^4 & -16x^3 & -30x^2 & & \\ \hline & & -4x^3 & -31x^2 & -52x & \\ & & 4x^3 & +32x^2 & +60x & \\ \hline & & & x^2 & +8x & +15 \\ & & & -x^2 & -8x & -15 \\ \hline & & & & & 0 \end{array}$$

So, $2x^4 + 12x^3 - x^2 - 52x + 15 = (x+3)(x+5)(2x^2 - 4x + 1)$.

We should check whether $2x^2 - 4x + 1$ is reducible or not. There's not an obvious way to factor it, but we can use the quadratic equation. This gives us roots $\dfrac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$. So, we have two more linear factors.

Specifically:
$2x^4 + 12x^3 - x^2 - 52x + 15 = (x+3)(x+5)\left(x - (2+\sqrt{2})\right)\left(x - (2-\sqrt{2})\right)$.

S-7: The goal of partial fraction decomposition is to write our integrand in a form that is easy to integrate. The antiderivative of (1) can be easily determined with the substitution $u = (ax+b)$. It's less clear how to find the antiderivative of (2).

S-8: The integrand is a rational function, so it's a candidate for partial fraction. We quickly rule out any obvious substitution or integration by parts, so we go ahead with the decomposition.

We start by expressing the integrand, i.e. the fraction $\frac{1}{x+x^2} = \frac{1}{x(1+x)}$, as a linear combination of the simpler fractions $\frac{1}{x}$ and $\frac{1}{x+1}$ (which we already know how to integrate). We will have

$$\frac{1}{x+x^2} = \frac{1}{x(1+x)} = \frac{a}{x} + \frac{b}{x+1} = \frac{a(x+1)+bx}{x(1+x)}$$

The fraction on the left hand side is the same as the fraction on the right hand side if and only if the numerator on the left hand side, which is $1 = 0x + 1$, is equal to the numerator on the right hand side, which is $a(x+1) + bx = (a+b)x + a$. This in turn is the case if and only of $a = 1$ (i.e. the constant terms are the same in the two numerators) and $a + b = 0$ (i.e. the coefficients of x are the same in the two numerators). So $a = 1$ and $b = -1$. Now we can easily evaluate the integral

$$\begin{aligned}\int_1^2 \frac{\mathrm{d}x}{x + x^2} &= \int_1^2 \frac{\mathrm{d}x}{x(x+1)} = \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1}\right) \mathrm{d}x = \Big[\ln x - \ln(x+1)\Big]_1^2 \\ &= \ln 2 - \ln\frac{3}{2} = \ln\frac{4}{3}\end{aligned}$$

S-9: This integrand is a rational function, with no obvious substitution. This sure looks like a partial fraction problem. Let's go through our protocol.

- The degree of the numerator $x - 13$ is one, which is strictly smaller than the degree of the denominator $x^2 - x - 6$, which is two. So we don't need long division to pull out a polynomial.
- Next we factor the denominator.

  $$x^2 - x - 6 = (x-3)(x+2)$$

- Next we find the partial fraction decomposition of the integrand. It is of the form

  $$\frac{x-13}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

  To find A and B, using the sneaky method, we cross multiply by the denominator.

  $$x - 13 = A(x+2) + B(x-3)$$

  Now we can find A by evaluating at $x = 3$

  $$3 - 13 = A(3+2) + B(3-3) \implies A = -2$$

  and find B by evaluating at $x = -2$.

  $$-2 - 13 = A(-2+2) + B(-2-3) \implies B = 3$$

  (Hmmm. A and B are nice round numbers. Sure looks like a rigged exam or homework question.) Our partial fraction decomposition is

  $$\frac{x-13}{(x-3)(x+2)} = \frac{-2}{x-3} + \frac{3}{x+2}$$

  As a check, we recombine the right hand side and make sure that it matches the left hand side.

  $$\frac{-2}{x-3} + \frac{3}{x+2} = \frac{-2(x+2) + 3(x-3)}{(x-3)(x+2)} = \frac{x-13}{(x-3)(x+2)}$$

- Finally, we evaluate the integral.

$$\int \frac{x-13}{x^2-x-6}\mathrm{d}x = \int \left(\frac{-2}{x-3}+\frac{3}{x+2}\right)\mathrm{d}x = -2\ln|x-3|+3\ln|x+2|+C$$

S-10: Again, this sure looks like a partial fraction problem. So let's go through our protocol.

- The degree of the numerator $5x+1$ is one, which is strictly smaller than the degree of the denominator x^2+5x+6, which is two. So we do not long divide to pull out a polynomial.
- Next we factor the denominator.

$$x^2+5x+6 = (x+2)(x+3)$$

- Next we find the partial fraction decomposition of the integrand. It is of the form

$$\frac{5x+1}{(x+2)(x+3)} = \frac{A}{x+2}+\frac{B}{x+3}$$

To find A and B, using the sneaky method, we cross multiply by the denominator.

$$5x+1 = A(x+3)+B(x+2)$$

Now we can find A by evaluating at $x=-2$

$$-10+1 = A(-2+3)+B(-2+2) \implies A=-9$$

and find B by evaluating at $x=-3$.

$$-15+1 = A(-3+3)+B(-3+2) \implies B=14$$

So our partial fraction decomposition is

$$\frac{5x+1}{(x+2)(x+3)} = \frac{-9}{x+2}+\frac{14}{x+3}$$

As a check, we recombine the right hand side and make sure that it matches the left hand side

$$\frac{-9}{x+2}+\frac{14}{x+3} = \frac{-9(x+3)+14(x+2)}{(x+2)(x+3)} = \frac{5x+1}{(x+2)(x+3)}$$

- Finally, we evaluate the integral

$$\int \frac{5x+1}{x^2+5x+6}\mathrm{d}x = \int \left(\frac{-9}{x+2}+\frac{14}{x+3}\right)\mathrm{d}x = -9\ln|x+2|+14\ln|x+3|+C$$

S-11: We have a rational function with no obvious substitution, so let's use partial fraction decomposition.

- Since the degree of the numerator is the same as the degree of the denominator, we need to pull out a polynomial.

$$\begin{array}{r|l} & 5 \\ \cline{2-2} x^2-1 & 5x^2-3x-1 \\ & -5x^2+5 \\ \cline{2-2} & -3x+4 \end{array}$$

That is,

$$\int \frac{5x^2-3x-1}{x^2-1}\mathrm{d}x = \int \left(5+\frac{-3x+4}{x^2-1}\right)\mathrm{d}x = 5x + \int \frac{-3x+4}{x^2-1}\mathrm{d}x$$

- Again, there's no obvious substitution for the new integrand, so we want to use partial fraction. The denominator factors as $(x-1)(x+1)$, so our decomposition has this form:

$$\frac{-3x+4}{x^2-1} = \frac{-3x+4}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{(A+B)x+(A-B)}{(x-1)(x+1)}$$

So, (1) $A+B=-3$ and (2) $A-B=4$.

- We solve (2) for A in terms of B, namely $A=4+B$. Plugging this into (1), we see $(4+B)+B=-3$. So, $B=-\frac{7}{2}$, and $A=\frac{1}{2}$.
- Now we can write our integral in a friendlier form and evaluate.

$$\begin{aligned}\int \frac{5x^2-3x-1}{x^2-1}\mathrm{d}x = &= 5x + \int \frac{-3x+4}{x^2-1}\mathrm{d}x = 5x + \int \frac{1/2}{x-1} - \frac{7/2}{x+1}\,\mathrm{d}x \\ &= 5x + \frac{1}{2}\ln|x-1| - \frac{7}{2}\ln|x+1| + C\end{aligned}$$

S-12: The integrand is a rational function with no obvious substitution, so we'll use a partial fraction decomposition.

- Since the numerator has strictly smaller degree than the denominator, we don't need to start off with a long division.
- We do, however, need to factor the denominator. We can immediately pull out x^2; the remaining part is $x^2-2x+1=(x-1)^2$.
- Now we can perform our partial fraction decomposition.

$$\frac{x^2+2x-1}{x^4-2x^3+x^2} = \frac{x^2+2x-1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Multiply both sides by the original denominator.

$$x^2+2x-1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2 \tag{1}$$

To be sneaky, we set $x = 0$, and find:

$$-1 = B$$

We also set $x = 1$, and find:

$$2 = D$$

We use B and D to simplify Equation (1).

$$\begin{aligned}
x^2 + 2x - 1 &= Ax(x-1)^2 - 1(x-1)^2 + Cx^2(x-1) + 2x^2 \\
0 &= Ax(x-1)^2 + Cx^2(x-1) \\
&= x(x-1)[(A+C)x - A] \\
\text{So,} \qquad 0 &= (A+C)x - A
\end{aligned}$$

That is, $A = C = 0$.

- Now we can evaluate our integral.

$$\begin{aligned}
\int \frac{x^2+2x-1}{x^4-2x^3+x^2}\,\mathrm{d}x &= \int \left(\frac{-1}{x^2} + \frac{2}{(x-1)^2} \right) \mathrm{d}x \\
&= \frac{1}{x} - \frac{2}{x-1} + C
\end{aligned}$$

S-13: **Solution 1:** The denominator has a repeated linear factor. The denominators for our expansion will be $3x-5$, $(3x-5)^2$, and $(3x-5)^3$. The degree of the numerator is less than the degree of the denominator, so we don't need to use long division.

Repeated linear factors don't usually lend themselves very well to the "sneaky method."

$$\begin{aligned}
\frac{9x^2+36x+35}{(3x+5)^3} &= \frac{A}{3x+5} + \frac{B}{(3x+5)^2} + \frac{C}{(3x+5)^3} \\
9x^2+36x+35 &= A(3x+5)^2 + B(3x+5) + C \\
&= A(9x^2+30x+25) + 3Bx + 5B + C \\
&= (9A)x^2 + (30A+3B)x + (25A+5B+C)
\end{aligned}$$

Matching up coefficients,

$$\begin{aligned}
9 &= 9A \implies A = 1 \\
36 &= (30A+3B) \implies B = 2 \\
35 &= 25A+5B+C \implies C = 0 \\
\int \frac{9x^2+36x+35}{(3x+5)^3} &= \int \left(\frac{1}{3x+5} + \frac{2}{(3x+5)^2} \right) \mathrm{d}x \\
&= \frac{1}{3}\ln|3x+5| - \frac{2}{3(3x+5)} + C
\end{aligned}$$

Solution 2: You might have noticed that $9x^2 + 36x + 35 = (3x+5)(3x+7)$.

$$\begin{aligned}
\frac{9x^2+36x+35}{(3x+5)^2} &= \frac{(3x+5)(3x+7)}{(3x+5)^3} \\
&= \frac{3x+7}{(3x+5)^2} = \frac{A}{3x+5} + \frac{B}{(3x+5)^2} \\
3x+7 &= A(3x+5) + B = 3Ax + (5A+B) \\
3 &= 3A \implies A = 1 \\
7 &= 5a + B \implies B = 2 \\
\int \frac{9x^2+36x+35}{(3x+5)^2} \mathrm{d}x &= \int \left(\frac{1}{3x+5} + \frac{2}{(3x+5)^2} \right) \mathrm{d}x \\
&= \frac{1}{3} \ln|3x+5| - \frac{2}{3(3x+5)} + C
\end{aligned}$$

S-14:

$$\begin{aligned}
\frac{8x^2-27x+29}{(x-4)(2x-1)^2} &= \frac{A}{x-4} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2} \\
&= \frac{A(2x-1)^2 + B(x-4)(2x-1) + C(x-4)}{(x-4)(2x-1)^2} \\
8x^2 - 27x + 29 &= A(2x-1)^2 + B(x-4)(2x-1) + C(x-4) \\
&= A(4x^2 - 4x + 1) + B(2x^2 - 9x + 4) + C(x-4) \\
&= (4A+2B)x^2 + (-4A - 9B + C)x + (A + 4B - 4C) \\
8 &= 4A + 2B \implies B = 4 - 2A \\
-27 &= (-4A - 9B + C) \implies C = 9 - 14A \\
29 &= (A + 4B - 4C) \implies A = 1 \implies B = 2,\ C = -5 \\
\int \frac{8x^2-27x+29}{(x-4)(2x-1)^2} \mathrm{d}x &= \int \left(\frac{1}{x-4} + \frac{2}{2x-1} - \frac{5}{(2x-1)^2} \right) \mathrm{d}x \\
&= \ln|x-4| + \ln|2x-1| + \frac{5}{2(2x-1)} + C
\end{aligned}$$

S-15:

$$\begin{aligned}
\frac{13x^3 - 20x^2 + x - 6}{(x^2-1)^2} &= \frac{13x^3 - 20x^2 + x - 6}{[(x-1)(x+1)]^2} \\
= \frac{13x^3 - 20x^2 + x - 6}{(x-1)^2(x+1)^2} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} \\
13x^3 - 20x^2 + x - 6 &= A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2 \\
\text{If } x = 1: \quad -12 &= B(2^2) \implies B = -3 \\
\text{If } x = -1: \quad -40 &= D((-2)^2) \implies D = -10
\end{aligned}$$

Those were the really good numbers for the sneaky method. Now that they're used up, we'll choose a few more values of x.

$$\begin{aligned}
\text{If } x = 0:&\quad -6 = A(-1) - 3(1) + C(1) - 10 \implies C = 7 + A\\
\text{If } x = 2:&\quad 20 = A(3^2) + B(3^2) + C(3) + D \implies A = 3,\ C = 10
\end{aligned}$$

$$\begin{aligned}
\int \frac{13x^3 - 20x^2 + x - 6}{(x^2-1)^2}\mathrm{d}x &= \int \left(\frac{3}{x-1} - \frac{3}{(x-1)^2} + \frac{10}{x+1} - \frac{10}{(x+1)^2} \right) \mathrm{d}x\\
&= 3\ln|x-1| + \frac{3}{x-1} + 10\ln|x+1| + \frac{10}{x+1} + C
\end{aligned}$$

S-16:

$$\begin{aligned}
\frac{10x^2 - 68x + 102}{(x-5)(x-3)^2} &= \frac{A}{x-5} + \frac{B}{x-3} + \frac{C}{(x-3)^2}\\
10x^2 - 68x + 102 &= A(x-3)^2 + B(x-5)(x-3) + C(x-5)\\
\text{If } x = 3, \quad -12 &= -2C \implies C = 6\\
\text{If } x = 5, \quad 12 &= 4C \implies A = 3\\
\text{If } x = 4, \quad -40 &= A - B - C \implies B = 7
\end{aligned}$$

$$\begin{aligned}
\int \frac{10x^2 - 68x + 102}{(x-5)(x-3)^2}\mathrm{d}x &= \int \left(\frac{3}{x-5} + \frac{7}{x-3} + \frac{6}{(x-3)^2} \right) \mathrm{d}x\\
&= 3\ln|x-5| + 7\ln|x-3| - \frac{6}{x-3} + C
\end{aligned}$$

S-17: Our integrand is a rational function with no obvious substitution, so we'll use the method of partial fractions.

- The degree of the numerator is less than the degree of the denominator.
- We need to factor the denominator. The first two terms have the same ratio as the last two terms.

$$\begin{aligned}
2x^3 - x^2 - 8x + 4 &= x^2(2x-1) - 4(2x-1)\\
&= (x^2-4)(2x-1)\\
&= (x-2)(x+2)(2x-1)
\end{aligned}$$

- Now we find our partial fraction decomposition.

$$\frac{3x^2 - 4x - 10}{2x^3 - x^2 - 8x + 4} = \frac{3x^2 - 4x - 10}{(x-2)(x+2)(2x-1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{2x-1}$$

  Multiply both sides by the original denominator.

$$3x^2 - 4x - 10 = A(x+2)(2x-1) + B(x-2)(2x-1) + C(x-2)(x+2)$$

Distinct linear factors is the best possible scenario for the sneaky method. First, let's set $x = 2$.

$$\begin{aligned} 3(4) - 4(2) - 10 &= A(4)(3) + B(0) + C(0) \\ A &= -\frac{1}{2} \end{aligned}$$

Now, let $x = -2$.

$$\begin{aligned} 3(4) - 4(-2) - 10 &= A(0) + B(-4)(-5) + C(0) \\ B &= \frac{1}{2} \end{aligned}$$

Finally, let $x = \frac{1}{2}$.

$$\begin{aligned} \frac{3}{4} - 2 - 10 &= A(0) + B(0) + C\left(-\frac{3}{2}\right)\left(\frac{5}{2}\right) \\ C &= 3 \end{aligned}$$

- Now we can evaluate our integral in its new form.

$$\begin{aligned} \int \frac{3x^2 - 4x - 10}{2x^3 - x^2 - 8x + 4}\, \mathrm{d}x &= \int \left(\frac{-1/2}{x-2} + \frac{1/2}{x+2} + \frac{3}{2x-1}\right) \mathrm{d}x \\ &= -\frac{1}{2}\ln|x-2| + \frac{1}{2}\ln|x+2| + \frac{3}{2}\ln|2x-1| + C \\ &= \frac{1}{2}\ln\left|\frac{x+2}{x-2}\right| + \frac{3}{2}\ln|2x-1| + C \end{aligned}$$

S-18: We'll first do a partial fraction decomposition. The sneaky way is to temporarily rename x^2 to y. Then $x^4 + x^2 = y^2 + y$ and

$$\frac{1}{x^4 + x^2} = \frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

as we found in Question 8. Now we restore y to x^2.

$$\int \frac{1}{x^4 + x^2}\, \mathrm{d}x = \int \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) \mathrm{d}x = -\frac{1}{x} - \arctan x + C$$

In this problem, it's crucial to remember that the method of partial fractions is algebra – not calculus. When we set $x^2 = y$, we were not using the substitution rule (and therefore didn't need to think about $\mathrm{d}y$). We used y to figure out an equivalent but nicer form of the integrand. Then it was necessary to replace y with x^2 *before* integrating.

The more general method of partial fraction accounts for irreducible quadratic terms (such as $x^2 + 1$) in the denominator. In this example, we got around the need for using that portion of the theory.

S-19: We follow the example in the text.

$$\int \csc x \, dx = \int \frac{1}{\sin x} \, dx = \int \frac{\sin x}{\sin^2 x} \, dx = \int \frac{\sin x}{1-\cos^2 x} \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$.

$$= \int \frac{-1}{1-u^2} \, du = \int \frac{-1}{(1+u)(1-u)} \, du$$

We see an opportunity for partial fraction.

$$\frac{-1}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u}$$

Multiply both sides by the original denominator.

$$-1 = A(1-u) + B(1+u)$$

Let $u = 1$.

$$-1 = 2B \qquad \Rightarrow B = -\frac{1}{2}$$

Let $u = -1$.

$$-1 = 2A \qquad \Rightarrow A = -\frac{1}{2}$$

We can now re-write our integral.

$$\begin{aligned}
\int \csc x \, dx &= \int \frac{-1}{(1+u)(1-u)} \, du = \int \left(\frac{-1/2}{1+u} + \frac{-1/2}{1-u} \right) du \\
&= -\frac{1}{2} \ln|1+u| + \frac{1}{2} \ln|1-u| + C \\
&= \frac{1}{2} \ln \left| \frac{1-u}{1+u} \right| + C \\
&= \frac{1}{2} \ln \left| \frac{1-\cos x}{1+\cos x} \right| + C
\end{aligned}$$

Remark: Elsewhere in the text, and in many tables of integrals, the antiderivative of cosecant is given as $\ln|\csc x - \cot x|$. We show that this is equivalent to our result.

$$\begin{aligned}
\ln|\csc x - \cot x| &= \frac{1}{2} \ln \left| (\csc x - \cot x)^2 \right| = \frac{1}{2} \ln \left| \csc^2 x - 2 \csc x \cot x + \cot^2 x \right| \\
&= \frac{1}{2} \ln \left| \frac{1}{\sin^2 x} - \frac{2\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \right| \\
&= \frac{1}{2} \ln \left| \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x} \right| = \frac{1}{2} \ln \left| \frac{(1-\cos x)^2}{1-\cos^2 x} \right| \\
&= \frac{1}{2} \ln \left| \frac{(1-\cos x)^2}{(1-\cos x)(1+\cos x)} \right| = \frac{1}{2} \ln \left| \frac{1-\cos x}{1+\cos x} \right|
\end{aligned}$$

S-20: We follow the example in the text.

$$\int \csc^3 x \, dx = \int \frac{1}{\sin^3 x} \, dx = \int \frac{\sin x}{\sin^4 x} \, dx = \int \frac{\sin x}{(1-\cos^2 x)^2} \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$.

$$= \int \frac{-1}{(1-u^2)^2} \, du$$

In Question 19, we saw $\frac{1}{1-u^2} = \frac{1/2}{1+u} + \frac{1/2}{1-u}$, so

$$\begin{aligned}
\int \frac{-1}{(1-u^2)^2} \, du &= -\int \left(\frac{1}{1-u^2}\right)^2 du = -\int \left(\frac{1/2}{1+u} + \frac{1/2}{1-u}\right)^2 du \\
&= -\frac{1}{4}\int \left(\frac{1}{(1+u)^2} + \frac{2}{1-u^2} + \frac{1}{(1-u)^2}\right) du \\
&= -\frac{1}{4}\int \left(\frac{1}{(1+u)^2} + \frac{1}{1+u} + \frac{1}{1-u} + \frac{1}{(1-u)^2}\right) du \\
&= -\frac{1}{4}\left(-\frac{1}{1+u} + \ln|1+u| - \ln|1-u| + \frac{1}{1-u}\right) + C \\
&= -\frac{1}{4}\left(\frac{2u}{1-u^2} + \ln\left|\frac{1+u}{1-u}\right|\right) + C \\
&= \frac{-u}{2(1-u^2)} + \frac{1}{4}\ln\left|\frac{1-u}{1+u}\right| + C \\
&= \frac{-\cos x}{2\sin^2 x} + \frac{1}{4}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C
\end{aligned}$$

Remark: In Example A.8.4 of the text, and in many tables of integrals, the antiderivative of $\csc^3 x$ is given as $-\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x| + C$. This is equivalent to our result. Recall in the remark after the solution to Question 19, we saw $\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| = \ln|\csc x - \cot x|$.

$$\begin{aligned}
-\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x| &= -\frac{1}{2}\cot x \csc x + \frac{1}{4}\ln\left|\frac{1-\cos x}{1+\cos x}\right| \\
&= -\frac{1}{2}\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right) + \frac{1}{4}\ln\left|\frac{1-\cos x}{1+\cos x}\right| \\
&= \frac{-\cos x}{2\sin^2 x} + \frac{1}{4}\ln\left|\frac{1-\cos x}{1+\cos x}\right|
\end{aligned}$$

S-21: If our denominator were all sines, we could use the substitution $x = \sin\theta$. To that end, we apply the identity $\cos^2\theta = 1 - \sin^2\theta$.

$$\int \frac{\cos\theta}{3\sin\theta + \cos^2\theta - 3} \, d\theta = \int \frac{\cos\theta}{3\sin\theta + 1 - \sin^2\theta - 3} \, d\theta = \int \frac{\cos\theta}{3\sin\theta - \sin^2\theta - 2} \, d\theta$$

We use the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\, \mathrm{d}\theta$.

$$= \int \frac{1}{3x - x^2 - 2}\, \mathrm{d}x = \int \frac{-1}{x^2 - 3x + 2}\, \mathrm{d}x = \int \frac{-1}{(x-1)(x-2)}\, \mathrm{d}x$$

Now we can find a partial fraction decomposition.

$$\begin{aligned}\frac{-1}{(x-1)(x-2)} &= \frac{A}{x-1} + \frac{B}{x-2}\\ -1 &= A(x-2) + B(x-1)\end{aligned}$$

Setting $x = 1$ and $x = 2$, we see

$$A = 1, \quad B = -1$$

Now, we can evaluate our integral.

$$\begin{aligned}\int \frac{\cos\theta}{3\sin\theta + \cos^2\theta - 3}\, \mathrm{d}\theta &= \int \frac{-1}{(x-1)(x-2)}\, \mathrm{d}x = \int \left(\frac{1}{x-1} - \frac{1}{x-2}\right)\, \mathrm{d}x\\ &= \ln|x-1| - \ln|x-2| + C = \ln\left|\frac{x-1}{x-2}\right| + C\\ &= \ln\left|\frac{\sin\theta - 1}{\sin\theta - 2}\right| + C\end{aligned}$$

S-22: This looks a lot like a rational function, but with the function e^t in place of the variable. So, we would like to make the substitution $x = e^t$, $\mathrm{d}x = e^t \mathrm{d}t$. Then $\mathrm{d}t = \frac{1}{e^t}\, \mathrm{d}x = \frac{1}{x}\, \mathrm{d}x$.

$$\begin{aligned}\int \frac{1}{e^{2t} - e^t - 2}\, \mathrm{d}t &= \int \frac{1}{x\,(x^2 - x - 2)} \mathrm{d}x\\ &= \int \frac{1}{x(x+1)(x-2)} \mathrm{d}x\end{aligned}$$

Now we can use partial fraction decomposition.

$$\begin{aligned}\frac{1}{x(x+1)(x-2)} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}\\ 1 &= A(x+1)(x-2) + Bx(x-2) + Cx(x+1)\end{aligned}$$

This is a good candidate for the sneaky method.

$$\begin{aligned}x = 0 &\implies 1 = A(1)(-2) + 0 + 0\\ x = -1 &\implies 1 = 0 + B(-1)(-3) + 0\\ x = 2 &\implies 1 = 0 + 0 + C(2)(3)\end{aligned}$$

All together we see $A = -\frac{1}{2}$, $B = \frac{1}{3}$, and $C = \frac{1}{6}$. Now we can evaluate our integral.

$$\begin{aligned}\int \frac{1}{e^{2t} - e^t - 2}\,\mathrm{d}t &= \int \frac{1}{x(x+1)(x-2)}\mathrm{d}x \\ &= \int \left(\frac{-1/2}{x} + \frac{1/3}{x+1} + \frac{1/6}{x-2} \right)\,\mathrm{d}x \\ &= -\frac{1}{2}\ln|x| + \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x-2| + C\end{aligned}$$

Now we undo the substitution $x = e^t$.

$$\begin{aligned}&= -\frac{1}{2}\ln\left|e^t\right| + \frac{1}{3}\ln\left|e^t+1\right| + \frac{1}{6}\ln\left|e^t-2\right| + C \\ &= -\frac{t}{2} + \frac{1}{3}\ln\left(e^t+1\right) + \frac{1}{6}\ln\left(e^t-2\right) + C\end{aligned}$$

S-23:

Solution 1: We use the substitution $u = \sqrt{1+e^x}$.
Then $\mathrm{d}u = \dfrac{e^x}{2\sqrt{1+e^x}}\mathrm{d}x$, so $\mathrm{d}x = \dfrac{2u}{u^2-1}\mathrm{d}u$.

$$\begin{aligned}\int \sqrt{1+e^x}\,\mathrm{d}x &= \int u \cdot \frac{2u}{u^2-1}\mathrm{d}u = \int \frac{2u^2}{u^2-1}\mathrm{d}u \\ &= \int \frac{2(u^2-1)+2}{u^2-1}\mathrm{d}u = \int \left(2 + \frac{2}{u^2-1}\right)\mathrm{d}u\end{aligned}$$

We use a partial fraction decomposition on the fractional part of the integrand.

$$\begin{aligned}&\frac{2}{u^2-1} = \frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{(A+B)u + (A-B)}{(u-1)(u+1)} \\ &A+B = 0, \quad A-B = 2 \\ &A = 1, \quad B = -1\end{aligned}$$

$$\begin{aligned}\int \sqrt{1+e^x}\,\mathrm{d}x &= \int \left(2 + \frac{2}{u^2-1}\right)\mathrm{d}u = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)\,\mathrm{d}u \\ &= 2u + \ln|u-1| - \ln|u+1| + C = 2u + \ln\left|\frac{u-1}{u+1}\right| + C \\ &= 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C\end{aligned}$$

Solution 2: It might not occur to us right away to use the fruitful substitution in Solution 1. More realistically, we might start with the "inside function," $u = 1+e^x$. Then $\mathrm{d}u = e^x\,\mathrm{d}x$, so $\mathrm{d}x = \frac{1}{u-1}\mathrm{d}u$.

$$\int \sqrt{1+e^x}\,\mathrm{d}x = \int \frac{\sqrt{u}}{u-1}\mathrm{d}u$$

This isn't quite a rational function, because we have a square root on top. If we could turn it into a rational function, we could use partial fraction. To that end, let $w = \sqrt{u}$, $\mathrm{d}w = \frac{1}{2\sqrt{u}}\mathrm{d}u$, so $\mathrm{d}u = 2w\mathrm{d}w$.

$$\begin{aligned}
&= \int \frac{w}{w^2-1} 2w\mathrm{d}w = \int \frac{2w^2}{w^2-1}\mathrm{d}w \\
&= \int \frac{2(w^2-1)+2}{w^2-1}\mathrm{d}w = \int 2 + \frac{2}{w^2-1}\mathrm{d}w
\end{aligned}$$

Now we can use partial fraction decomposition.

$$\begin{aligned}
&\frac{2}{w^2-1} = \frac{2}{(w-1)(w+1)} = \frac{A}{w-1} + \frac{B}{w+1} = \frac{(A+B)w + (A-B)}{(w-1)(w+1)} \\
&A + B = 0, \quad A - B = 2 \\
&A = 1, \quad B = -1
\end{aligned}$$

This allows us to antidifferentiate.

$$\begin{aligned}
\int \sqrt{1+e^x}\,\mathrm{d}x &= \int \left(2 + \frac{2}{w^2-1}\right)\mathrm{d}w = \int \left(2 + \frac{1}{w-1} - \frac{1}{w+1}\right)\,\mathrm{d}w \\
&= 2w + \ln|w-1| - \ln|w+1| + C \\
&= 2w + \ln\left|\frac{w-1}{w+1}\right| + C \\
&= 2\sqrt{u} + \ln\left|\frac{\sqrt{u}-1}{\sqrt{u}+1}\right| + C \\
&= 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C
\end{aligned}$$

Remark: we also evaluated this integral using trigonometric substitution in Section 1.9, Question 26. In that question, we found the antiderivative to be $2\sqrt{1+e^x} + 2\ln|1-\sqrt{1+e^x}| - x + C$. These expressions are equivalent:

$$\begin{aligned}
\ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| &= \ln\left|\sqrt{1+e^x}-1\right| + \ln\left|\frac{1}{\sqrt{1+e^x}+1}\right| \\
&= \ln\left|\sqrt{1+e^x}-1\right| + \ln\left|\left(\frac{1}{\sqrt{1+e^x}+1}\right)\left(\frac{1-\sqrt{1+e^x}}{1-\sqrt{1+e^x}}\right)\right| \\
&= \ln\left|\sqrt{1+e^x}-1\right| + \ln\left|\frac{1-\sqrt{1+e^x}}{1-(1+e^x)}\right| \\
&= \ln\left|\sqrt{1+e^x}-1\right| + \ln\left|\frac{1-\sqrt{1+e^x}}{-e^x}\right| \\
&= \ln\left|\sqrt{1+e^x}-1\right| + \ln\left|1-\sqrt{1+e^x}\right| - \ln|-e^x| \\
&= 2\ln\left|\sqrt{1+e^x}-1\right| - x
\end{aligned}$$

S-24: In order to find the area between the curves, we need to know which one is on top, and which on the bottom. Let's start by finding where they meet.

$$\begin{aligned}
\frac{4}{3+x^2} &= \frac{2}{x(x+1)} \\
2x^2+2x &= 3+x^2 \\
x^2+2x-3 &= 0 \\
(x-1)(x+3) &= 0
\end{aligned}$$

In the interval $[\frac{1}{4}, 3]$, the curves only meet at $x = 1$. So, to find which is on top and on bottom in the intervals $[\frac{1}{4}, 1)$ and $(1, 3]$, it suffices to check some point in each interval.

x	$\frac{4}{3+x^2}$	$\frac{2}{x(x+1)}$	Top:
$1/2$	$16/13$	$8/3$	$\frac{2}{x(x+1)}$
2	$4/7$	$1/3$	$\frac{4}{3+x^2}$

So, $\frac{2}{x(x+1)}$ is the top function when $\frac{1}{4} \leqslant x < 1$, and $\frac{4}{3+x^2}$ is the top function when $1 < x \leqslant 3$. Then the area we want to find is:

$$\text{Area} = \int_{\frac{1}{4}}^{1} \left(\frac{2}{x(x+1)} - \frac{4}{3+x^2}\right) \,\mathrm{d}x + \int_{1}^{3} \left(\frac{4}{3+x^2} - \frac{2}{x(x+1)}\right) \,\mathrm{d}x$$

We'll need to antidifferentiate both these functions. We can antidifferentiate $\dfrac{2}{x(x+1)}$ using partial fraction decomposition.

$$\begin{aligned}
\frac{2}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} = \frac{(A+B)x + A}{x(x+1)} \\
& A = 2, \quad B = -2 \\
\int \frac{2}{x(x+1)} \,\mathrm{d}x &= \int \left(\frac{2}{x} - \frac{2}{x+1}\right) \,\mathrm{d}x = 2\ln|x| - 2\ln|x+1| + C \\
&= 2\ln\left|\frac{x}{x+1}\right| + C
\end{aligned}$$

We can antidifferentiate $\dfrac{4}{3+x^2}$ using the substitution $u = \frac{x}{\sqrt{3}}$, $\mathrm{d}u = \frac{1}{\sqrt{3}}\,\mathrm{d}x$.

$$\begin{aligned}
\int \frac{4}{3+x^2} \,\mathrm{d}x &= \int \frac{4}{3\left(1+\left(\frac{x}{\sqrt{3}}\right)^2\right)} \,\mathrm{d}x = \int \frac{4\sqrt{3}}{3\,(1+u^2)} \,\mathrm{d}u \\
&= \frac{4}{\sqrt{3}} \arctan u + C = \frac{4}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + C
\end{aligned}$$

Now, we can find our area.

$$\begin{aligned}
\text{Area} &= \int_{\frac{1}{4}}^{1} \left(\frac{2}{x(x+1)} - \frac{4}{3+x^2} \right) \, \mathrm{d}x + \int_{1}^{3} \left(\frac{4}{3+x^2} - \frac{2}{x(x+1)} \right) \, \mathrm{d}x \\
&= \left[2\ln\left|\frac{x}{x+1}\right| - \frac{4}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) \right]_{1/4}^{1} + \left[\frac{4}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) - 2\ln\left|\frac{x}{x+1}\right| \right]_{1}^{3} \\
&= \left(2\ln\frac{1}{2} - \frac{4}{\sqrt{3}} \cdot \frac{\pi}{6} - 2\ln\frac{1}{5} + \frac{4}{\sqrt{3}} \arctan\frac{1}{4\sqrt{3}} \right) + \\
&\quad \left(\frac{4}{\sqrt{3}} \cdot \frac{\pi}{3} - 2\ln\frac{3}{4} - \frac{4}{\sqrt{3}} \cdot \frac{\pi}{6} + 2\ln\frac{1}{2} \right) \\
&= 2\ln\frac{5}{3} + \frac{4}{\sqrt{3}} \arctan\frac{1}{4\sqrt{3}}
\end{aligned}$$

S-25: (a) To antidifferentiate $\dfrac{1}{t^2-9}$, we use a partial fraction decomposition.

$$\begin{aligned}
\frac{1}{t^2-9} &= \frac{1}{(t-3)(t+3)} = \frac{A}{t-3} + \frac{B}{t+3} = \frac{(A+B)t + 3(A-B)}{(t-3)(t+3)} \\
A+B &= 0, \quad A-B = \frac{1}{3} \\
A &= \frac{1}{6}, \quad B = -\frac{1}{6} \\
F(x) &= \int_{1}^{x} \frac{1}{t^2-9} \, \mathrm{d}x = \int_{1}^{x} \left(\frac{1/6}{t-3} - \frac{1/6}{t+3} \right) \, \mathrm{d}x \\
&= \left[\frac{1}{6}\ln|t-3| - \frac{1}{6}\ln|t+3| \right]_{1}^{x} \\
&= \left(\frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| - \frac{1}{6}\ln 2 + \frac{1}{6}\ln 4 \right) \\
&= \frac{1}{6}\left(\ln\left| 2 \cdot \frac{x-3}{x+3} \right| \right)
\end{aligned}$$

(b) Rather than differentiate our answer from (a), we use the Fundamental Theorem of Calculus Part 1 to conclude

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left\{ \int_{1}^{x} \frac{1}{t^2-9} \mathrm{d}t \right\} = \frac{1}{x^2-9}$$

S-26: If $u^5 = x$, then $5u^4 \mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}
\int \frac{1}{10x^{6/5} + 5x} \mathrm{d}x &= \int \frac{1}{10\left(u^5\right)^{6/5} + 5u^5} \cdot 5u^4 \mathrm{d}u \\
&= \frac{u^4}{2u^6 + u^5} \mathrm{d}u \\
&= \frac{1}{2u^2 + u} \mathrm{d}u \\
&= \frac{1}{u(2u+1)} \mathrm{d}u
\end{aligned}$$

Now we see an opportunity for our current favourite integration technique.

$$\begin{aligned}
\frac{1}{u(2u+1)} &= \frac{A}{u} + \frac{B}{2u+1} = \frac{A(2u+1) + Bu}{u(2u+1)} \\
1 &= A(2u+1) + Bu \\
\text{if } u = 0: \qquad 1 &= A \\
\text{if } u = -\tfrac{1}{2}: \qquad 1 &= -\frac{1}{2}B \implies B = -2 \\
\int \frac{1}{u(2u+1)} \mathrm{d}u &= \int \left(\frac{1}{u} - \frac{2}{2u+1} \right) \mathrm{d}u \\
&= \ln|u| - \ln|2u+1| + C \\
&= \ln\left|x^{1/5}\right| - \ln\left|2x^{1/5} + 1\right| \\
&= \frac{1}{5}\ln|x| - \ln\left|2x^{1/5} + 1\right|
\end{aligned}$$

S-27: If $x = u^6$ then $\mathrm{d}x = 6u^5 \mathrm{d}u$

$$\begin{aligned}
\int \frac{1}{x^{3/2} - x^{4/3}} \mathrm{d}x &= \int \frac{1}{\left(u^6\right)^{3/2} - \left(u^6\right)^{4/3}} \cdot 6u^5 \mathrm{d}u \\
&= \int \frac{6u^5}{u^9 - u^8} \mathrm{d}u \\
&= \int \frac{6}{u^4 - u^3} \mathrm{d}u
\end{aligned}$$

What time is it? Yeah, you got it — it's partial fractions time!

$$
\begin{aligned}
\frac{6}{u^4-u^3} = \frac{6}{u^3(u-1)} &= \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{u-1} \\
&= \frac{Au^2(u-1) + Bu(u-1) + C(u-1) + Du^3}{u^3(u-1)} \\
6 &= Au^2(u-1) + Bu(u-1) + C(u-1) + Du^3 \\
\text{If } u = 0, \qquad 6 &= C(-1) \implies C = -6 \\
\text{So,} \qquad 6 &= Au^2(u-1) + Bu(u-1) + Du^3 - 6u + 6 \\
&= (A+D)u^3 + (-A+B)u^2 + (-B-6)u + 6 \\
&\implies \begin{cases} A + D &= 0 \\ -A + B &= 0 \\ -B - 6 &= 0 \end{cases} \\
&\implies \begin{cases} D &= 6 \\ A &= -6 \\ B &= -6 \end{cases} \\
\int \frac{6}{u^4-u^3}\mathrm{d}u &= \int \left(\frac{-6}{u} - \frac{6}{u^2} - \frac{6}{u^3} + \frac{6}{u-1} \right) \mathrm{d}u \\
&= -6\ln|u| + \frac{6}{u} + \frac{3}{u^2} + 6\ln|u-1| + C \\
&= -6\ln\left|x^{1/6}\right| + \frac{6}{x^{1/6}} + \frac{3}{\left(x^{1/6}\right)^2} + 6\ln\left|x^{1/6} - 1\right| + C \\
&= -\ln|x| + \frac{6}{x^{1/6}} + \frac{3}{x^{1/3}} + 6\ln\left|x^{1/6} - 1\right| + C
\end{aligned}
$$

S-28: Taking our cue from the previous example, we note using $x = u^3$, $\mathrm{d}x = 3u^2\mathrm{d}u$ will get rid of the fractional power.

$$\begin{aligned}
\int \frac{1}{x+3x^{2/3}+2x^{1/3}}\mathrm{d}x &= \int \frac{1}{u^3+3\left(u^3\right)^{2/3}+2\left(u^3\right)^{1/3}}\cdot 3u^2\mathrm{d}u\\
&= \int \frac{3u^2}{u^3+3u^2+2u}\mathrm{d}u\\
&= \int \frac{3u}{u^2+3u+2}\mathrm{d}u\\
&= \int \frac{3u}{(u+1)(u+2)}\mathrm{d}u\\
\frac{3u}{(u+1)(u+2)} &= \frac{A}{u+1}+\frac{B}{u+2} = \frac{A(u+1)+B(u+2)}{(u+1)(u+2)}\\
3u &= A(u+2)+B(u+1)\\
\text{If } u=-1, \quad -3 &= A\\
\text{If } u=-2, \quad -6 &= B(-1) \implies B=6\\
\int \frac{3u}{(u+1)(u+2)}\mathrm{d}u &= \int \left(\frac{-3}{u+1}+\frac{6}{u+2}\right)\mathrm{d}u\\
&= -3\ln|u+1|+6\ln|u+2|+C\\
&= -3\ln\left|x^{1/3}+1\right|+6\ln\left|x^{1/3}+2\right|+C
\end{aligned}$$

S-29: The equation $u=\sqrt{x-5}$ leads us to $x=5+u^2$, and so $\mathrm{d}x=2u\mathrm{d}u$.

$$\int \frac{\sqrt{x-5}}{x-7}\mathrm{d}x = \int \frac{\sqrt{u^2}}{u^2-2}\cdot 2u\mathrm{d}u = \int \frac{2u^2}{u^2-2}\mathrm{d}u$$

Since the degree of the numerator is not smaller than the degree of the denominator, we can use long division to get our integrand into a form suitable for partial fraction decomposition.

$$\begin{aligned}
\frac{2u^2}{u^2-2} &= \frac{2(u^2-2)+4}{u^2-2} = 2+\frac{4}{u^2-2}\\
\frac{4}{u^2-2} = \frac{4}{(u-\sqrt{2})(u+\sqrt{2})} &= \frac{A}{u-\sqrt{2}}+\frac{B}{u+\sqrt{2}}\\
4 &= A(u+\sqrt{2})+B(u-\sqrt{2})\\
\text{If } u=-\sqrt{2}, \quad 4 &= -2\sqrt{2}B \implies B=\sqrt{2}\\
\text{If } u=\sqrt{2}, \quad 4 &= 2\sqrt{2}A \implies A=\sqrt{2}\\
\int\left(2+\frac{4}{u^2-2}\right)\mathrm{d}u &= \int\left(2+\frac{\sqrt{2}}{u-\sqrt{2}}-\frac{\sqrt{2}}{u+\sqrt{2}}\right)\mathrm{d}u\\
&= 2u+\sqrt{2}\ln|u-\sqrt{2}|-\sqrt{2}\ln|u+\sqrt{2}|+C\\
&= 2\sqrt{x-5}+\sqrt{2}\ln\left|\sqrt{x-5}-\sqrt{2}\right|-\sqrt{2}\ln\left|\sqrt{x-5}+\sqrt{2}\right|+C
\end{aligned}$$

Solutions to Exercises 3.9 — Jump to TABLE OF CONTENTS

S-1: The absolute error is the difference between the two values:

$$|1.387 - 1.5| = 0.113$$

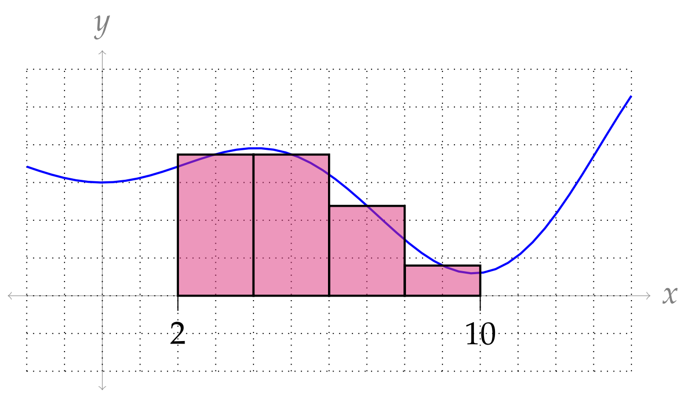
The relative error is the absolute error divided by the exact value:

$$\frac{0.113}{1.387} \approx 0.08147$$

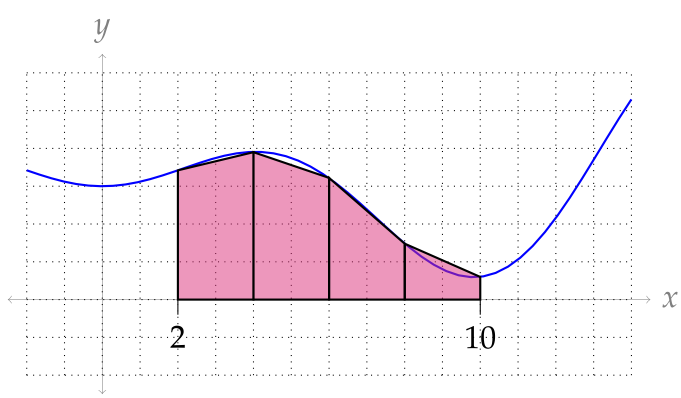
The percent error is 100 times the relative error:

$$\approx 8.147\%$$

S-2: Midpoint rule:



Trapezoidal rule:

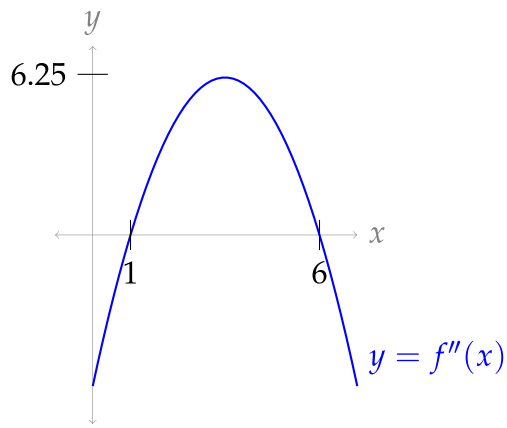


S-3:

(a) Differentiating, we find $f''(x) = -x^2 + 7x - 6$. Since $f''(x)$ is quadratic, we have a pretty good idea of what it looks like.

- It factors as $f(x) = -(x-6)(x-1)$, so its two roots are at $x = 6$ and $x = 1$.
- The "flat part" of the parabola is at $x = 3.5$ (since this is exactly half way between $x = 1$ and $x = 6$; alternately, we can check that $f'''(3.5) = 0$).
- Since the coefficient of x^2 is negative, $f(x)$ is increasing from $-\infty$ to 3.5, then decreasing from 3.5 to ∞.

Therefore, over the interval $[1, 6]$, the largest positive value of $f''(x)$ occurs when $x = 3.5$, and this is $f''(3.5) = -(3.5-6)(3.5-1) = 6.25$.



So, we take $M = 6.25$.

(b) We differentiate further to find $f^{(4)}(x) = -2$. This is constant everywhere, so we take $L = |-2| = 2$.

S-4: Let's start by differentiating.

$$\begin{aligned}
f(x) &= x\sin x + 2\cos x \\
f'(x) &= x\cos x + \sin x - 2\sin x = x\cos x - \sin x \\
f''(x) &= -x\sin x + \cos x - \cos x = -x\sin x
\end{aligned}$$

For any value of x, $|\sin x| \leqslant 1$. When $-3 \leqslant x \leqslant 2$, then $|x| \leqslant 3$. So, it is true (and not unreasonably sloppy) that

$$f''(x) \leqslant 3$$

whenever x is in the interval $[-3, 2]$. So, we can take $M = 3$.

Note that $|f''(x)|$ is actually *smaller* than 3 whenever x is in the interval $[-3, 2]$, because when $x = -3$, $\sin x \neq 1$. In fact, since 3 is pretty close to π, $\sin 3$ is pretty small. (The actual maximum value of $|f''(x)|$ when $-3 \leqslant x \leqslant 2$ is about 1.8.) However, we find parameters like M for the purpose of computing error bounds. There is often not much to be gained from taking the time to find the actual maximum of a function, so we content ourselves with reasonable upper bounds. Question 31 has a further investigation of "sloppy" bounds like this.

S-5:

(a) Let $f(x) = \cos x$. Then $f^{(4)}(x) = \cos x$, so $|f^{(4)}(x)| \leqslant 1$ when $-\pi \leqslant x \leqslant \pi$. So, using $L = 1$, we find the upper bound of the error using Simpson's rule with $n = 4$ is:

$$\frac{L(b-a)^5}{180n^4} = \frac{(2\pi)^5}{180 \cdot 4^4} = \frac{\pi^5}{180 \cdot 8} \approx 0.2$$

The error bound comes from Theorem 3.9.12 in the text. We used a calculator to find the approximate decimal value.

(b) We use the general form of Simpson's rule (Equation 3.9.9 in the text) with $\Delta x = \frac{b-a}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$.

$$\begin{aligned} A &\approx \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right) \\ &= \frac{\pi/2}{3}\left(f(-\pi) + 4f(\tfrac{-\pi}{2}) + 2f(0) + 4f(\tfrac{\pi}{2}) + f(\pi)\right) \\ &= \frac{\pi}{6}\left(-1 + 4(0) + 2(1) + 4(0) - 1\right) = 0 \end{aligned}$$

(c) To find the actual error in our approximation, we compare the approximation from (b) to the exact value of A. In fact, $A = 0$: this is a fact you've probably seen before by considering the symmetry of cosine, but it's easy enough to calculate:

$$A = \int_{-\pi}^{\pi} \cos x \, \mathrm{d}x = \sin \pi - \sin(-\pi) = 0$$

So, our approximation was exactly the same as our exact value. The absolute error is 0.

Remark: the purpose of this question was to remind you that the error bounds we calculate are not (usually) the same as the actual error. Often our approximations are better than we give them credit for. In normal circumstances, we would be approximating an integral precisely to avoid evaluating it exactly, so we wouldn't find our exact error. The bound is a quick way of ensuring that our approximation is not *too* far off.

S-6: Using Theorem 3.9.12 in the text, the error using the trapezoidal rule as described is *at most*

$$\frac{M(b-a)^3}{12 \cdot n^2} = \frac{M}{48} \leqslant \frac{3}{48} = \frac{1}{16}.$$

So, we're really being asked to find a function with the maximum possible error using the trapezoidal rule, given its second derivative.

With that in mind, our function should have the largest second derivative possible: let's set $f''(x) = 3$ for every x. Then:

$$\begin{aligned} f''(x) &= 3 \\ f'(x) &= 3x + C \\ f(x) &= \frac{3}{2}x^2 + Cx + D \end{aligned}$$

for some constants C and D. Now we can find the exact and approximate values of $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$.

$$\text{Exact:}\qquad \int_0^1 f(x)\,\mathrm{d}x = \int_0^1 \left(\frac{3}{2}x^2 + Cx + D\right)\,\mathrm{d}x$$
$$= \left[\frac{1}{2}x^3 + \frac{C}{2}x^2 + Dx\right]_0^1$$
$$= \frac{1}{2} + \frac{C}{2} + D$$
$$\text{Approximate:}\qquad \int_0^1 f(x)\,\mathrm{d}x \approx \Delta x\left[\frac{1}{2}f(0) + f(\tfrac{1}{2}) + \frac{1}{2}f(1)\right]$$
$$= \frac{1}{2}\left[\frac{1}{2}(D) + \left(\frac{3}{8} + \frac{C}{2} + D\right) + \frac{1}{2}\left(\frac{3}{2} + C + D\right)\right]$$
$$= \frac{1}{2}\left[\frac{9}{8} + C + 2D\right]$$
$$= \frac{9}{16} + \frac{C}{2} + D$$

So, the absolute error associated with the trapezoidal approximation is:

$$\left|\left(\frac{1}{2} + \frac{C}{2} + D\right) - \left(\frac{9}{16} + \frac{C}{2} + D\right)\right| = \frac{1}{16}$$

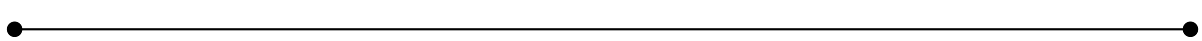
So, for any constants C and D, $f(x) = \frac{3}{2}x^2 + Cx + D$ has the desired error.

Remark: contrast this question with Question 5. In this problem, our absolute error was exactly as bad as the bound predicted, but sometimes it is much better. The thing to remember is that, in general, we *don't know* our absolute error. We only guarantee that it's not any worse than some worst-case-scenario bound.

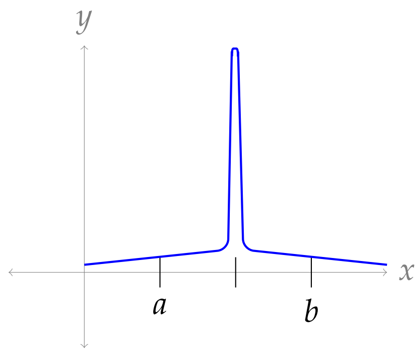
S-7: Under any reasonable assumptions[12], my mother is older than I am.

S-8: (a) Since both expressions are positive, and $\frac{1}{24} \leqslant \frac{1}{12}$, the inequality is true.
(b) False. The reasoning is the same as in Question 7. The *error bound* given by Theorem 1.11.12 is always better for the trapezoid rule, but this doesn't necessarily mean the *error* is better.

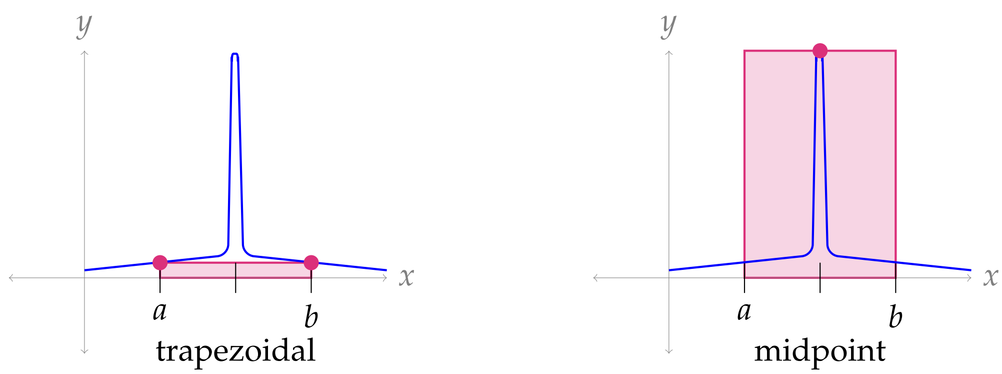
To see how the trapezoid approximation could be better than the corresponding midpoint approximation in some cases, consider the function $f(x)$ sketched below.



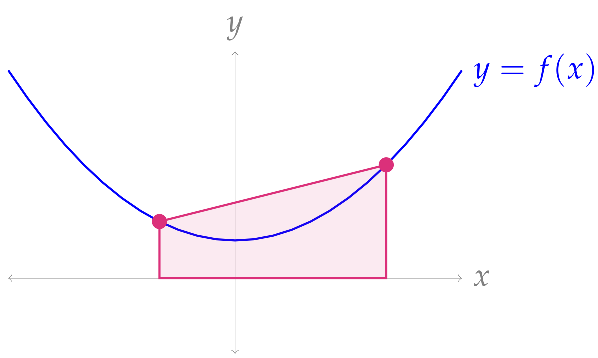
12 Anyone caught trying to come up with a scenario in which I am older than my mother will be sent to maximum security grad school.



The trapezoidal approximation of $\int_a^b f(x)\,dx$ with $n = 1$ misses the thin spike, and gives a mild underapproximation. By contrast, the midpoint approximation with $n = 1$ takes the spike as the height of the entire region, giving a vast overapproximation.



S-9: True. Because $f(x)$ is positive and concave up, the graph of $f(x)$ is always below the top edges of the trapezoids used in the trapezoidal rule.



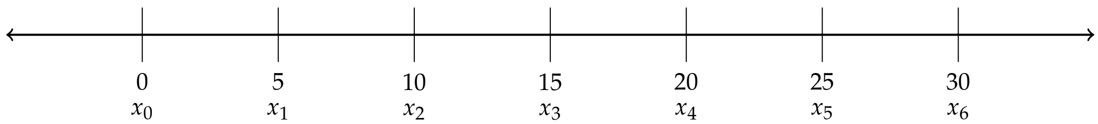
S-10: According to Theorem 3.9.12 in the text, the error associated with the Simpson's rule approximation is no more than $\dfrac{L}{180}\dfrac{(b-a)^5}{n^4}$, where L is a constant such that $|f^{(4)}(x)| \leqslant L$ for all x in $[a, b]$. If $L = 0$, then the error is no more than 0 regardless of a, b, or n–that is, the approximation is exact.

Any polynomial $f(x)$ of degree at most 3 has $f^{(4)}(x) = 0$ for all x. So, any polynomial of degree at most 3 is an acceptable answer. For example, $f(x) = 5x^3 - 27$, or $f(x) = x^2$.

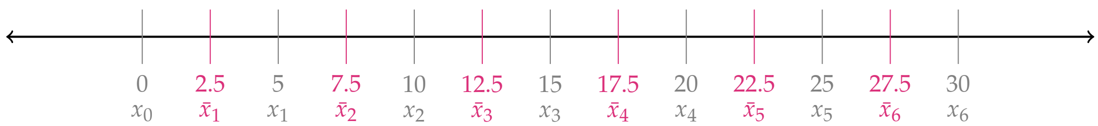
S-11:

- For all three approximations, $\Delta x = \dfrac{b-a}{n} = \dfrac{30-0}{6} = 5$.

- For the trapezoidal rule and Simpson's rule, the x-values where we evaluate $\dfrac{1}{x^3+1}$ start at $x = a = 0$ and move up by $\Delta x = 5$: $x_0 = 0$, $x_1 = 5$, $x_2 = 10$, $x_3 = 15$, $x_4 = 20$, $x_5 = 25$, and $x_6 = 30$.



- For the midpoint rule, the x-values where we evaluate $\dfrac{1}{x^3+1}$ start at $x = 2.5 = \frac{x_0+x_1}{2}$ and move up by $\Delta x = 5$: $\bar{x}_1 = 2.5$, $\bar{x}_2 = 7.5$, $\bar{x}_3 = 12.5$, $\bar{x}_4 = 17.5$, $\bar{x}_5 = 22.5$, and $\bar{x}_6 = 27.5$.



- Following Equation 3.9.2 in the text, the midpoint rule approximation is:

$$\begin{aligned}\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x &\approx \Big[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\Big]\Delta x \\ &= \left[\tfrac{1}{(2.5)^3+1} + \tfrac{1}{(7.5)^3+1} + \tfrac{1}{(12.5)^3+1} + \tfrac{1}{(17.5)^3+1} + \tfrac{1}{(22.5)^3+1} + \tfrac{1}{(27.5)^3+1}\right]5\end{aligned}$$

- Following Equation 3.9.6 in the text, the trapezoidal rule approximation is:

$$\begin{aligned}\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x &\approx \Big[\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)\Big]\Delta x \\ &= \left[\frac{1/2}{0^3+1} + \frac{1}{5^3+1} + \frac{1}{10^3+1} + \frac{1}{15^3+1} + \frac{1}{20^3+1} + \frac{1}{25^3+1} + \frac{1/2}{30^3+1}\right]5\end{aligned}$$

- Following Equation 3.9.9 in the text, the Simpson's rule approximation is:

$$\begin{aligned}\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x &\approx \Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\Big]\frac{\Delta x}{3} \\ &= \left[\frac{1}{0^3+2} + \frac{4}{5^3+1} + \frac{2}{10^3+1} + \frac{4}{15^3+1} + \frac{2}{20^3+1} + \frac{4}{25^3+1} + \frac{1}{30^3+1}\right]\frac{5}{3}\end{aligned}$$

S-12: By Equation 3.9.2 in the text, the midpoint rule approximation to $\int_a^b f(x)\,\mathrm{d}x$ with $n = 3$ is

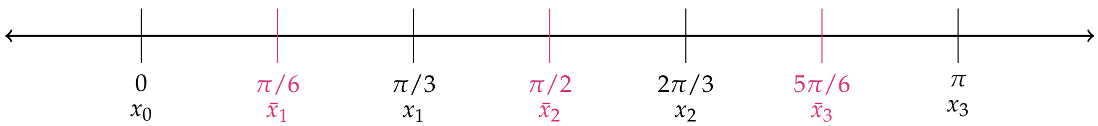
$$\int_a^b f(x)\,\mathrm{d}x \approx \big[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3)\big]\Delta x$$

where $\Delta x = \frac{b-a}{3}$ and

$$\begin{array}{llll} x_0 = a & x_1 = a + \Delta x & x_2 = a + 2\Delta x & x_3 = b \\ & \bar{x}_1 = \frac{x_0+x_1}{2} & \bar{x}_2 = \frac{x_1+x_2}{2} & \bar{x}_3 = \frac{x_2+x_3}{2} \end{array}$$

For this problem, $a = 0$, $b = \pi$ and $f(x) = \sin x$, so that $\Delta x = \frac{\pi}{3}$ and

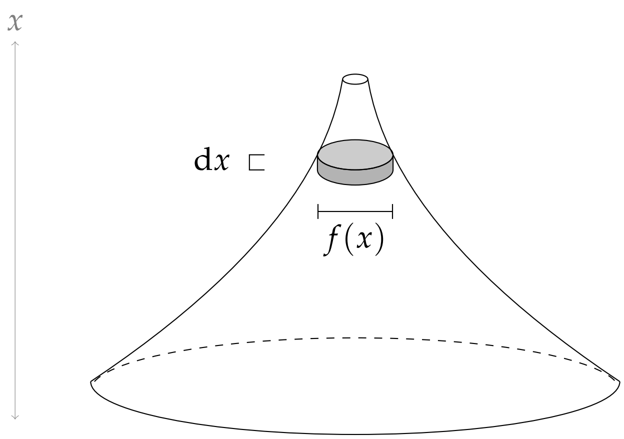
$$\begin{array}{llll} x_0 = 0 & x_1 = \frac{\pi}{3} & x_2 = \frac{2\pi}{3} & x_3 = \pi \\ & \bar{x}_1 = \frac{\pi}{6} & \bar{x}_2 = \frac{\pi}{2} & \bar{x}_3 = \frac{5\pi}{6} \end{array}$$



Therefore,

$$\int_0^{\pi} \sin x \, dx \approx \left[\sin\frac{\pi}{6} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{6}\right]\frac{\pi}{3} = \left[\frac{1}{2} + 1 + \frac{1}{2}\right]\frac{\pi}{3} = \frac{2\pi}{3}$$

S-13: Let $f(x)$ denote the diameter at height x. To approximate the volume of the solid, we slice it into thin horizontal "pancakes", which in this case are circular.



- We are told that the pancake at height x is a circular disk of diameter $f(x)$ and so
- has cross-sectional area $\pi\left(\frac{f(x)}{2}\right)^2$ and thickness dx and hence
- has volume $\pi\left(\frac{f(x)}{2}\right)^2 dx$.

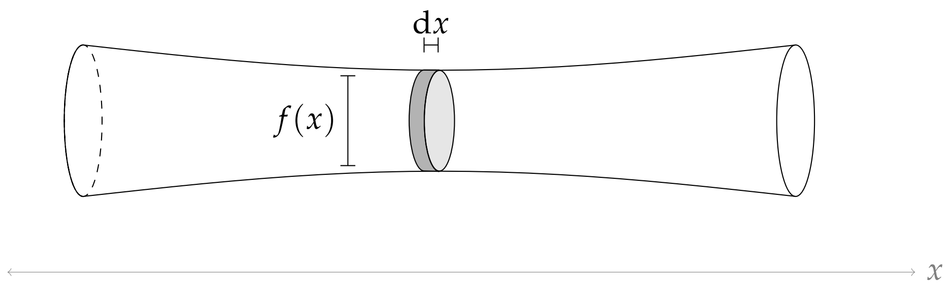
Hence the volume of V is

$$\begin{aligned} \int_0^{40} \pi\left[\frac{f(x)}{2}\right]^2 dx &\approx \frac{\pi}{4}10\left[\tfrac{1}{2}f(0)^2 + f(10)^2 + f(20)^2 + f(30)^2 + \tfrac{1}{2}f(40)^2\right] \\ &= \frac{\pi}{4}10\left[\tfrac{1}{2}24^2 + 16^2 + 10^2 + 6^2 + \tfrac{1}{2}4^2\right] \\ &= 688 \times 2.5\pi = 1720\pi \approx 5403.5 \end{aligned}$$

where we have approximated the integral using the trapezoidal rule with $\Delta x = 10$, and used a calculator to get a decimal approximation.

S-14: Let $f(x)$ be the diameter a distance x from the left end of the log. If we slice our log into thin disks, the disks x metres from the left end of the log has

- radius $\frac{f(x)}{2}$,
- width $\mathrm{d}x$, and so
- volume $\pi\left(\frac{f(x)}{2}\right)^2 \mathrm{d}x = \frac{\pi}{4}f(x)^2\,\mathrm{d}x$.

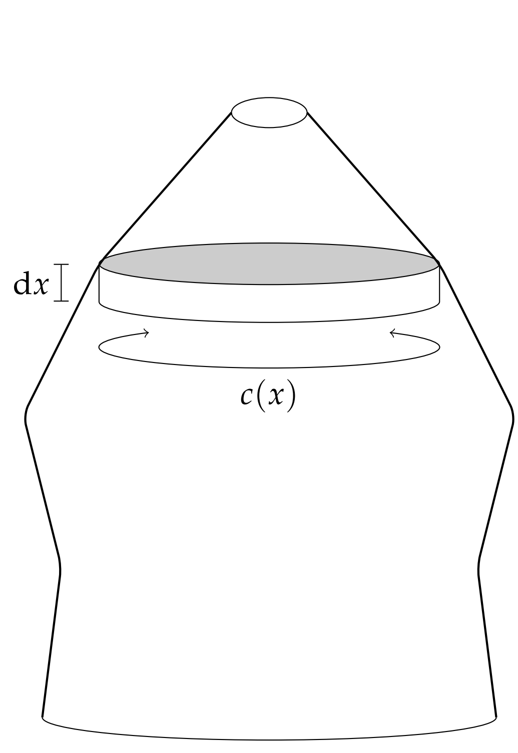


Using Simpson's Rule with $\Delta x = 1$, the volume of the log is:

$$\begin{aligned} V = \int_0^6 \frac{\pi}{4}f(x)^2\,\mathrm{d}x &\approx \frac{\pi}{4}\frac{1}{3}\left[f(0)^2 + 4f(1)^2 + 2f(2)^2 + 4f(3)^2 + 2f(4)^2 + 4f(5)^2 + f(6)^2\right] \\ &= \frac{\pi}{12}\left[1.2^2 + 4(1)^2 + 2(0.8)^2 + 4(0.8)^2 + 2(1)^2 + 4(1)^2 + 1.2^2\right] \\ &= \frac{\pi}{12}(16.72) \\ &\approx 4.377\ \mathrm{m}^3 \end{aligned}$$

where we used a calculator to approximate the decimal value.

S-15: At height x metres, let the circumference of the tree be $c(x)$. The corresponding radius is $\dfrac{c(x)}{2\pi}$, so the corresponding cross–sectional area is $\pi\left(\dfrac{c(x)}{2\pi}\right)^2 = \dfrac{c(x)^2}{4\pi}$.



The height of a very thin cross–sectional disk is dx, so the volume of a cross-sectional disk is $\dfrac{c(x)^2}{4\pi}\,\mathrm{d}x$. Therefore, total volume of the tree is:

$$\begin{aligned}\int_0^8 \frac{c(x)^2}{4\pi}\,\mathrm{d}x &\approx \frac{1}{4\pi}\frac{2}{3}\Big[c(0)^2+4c(2)^2+2c(4)^2+4c(6)^2+c(8)^2\Big]\\ &= \frac{1}{6\pi}\Big[1.2^2+4(1.1)^2+2(1.3)^2+4(0.9)^2+0.2^2\Big]\\ &= \frac{12.94}{6\pi}\approx 0.6865\end{aligned}$$

where we used Simpson's rule with $\Delta x = 2$ and $n = 4$ to approximate the value of the integral based on the values of $c(x)$ given in the table.

S-16: For both approximations, $\Delta x = 10$ and $n = 6$.

(a) The Trapezoidal Rule gives

$$\begin{aligned}V = \int_0^{60} A(h)\,\mathrm{d}h &\approx 10\Big[\tfrac{1}{2}A(0)+A(10)+A(20)+A(30)+A(40)+A(50)+\tfrac{1}{2}A(60)\Big]\\ &= 363{,}500\end{aligned}$$

(b) Simpson's Rule gives

$$\begin{aligned}V = \int_0^{60} A(h)\,\mathrm{d}h &\approx \frac{10}{3}\Big[A(0)+4A(10)+2A(20)+4A(30)+2A(40)+4A(50)+A(60)\Big]\\ &= 367{,}000\end{aligned}$$

S-17: Call the curve in the graph $y = f(x)$. It looks like

$$f(2) = 3 \qquad f(3) = 8 \qquad f(4) = 7 \qquad f(5) = 6 \qquad f(6) = 4$$

We're estimating $\int_2^6 f(x)\,\mathrm{d}x$ with $n = 4$, so $\Delta x = \frac{6-2}{4} = 1$.

(a) The trapezoidal rule gives

$$T_4 = \left[\frac{3}{2} + 8 + 7 + 6 + \frac{4}{2}\right] \times 1 = \frac{49}{2}$$

(b) Simpson's rule gives

$$S_4 = \frac{1}{3}\big[3 + 4 \times 8 + 2 \times 7 + 4 \times 6 + 4\big] \times 1 = \frac{77}{3}$$

S-18: Let $f(x) = \sin(x^2)$. Then $f'(x) = 2x\cos(x^2)$ and

$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2).$$

Since $|x^2| \leqslant 1$ when $|x| \leqslant 1$, and $|\sin\theta| \leqslant 1$ and $|\cos\theta| \leqslant 1$ for all θ, we have

$$\left|2\cos(x^2) - 4x^2\sin(x^2)\right| \leqslant 2|\cos(x^2)| + 4x^2|\sin(x^2)| \leqslant 2 \times 1 + 4 \times 1 \times 1 = 2 + 4 = 6$$

We can therefore choose $M = 6$, and it follows that the error is at most

$$\frac{M[b-a]^3}{24n^2} \leqslant \frac{6 \cdot [1-(-1)]^3}{24 \cdot 1000^2} = \frac{2}{10^6} = 2 \cdot 10^{-6}$$

S-19: Setting $f(x) = 2x^4$ and $b - a = 1 - (-2) = 3$, we compute $f''(x) = 24x^2$. The largest value of $24x^2$ on the interval $[-2, 1]$ occurs at $x = -2$, so we can take $M = 24 \cdot (-2)^2 = 96$. Thus the total error for the midpoint rule with $n = 60$ points is bounded by

$$\frac{M(b-a)^3}{24n^2} = \frac{96 \times 3^3}{24 \times 60 \times 60} = \frac{3}{100}$$

That is: we are guaranteed our absolute error is certainly no more[13] than $\frac{3}{100}$, and using the bound stated in the problem we cannot give a better guarantee. (The second part of the previous sentence comes from the fact that we used the smallest possible M: if we had used a larger value of M, we would still have some true statement about the error, for example "the error is no more than $\frac{5}{100}$," but it would not be the *best* true statement we could make.)

13 This is what the error bound always tells us.

S-20: (a) Since $a = 0$, $b = 2$ and $n = 6$, we have $\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$, and so $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$, $x_4 = \frac{4}{3}$, $x_5 = \frac{5}{3}$, and $x_6 = 2$. Since Simpson's Rule with $n = 6$ in general is

$$\frac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\big],$$

the desired approximation is

$$\frac{1/3}{3}\left((-3)^5 + 4\Big(\frac{1}{3} - 3\Big)^5 + 2\Big(\frac{2}{3} - 3\Big)^5 + 4(-2)^5 + 2\Big(\frac{4}{3} - 3\Big)^5 + 4\Big(\frac{5}{3} - 3\Big)^5 + (-1)^5\right)$$

(b) Here $f(x) = (x-3)^5$, which has derivatives

$$f'(x) = 5(x-3)^4 \qquad\qquad f''(x) = 20(x-3)^3$$
$$f^{(3)}(x) = 60(x-3)^2 \qquad\qquad f^{(4)}(x) = 120(x-3).$$

For $0 \leqslant x \leqslant 2$, $(x-3)$ runs from -3 to -1, so the maximum absolute values are found at $x = 0$, giving $M = 20 \cdot |0-3|^3 = 540$ and $L = 120 \cdot |0-3| = 360$. Consequently, for the Midpoint Rule with $n = 100$,

$$|E_M| \leqslant \frac{M(b-a)^3}{24n^2} = \frac{540 \times 2^3}{24 \times 10^4} = \frac{180}{10^4};$$

whereas for Simpson's Rule with $n = 10$,

$$|E_S| \leqslant \frac{360 \times 2^5}{180 \times 10^4} = \frac{64}{10^4}.$$

Since $64 < 180$, Simpson's Rule results in a smaller error bound.

S-21: In general the error in approximating $\int_a^b f(x)\,\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\frac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $L \geqslant |f^{(4)}(x)|$ for all $a \leqslant x \leqslant b$. In this case, $a = 1$, $b = 5$, $n = 4$ and $f(x) = \frac{1}{x}$. We need to find L, so we differentiate.

$$f'(x) = -\frac{1}{x^2} \qquad f''(x) = \frac{2}{x^3} \qquad f^{(3)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(x) = \frac{24}{x^5}$$

and

$$\left|f^{(4)}(x)\right| \leqslant 24 \text{ for all } x \geqslant 1$$

So we may take $L = 24$ and $\Delta x = \frac{5-1}{4} = 1$, which leads to

$$|\text{Error}\,| \leqslant \frac{24(5-1)}{180}(1)^4 = \frac{24}{45} = \frac{8}{15}$$

S-22: In general, the error in approximating $\int_a^b f(x)\,\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\dfrac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \dfrac{b-a}{n}$ and $L \geqslant |f^{(4)}(x)|$ for all $a \leqslant x \leqslant b$. In this case, $a = 0$, $b = 1$, $n = 6$ and $f(x) = e^{-2x} + 3x^3$. We need to find L, so we differentiate.

$$f'(x) = -2e^{-2x} + 9x^2 \qquad f''(x) = 4e^{-2x} + 18x \qquad f^{(3)}(x) = -8e^{-2x} + 18 \qquad f^{(4)}(x) = 16e^{-2x}$$

Since $e^{-2x} = \dfrac{1}{e^{2x}}$, we see $f^{(4)}(x)$ is a positive, decreasing function. So, its maximum occurs when x is as small as possible. In the interval $[0, 1]$, that means $x = 0$.

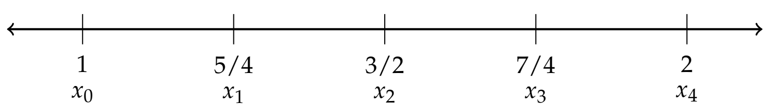
$$\left|f^{(4)}(x)\right| \leqslant f(0) = 16 \text{ for all } x \geqslant 0$$

So, we take $L = 16$ and $\Delta x = \frac{1-0}{6} = \frac{1}{6}$.

$$|\text{Error}\,| \leqslant \frac{L(b-a)}{180}(\Delta x)^4 = \frac{16(1-0)}{180}(1/6)^4 = \frac{16}{180 \times 6^4} = \frac{1}{180 \times 3^4} = \frac{1}{14580}$$

S-23: For both approximations, $a = 1$, $b = 2$, $n = 4$, $f(x) = \frac{1}{x}$ and $\Delta x = \frac{b-a}{n} = \frac{1}{4}$.

Then $x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4}$, and $x_4 = 2$.



(a)

$$\begin{aligned}
T_4 =& \Delta x \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right] \\
=& \Delta x \left[\frac{1}{2}f(1) + f(5/4) + f(3/2) + f(7/4) + \frac{1}{2}f(2)\right] \\
=& \ \frac{1}{4}\left[\left(\frac{1}{2} \times 1\right) + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \left(\frac{1}{2} \times \frac{1}{2}\right)\right]
\end{aligned}$$

(b)

$$\begin{aligned}
S_4 &= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\
&= \frac{\Delta x}{3}\left[f(1) + 4f(5/4) + 2f(3/2) + 4f(7/4) + f(2)\right] \\
&= \frac{1}{12}\left[1 + \left(4 \times \frac{4}{5}\right) + \left(2 \times \frac{2}{3}\right) + \left(4 \times \frac{4}{7}\right) + \frac{1}{2}\right]
\end{aligned}$$

(c) In this case, $a = 1$, $b = 2$, $n = 4$ and $f(x) = \frac{1}{x}$. We need to find L, so we differentiate.

$$f'(x) = -\frac{1}{x^2} \qquad f''(x) = \frac{2}{x^3} \qquad f^{(3)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(x) = \frac{24}{x^5}$$

So,

$$\left|f^{(4)}(x)\right| \leqslant 24 \text{ for all } x \text{ in the interval } [1,2]$$

We take $L = 24$.

$$|\text{Error }| \leqslant \frac{L(b-a)^5}{180 \times n^4} \leqslant \frac{24(2-1)^5}{180 \times 4^4} = \frac{24}{180 \times 4^4} = \frac{3}{5760}$$

S-24: Set $a = 0$ and $b = 8$. Since we have information about $s(x)$ when x is 0, 2, 4, 6, and 8, we set $\Delta x = \frac{b-a}{n} = 2$, so $n = 4$. (Recall with the trapezoid rule and Simpson's rule, $n = 4$ intervals actually uses the value of the function at 5 points.)

We could perform the trapezoidal approximations with fewer intervals, for example $n = 2$, but this would involve ignoring some of the points we're given. Since the question asks for the best estimation we can give, we use $n = 4$ intervals and no fewer.

(a)

$$\begin{aligned}
T_4 &= \Delta x \left[\frac{1}{2}s(0) + s(2) + s(4) + s(6) + \frac{1}{2}s(8)\right] \\
&= 2\left[\frac{1.00664}{2} + 1.00543 + 1.00435 + 1.00331 + \frac{1.00233}{2}\right] \\
&= 8.03515 \\
S_4 &= \frac{\Delta x}{3}\left[s(0) + 4s(2) + 2s(4) + 4s(6) + s(8)\right] \\
&= \frac{2}{3}\left[1.00664 + 4 \times 1.00543 + 2 \times 1.00435 + 4 \times 1.00331 + 1.00233\right] \\
&\approx 8.03509
\end{aligned}$$

(b) The information $\left|s^{(k)}(x)\right| \leqslant \dfrac{k}{1000}$, with $k = 2$, tells us $|s''(x)| \leqslant \frac{2}{1000}$ for all x in the interval $[0, 8]$. So, we take K_2 (also called M in your text) to be $\frac{2}{1000}$.

Then the absolute error associated with our trapezoid rule approximation is at most

$$\left|\int_a^b f(x)\,dx - T_n\right| \leqslant \frac{K_2(b-a)^3}{12n^2} \leqslant \frac{2}{1000} \cdot \frac{8^3}{12(4)^2} \leqslant 0.00533$$

For $k = 4$, we see $|s^{(4)}(x)| \leqslant \frac{4}{1000}$ for all x in the interval $[0, 8]$. So, we take K_4 (also called L in your text) to be $\frac{4}{1000}$.

Then the absolute error associated with our Simpson's rule approximation is at most

$$\left|\int_a^b f(x)\,dx - S_n\right| \leqslant \frac{K_4(b-a)^5}{180n^4} \leqslant \frac{4}{1000} \cdot \frac{8^5}{180(4)^4} \leqslant 0.00284$$

S-25: In this case, $a = 1$, $b = 4$. Since $-2 \leqslant f''(x) \leqslant 0$ over the relevant interval, we take $M = 2$. (Remember M is an upper bound on $|f''(x)|$, not $f''(x)$.) So we need n to obey

$$\frac{2(4-1)^3}{12n^2} \leqslant 0.001 \iff n^2 \geqslant \frac{2(3)^3}{12}1000 = \frac{27000}{6} = \frac{9000}{2} = 4500$$

One obvious allowed n is 100. Since $\sqrt{4500} \approx 67.01$, and n has to be a whole number, any $n \geqslant 68$ works.

S-26: Denote by $f(x)$ the width of the pool x feet from the left-hand end. From the sketch, $f(0) = 0$, $f(2) = 10$, $f(4) = 12$, $f(6) = 10$, $f(8) = 8$, $f(10) = 6$, $f(12) = 8$, $f(14) = 10$ and $f(16) = 0$.

A cross-section of the pool x feet from the left end is half of a circular disk with diameter $f(x)$ (so, radius $\frac{f(x)}{2}$) and thickness $\mathrm{d}x$. So, the volume of the part of the pool with x–coordinate running from x to $(x + \mathrm{d}x)$ is $\frac{1}{2}\pi\left(\frac{f(x)}{2}\right)^2 \mathrm{d}x = \frac{\pi}{8}[f(x)]^2 \mathrm{d}x$.

The total volume is given by the following integral.

$$\begin{aligned}
V &= \frac{\pi}{8}\int_0^{16} f(x)^2 \,\mathrm{d}x \\
&\approx \frac{\pi}{8}\cdot\frac{\Delta x}{3}\Big[f(0)^2 + 4f(2)^2 + 2f(4)^2 + 4f(6)^2 + 2f(8)^2 + 4f(10)^2 + 2f(12)^2 + 4f(14)^2 + f(16)^2\Big] \\
&= \frac{\pi}{8}\cdot\frac{2}{3}\Big[0 + 4(10)^2 + 2(12)^2 + 4(10)^2 + 2(8)^2 + 4(6)^2 + 2(8)^2 + 4(10)^2 + 0\Big] \\
&= \frac{472}{3}\pi \approx 494 \text{ ft}^3
\end{aligned}$$

S-27: (a) The Trapezoidal Rule with $n = 4$, $a = 0$, $b = 1$, and $\Delta x = \frac{1}{4}$ gives:

$$\begin{aligned}
W = 2\pi 10^{-6}\int_0^1 rg(r)\,\mathrm{d}r &\approx 2\pi 10^{-6}\Delta x\left[\frac{1}{2}x_0 g(x_0) + x_1 g(x_1) + x_2 g(x_2) + x_3 g(x_3) + \frac{1}{2}x_4 g(x_4)\right] \\
&= 2\pi 10^{-6}\,\frac{1}{4}\Big[\frac{1}{2}0g(0) + \frac{1}{4}g\Big(\frac{1}{4}\Big) + \frac{1}{2}g\Big(\frac{1}{2}\Big) + \frac{3}{4}g\Big(\frac{3}{4}\Big) + \frac{1}{2}g(1)\Big] \\
&= \pi 10^{-6}\,\frac{1}{2}\Big[\frac{8100}{4} + \frac{8144}{2} + \frac{3\cdot 8170}{4} + \frac{8190}{2}\Big] \\
&= \frac{32639\pi}{4\cdot 10^6} \approx 0.025635
\end{aligned}$$

(b) Using the product rule, the integrand $f(r) = 2\pi 10^{-6} rg(r)$ obeys

$$f''(r) = 2\pi 10^{-6}\frac{\mathrm{d}}{\mathrm{d}r}\big[g(r) + rg'(r)\big] = 2\pi 10^{-6}\big[2g'(r) + rg''(r)\big]$$

and hence, for $0 \leqslant r \leqslant 1$,

$$\big|f''(r)\big| \leqslant 2\pi 10^{-6}\big[2\times 200 + 1\times 150\big] = 1.1\pi 10^{-3}$$

So,

$$|\text{Error}| \leqslant \frac{1.1\pi 10^{-3}(1-0)^3}{12(4)^2} \leqslant 1.8\times 10^{-5}$$

S-28: (a) Let $f(x)=\frac{1}{x}$, $a=1$, $b=2$ and $\Delta x=\frac{b-a}{6}=\frac{1}{6}$. Using Simpson's rule:

$$\begin{aligned}\int_1^2 \frac{1}{x}\,\mathrm{d}x &\approx \frac{\Delta x}{3}\Big[f(1)+4f\Big(\frac{7}{6}\Big)+2f\Big(\frac{8}{6}\Big)+4f\Big(\frac{9}{6}\Big)+2f\Big(\frac{10}{6}\Big)+4f\Big(\frac{11}{6}\Big)+f(2)\Big]\\ &=\frac{1}{18}\Big[1+\frac{24}{7}+\frac{12}{8}+\frac{24}{9}+\frac{12}{10}+\frac{24}{11}+\frac{1}{2}\Big]\approx 0.6931698\end{aligned}$$

(b) The integrand is $f(x)=\frac{1}{x}$. The first four derivatives of $f(x)$ are:

$$f'(x)=-\frac{1}{x^2},\qquad f''(x)=\frac{2}{x^3},\qquad f^{(3)}(x)=-\frac{6}{x^4},\qquad f^{(4)}(x)=\frac{24}{x^5}$$

On the interval $1\leqslant x\leqslant 2$, the fourth derivative is never bigger in magnitude than $L=24$.

$$|E_n|\leqslant \frac{L(b-a)^5}{180n^4}=\frac{24(2-1)^5}{180n^4}=\frac{4}{30n^4}$$

So, we want an even number n such that

$$\begin{aligned}\frac{4}{30n^4}&\leqslant 0.00001=\frac{1}{10^5}\\ n^4&\geqslant\frac{40000}{3}\\ n&\geqslant\sqrt[4]{\frac{40000}{3}}\approx 10.7\end{aligned}$$

So, any even number greater than or equal to 12 will do.

S-29: (a) From the figure, we see that the magnitude of $|f''''(x)|$ never exceeds 310 for $0\leqslant x\leqslant 2$. So, the absolute error is bounded by

$$\frac{310(2-0)^5}{180\times 8^4}\leqslant 0.01345$$

(b) We want to choose n such that:

$$\begin{aligned}\frac{310(2-0)^5}{180\times n^4}&\leqslant 10^{-4}\\ n^4&\geqslant\frac{310\times 2^5}{180}10^4\\ n&\geqslant 10\sqrt[4]{\frac{310\times 32}{180}}\approx 27.2\end{aligned}$$

For Simpson's rule, n must be even, so any even integer obeying $n\geqslant 28$ will guarantee us the requisite accuracy.

S-30: Let $g(x) = \int_0^x \sin(\sqrt{t})\,\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $g'(x) = \sin(\sqrt{x})$. By its definition, $f(x) = g(x^2)$, so we use the chain rule to differentiate $f(x)$.

$$f'(x) = 2xg'(x^2) = 2x\sin x \qquad\qquad f''(x) = 2\sin x + 2x\cos x$$

Since $|\sin x|, |\cos x| \leqslant 1$, we have $|f''(x)| \leqslant 2 + 2|x|$ and, for $0 \leqslant t \leqslant 1$, $|f''(t)| \leqslant 4$. When the trapezoidal rule with n subintervals is applied, the resulting error E_n obeys

$$E_n \leqslant \frac{4(1-0)^3}{12n^2} = \frac{1}{3n^2}$$

We want an integer n such that

$$\begin{aligned}
\frac{1}{3n^2} &\leqslant 0.000005 \\
n^2 &\geqslant \frac{4}{12 \times 0.000005} \\
n &\geqslant \sqrt{\frac{1}{3 \times 0.000005}} \approx 258.2
\end{aligned}$$

Any integer $n \geqslant 259$ will do.

S-31:

(a) When $0 \leqslant x \leqslant 1$, then $x^2 \leqslant 1$ and $x + 1 \geqslant 1$, so $|f''(x)| = \dfrac{x^2}{|x+1|} \leqslant \dfrac{1}{1} = 1$.

(b) To find the maximum value of a function over a closed interval, we test the function's values at the endpoints of the interval and at its critical points inside the interval. The critical points are where the function's derivative is zero or it does not exist.

The function we're trying to maximize is $|f''(x)| = \frac{x^2}{|x+1|} = \frac{x^2}{x+1} = f''(x)$ (since our interval only contains nonnegative numbers). So, the critical points occur when $f'''(x) = 0$ or does not exist. We find $f'''(x)$ Using the quotient rule.

$$\begin{aligned}
f'''(x) &= \frac{(x+1)(2x) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \\
0 &= \frac{x(x+2)}{x+1} \\
0 &= x \quad \text{or} \quad x = -1 \quad \text{or} \quad x = -2
\end{aligned}$$

The only critical point in $[0, 1]$ is $x = 0$. So, the extrema of $f''(x)$ over $[0, 1]$ will occur at its endpoints. Indeed, since $f'''(x) \geqslant 0$ for all x in $[0, 1]$, $f''(x)$ is increasing over this interval, so its maximum occurs at $x = 1$. That is,

$$|f''(x)| \leqslant f''(1) = \frac{1}{2}$$

(c) The absolute error using the midpoint rule is at most $\dfrac{M(b-a)^3}{24n^2}$. Using $M=1$, if we want this to be no more than 10^{-5}, we find an acceptable value of n with the following calculation:

$$\begin{aligned} \frac{M(b-a)^3}{24n^2} &\leqslant 10^{-5} \\ \frac{1}{24n^2} &\leqslant 10^{-5} && (b-a=1,\ M=1) \\ \frac{10^5}{24} &\leqslant n^2 \\ n &\geqslant 65 \end{aligned}$$

(d) The absolute error using the midpoint rule is at most $\dfrac{M(b-a)^3}{24n^2}$. Using $M=\frac{1}{2}$, if we want this to be no more than 10^{-5}, we find an acceptable value of n with the following calculation:

$$\begin{aligned} \frac{M(b-a)^3}{24n^2} &\leqslant 10^{-5} \\ \frac{1}{48n^2} &\leqslant 10^{-5} && (b-a=1,\ M=\frac{1}{2}) \\ \frac{10^5}{48} &\leqslant n^2 \\ n &\geqslant 46 \end{aligned}$$

Remark: how accurate you want to be in these calculations depends a lot on your circumstances. Imagine, for instance, that you were finding M by hand, using this to find n by hand, then programming a computer to evaluate the approximation. For a simple integral like this, the difference between computing time for 65 intervals versus 46 is likely to be miniscule. So, there's not much to be gained by the extra work in (b). However, if your original sloppy M gave you something like $n = 1000000$, you might want to put some time into improving it, to shorten computation time. Moreover, if you were finding the approximation by hand, the difference between adding 46 terms and adding 65 terms would be considerable, and you would probably want to put in the effort up front to find the most accurate M possible.

S-32: Before we can take our Simpson's rule approximation of $\displaystyle\int_1^x \frac{1}{t}\,\mathrm{d}t$, we need to know how many intervals to use. That means we need to bound our error, which means we need to bound $\frac{\mathrm{d}^4}{\mathrm{d}t^4}\left\{\frac{1}{t}\right\}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{t}\right\} = -\frac{1}{t^2} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\frac{1}{t}\right\} = \frac{2}{t^3}$$
$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left\{\frac{1}{t}\right\} = -\frac{6}{t^4} \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4}\left\{\frac{1}{t}\right\} = \frac{24}{t^5}$$

So, over the interval $[1,3]$, $\left|\frac{\mathrm{d}^4}{\mathrm{d}t^4}\left\{\frac{1}{t}\right\}\right| \leqslant 24$.

Now, we can find an appropriate n to ensure our error will be be less than 0.1 for any x in $[1,3]$:

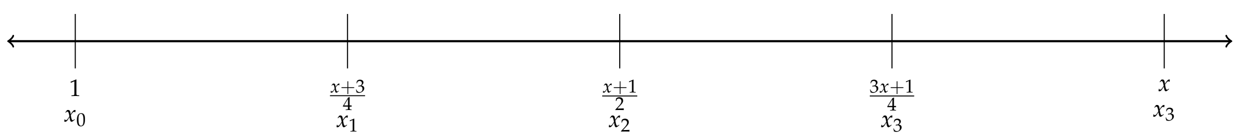
$$\begin{aligned}\frac{L(b-a)^5}{180n^4} &< 0.1\\ \frac{24(x-1)^5}{180n^4} &< \frac{1}{10}\\ n^4 &> \frac{24\cdot(x-1)^5}{18}\end{aligned}$$

Because $x-1 \leqslant 2$ for every x in $[1,3]$, if $n^4 > \frac{24\cdot 2^5}{18}$, then $n^4 > \frac{24\cdot(x-1)^5}{18}$ for every allowed x.

$$\begin{aligned}n^4 &> \frac{24\cdot 2^5}{18} = \frac{128}{3}\\ n &> \sqrt[4]{\frac{128}{3}} \approx 2.6\end{aligned}$$

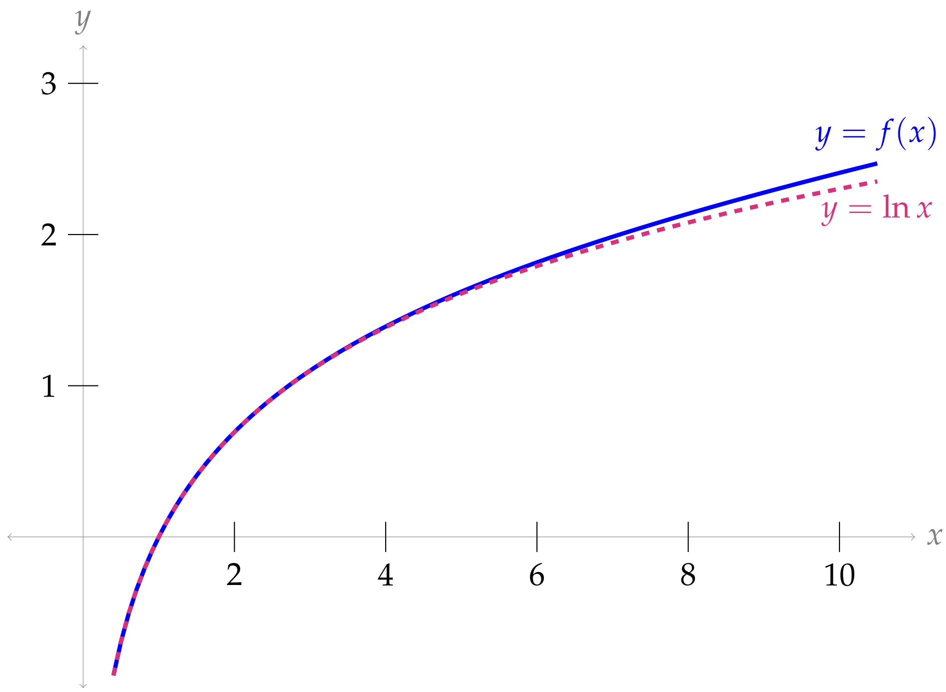
Since n must be even, $n=4$ is enough intervals to guarantee our error is not too high for any x in $[1,3]$. Now we find our Simpson's rule approximation with $n=4$, $a=1$, $b=x$, and $\Delta x = \frac{x-1}{4}$. The points where we evaluate $\frac{1}{t}$ are:

$$x_0 = 1 \qquad x_1 = 1+\frac{x-1}{4} = \frac{x+3}{4} \qquad x_2 = 1+2\frac{x-1}{4} = \frac{x+1}{2} \qquad x_3 = 1+3\frac{x-1}{4} = \frac{3x+1}{4} \qquad x_4 = 1+4\frac{x-1}{4} = x$$



$$\begin{aligned}\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t &\approx \frac{\Delta x}{3}\left[\frac{1}{x_0}+\frac{4}{x_1}+\frac{2}{x_2}+\frac{4}{x_3}+\frac{1}{x_4}\right]\\ &= \frac{x-1}{12}\left[1+\frac{16}{x+3}+\frac{4}{x+1}+\frac{16}{3x+1}+\frac{1}{x}\right]\\ &= f(x)\end{aligned}$$

Below is a graph of our approximation $f(x)$ and natural logarithm on the same axes. The natural logarithm function is shown red and dashed, while our approximating function is solid blue. Our approximation appears to be quite accurate for small, positive values of x.



S-33: First, we want a strategy for approximating $\arctan 2$. Our hints are that involves integrating $\frac{1}{1+x^2}$, which is the antiderivative of arctangent, and the number $\frac{\pi}{4}$, which is the same as $\arctan(1)$. With that in mind:

$$\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(2) - \arctan(1) = \arctan(2) - \frac{\pi}{4}$$

$$\text{So,} \qquad \arctan(2) = \frac{\pi}{4} + \int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x \tag{$*$}$$

We won't know the value of the integral exactly, but we'll have an approximation A bounded by some positive error bound ε. Then,

$$-\varepsilon \leqslant \left(\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x - A\right) \leqslant \varepsilon$$

$$A - \varepsilon \leqslant \left(\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x\right) \leqslant A + \varepsilon$$

$$\text{So, from } (*), \qquad \frac{\pi}{4} + A - \varepsilon \leqslant \arctan(2) \leqslant \frac{\pi}{4} + A + \varepsilon$$

Which approximation should we use? We're given the fourth derivative of $\frac{1}{1+x^2}$, which is the derivative we need for Simpson's rule. Simpson's rule is also usually quite efficient, and we're very interested in not adding up dozens of terms, so we choose Simpson's rule.

Now that we've chosen Simpson's rule, we should decide how many intervals to use. In order to bound our error, we need to find a bound for the fourth derivative. To that end, define $N(x) = 24(5x^4 - 10x^2 + 1)$. Then $N'(x) = 24(20x^3 - 20x) = 480x(x^2 - 1)$, which is positive over the interval $[1,2]$. So, $N(x) \leqslant N(2) = 24(5 \cdot 2^4 - 10 \cdot 2^2 + 1) = 984$ when $1 \leqslant x \leqslant 2$. Furthermore, let $D(x) = (x^2+1)^5$. If $1 \leqslant x \leqslant 2$, then $D(x) \geqslant 2^5$. Now we can find a reasonable value of L:

$$|f^{(4)}(x)| = \left|\frac{25(5x^4 - 10x^2 + 1)}{(x^2+1)^5}\right| = \left|\frac{N(x)}{D(x)}\right| \leqslant \frac{984}{2^5} = \frac{123}{4} = 30.75$$

So, we take $L = 30.75$.

We want $\left[\frac{\pi}{4} + A - \varepsilon, \frac{\pi}{4} + A + \varepsilon\right]$ to look something like $\left[\frac{\pi}{4} + 0.321, \frac{\pi}{4} + 0.323\right]$. Note ε is half the length of the first interval. Half the length of the second interval is $0.001 = \frac{1}{1000}$. So, we want a value of ε that is no larger than this. Now we can find our n:

$$\begin{aligned}
\frac{L(b-a)^5}{180 \cdot n^4} &\leqslant \frac{1}{1000} \\
\frac{30.75}{180 \cdot n^4} &\leqslant \frac{1}{1000} \\
n^4 &\geqslant \frac{30.75 \times 1000}{180} \\
n &\geqslant \sqrt[4]{\frac{30750}{180}} \approx 3.62
\end{aligned}$$

So, we choose $n = 4$), and are guaranteed that the absolute error in our approximation will be no more than $\dfrac{30.75}{180 \cdot 4^4} < 0.00067$.

Since $n = 4$, then $\Delta x = \dfrac{b-a}{n} = \dfrac{1}{4}$, so:

$$x_0 = 1 \qquad x_1 = \frac{5}{4} \qquad x_2 = \frac{3}{2} \qquad x_3 = \frac{7}{4} \qquad x_4 = 2$$

Now we can find our Simpson's rule approximation A:

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x &\approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\
&= \frac{1/4}{3}\left[f(1) + 4f(5/4) + 2f(3/2) + 4f(7/4) + f(2)\right] \\
&= \frac{1}{12}\left[\frac{1}{1+1} + \frac{4}{25/16+1} + \frac{2}{9/4+1} + \frac{4}{49/16+1} + \frac{1}{4+1}\right] \\
&= \frac{1}{12}\left[\frac{1}{2} + \frac{4 \cdot 16}{25+16} + \frac{2 \cdot 4}{9+4} + \frac{4 \cdot 16}{49+16} + \frac{1}{5}\right] \\
&= \frac{1}{12}\left[\frac{1}{2} + \frac{64}{41} + \frac{8}{13} + \frac{64}{65} + \frac{1}{5}\right] \\
&\approx 0.321748 = A
\end{aligned}$$

As we saw before, the error associated with this approximation is at most $\frac{30.75}{180 \cdot 4^4} < 0.00067 = \varepsilon$. So,

$$\begin{array}{rlclcl}
 & A - \varepsilon & \leqslant & \displaystyle\int_1^2 \frac{1}{1+x^2}\,dx & \leqslant & A + \varepsilon \\
\Rightarrow & 0.321748 - 0.00067 & \leqslant & \displaystyle\int_1^2 \frac{1}{1+x^2}\,dx & \leqslant & 0.321748 + 0.00067 \\
\Rightarrow & 0.321078 & \leqslant & \displaystyle\int_1^2 \frac{1}{1+x^2}\,dx & \leqslant & 0.322418 \\
\Rightarrow & 0.321 & \leqslant & \displaystyle\int_1^2 \frac{1}{1+x^2}\,dx & \leqslant & 0.323 \\
\Rightarrow & \frac{\pi}{4} + 0.321 & \leqslant & \displaystyle\int_1^2 \frac{1}{1+x^2}\,dx + \frac{\pi}{4} & \leqslant & \frac{\pi}{4} + 0.323 \\
\Rightarrow & \frac{\pi}{4} + 0.321 & \leqslant & \arctan(2) & \leqslant & \frac{\pi}{4} + 0.323
\end{array}$$

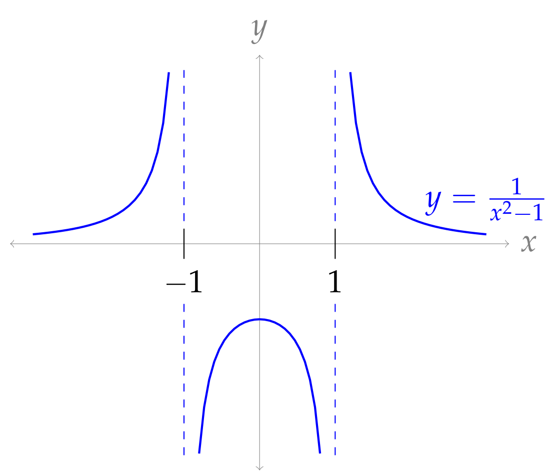
This is precisely what we wanted to show.

Solutions to Exercises 3.10 — Jump to TABLE OF CONTENTS

S-1: If $b = \pm\infty$, then our integral is improper because one limit is not a real number.

Furthermore, our integral will be improper if its domain of integration contains either of its infinite discontinuities, $x = 1$ and $x = -1$. Since one limit of integration is 0, the integral is improper if $b \geqslant 1$ or if $b \leqslant -1$.

Below, we've graphed $\frac{1}{x^2-1}$ to make it clearer why values of b in $(-1, 1)$ are the only values that *don't* result in an improper integral when the other limit of integration is $a = 0$.



S-2: Since the integrand is continuous for all real x, the only kind of impropriety available to us is to set $b = \pm\infty$.

S-3: For large values of x, |red dotted function| $\leqslant$ (blue solid function) and $0 \leqslant$ (blue solid function). If the blue solid function's integral converged, then the red dotted function's integral would as well (by the comparison test, Theorem 3.10.17 in the text). Since one integral converges and the other diverges, the blue solid function is $g(x)$ and the red dotted function is $f(x)$.

S-4: False. The inequality goes the "wrong way" for Theorem 3.10.17 in the text: the area under the curve $f(x)$ is finite, but the area under $g(x)$ could be much larger, even infinitely larger.

For example, if $f(x) = e^{-x}$ and $g(x) = 1$, then $0 \leqslant f(x) \leqslant g(x)$ and $\displaystyle\int_1^\infty f(x)\,dx$ converges, but $\displaystyle\int_1^\infty g(x)\,dx$ diverges.

S-5:

(a) Not enough information to decide. For example, consider $h(x) = 0$ versus $h(x) = -1$. In both cases, $h(x) \leqslant f(x)$. However, $\displaystyle\int_0^\infty 0\,dx$ converges to 0, while $\displaystyle\int_0^\infty -1\,dx$ diverges.

Note: if we had also specified $0 \leqslant h(x)$, then we would be able to conclude that $\int_0^\infty h(x)\,dx$ converges by the comparison test.

(b) Not enough information to decide. For example, consider $h(x) = f(x)$ versus $h(x) = g(x)$. In both cases, $f(x) \leqslant h(x) \leqslant g(x)$.

(c) $\displaystyle\int_0^\infty h(x)\,dx$ converges.

- From the given information, $|h(x)| \leqslant 2f(x)$.
- We claim $\displaystyle\int_0^\infty 2f(x)\,dx$ converges.
  - We can see this by writing $\displaystyle\int_0^\infty 2f(x)\,dx = 2\int_0^\infty f(x)\,dx$ and noting that the second integral converges.
  - Alternately, we can use the limiting comparison test, Theorem 3.10.22 in the text. Since $f(x) \geqslant 0$, $\displaystyle\int_0^\infty f(x)\,dx$ converges, and $\displaystyle\lim_{x\to\infty}\frac{2f(x)}{f(x)} = 2$ (the limit exists), we conclude $\displaystyle\int_0^\infty 2f(x)\,dx$ converges.

- So, comparing $h(x)$ to $2f(x)$, by the comparison test (Theorem 3.10.17 in the text) $\int_0^\infty h(x)\,\mathrm{d}x$ converges.

S-6: The denominator is zero when $x = 1$, but the numerator is not, so the integrand has a singularity (infinite discontinuity) at $x = 1$. Let's replace the limit $x = 1$ with a variable that creeps toward 1.

$$\int_0^1 \frac{x^4}{x^5-1}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t \frac{x^4}{x^5-1}\,\mathrm{d}x$$

To evaluate this integral we use the substitution $u = x^5$, $\mathrm{d}u = 5x^4\mathrm{d}x$. When $x = 0$ we have $u = 0$, and when $x = t$ we have $u = t^5$, so

$$\begin{aligned}\int_0^1 \frac{x^4}{x^5-1}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t \frac{x^4}{x^5-1}\,\mathrm{d}x &= \lim_{t\to 1^-}\int_{u=0}^{u=t^5} \frac{1}{5(u-1)}\,\mathrm{d}u \\ &= \lim_{t\to 1^-}\left(\left[\frac{1}{5}\ln|u-1|\right]_0^{t^5}\right) = \lim_{t\to 1^-}\frac{1}{5}\ln|t^5-1| = -\infty\end{aligned}$$

The limit diverges, so the integral diverges as well.

S-7: The denominator of the integrand is zero when $x = -1$, but the numerator is not. So, the integrand has a singularity (infinite discontinuity) at $x = -1$. This is the only "source of impropriety" in this integral, so we only need to make one break in the domain of integration.

$$\int_{-2}^{2} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x = \lim_{t\to -1^-}\int_{-2}^{t} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x + \lim_{t\to -1^+}\int_{t}^{2} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x$$

Let's start by considering the left limit.

$$\begin{aligned}\lim_{t\to -1^-}\int_{-2}^{t} \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x &= \lim_{t\to -1^-}\left(\left[-\frac{3}{(x+1)^{1/3}}\right|_{-2}^{t}\right) \\ &= \lim_{t\to -1^-}\left(-\frac{3}{(t+1)^{1/3}} + \frac{3}{(-1)^{1/3}}\right) = \infty\end{aligned}$$

Since this limit diverges, the integral diverges. (A similar argument shows that the second integral diverges. Either one of them diverging is enough to conclude that the original integral diverges.)

S-8: First, let's identify all "sources of impropriety." The integrand has a singularity when $4x^2 - x = 0$, that is, when $x(4x-1) = 0$, so at $x = 0$ and $x = \frac{1}{4}$. Neither of these are in our domain of integration, so the only "source of impropriety" is the unbounded domain of integration.

We could antidifferentiate this function (it looks like a nice candidate for a trig substitution), but is seems easier to use a comparison. For large values of x, the term x^2

will be much larger than x, so we might guess that our integral behaves similarly to $\int_1^\infty \frac{1}{\sqrt{4x^2}}\,dx = \int_1^\infty \frac{1}{2x}\,dx$.

For all $x \geqslant 1$, $\sqrt{4x^2 - x} \leqslant \sqrt{4x^2} = 2x$. So, $\frac{1}{\sqrt{4x^2-x}} \geqslant \frac{1}{2x}$. Note $\int_1^\infty \frac{1}{2x}\,dx$ diverges:

$$\lim_{t\to\infty}\int_1^t \frac{1}{2x}\,dx = \lim_{t\to\infty}\left(\frac{1}{2}\big[\ln x\big]_1^t\right) = \lim_{t\to\infty}\frac{1}{2}\ln t = \infty$$

So:

- $\frac{1}{2x}$ and $\frac{1}{\sqrt{4x^2-x}}$ are defined and continuous for all $x \geqslant 1$,
- $\frac{1}{2x} \geqslant 0$ for all $x \geqslant 1$,
- $\frac{1}{\sqrt{4x^2-x}} \geqslant \frac{1}{\sqrt{4x^2}} = \frac{1}{2x}$ for all $x \geqslant 1$, and
- $\int_1^\infty \frac{1}{2x}\,dx$ diverges.

By the comparison test, Theorem 3.10.17 in the text, the integral does not converge.

S-9: The integrand is positive everywhere. So, either the integral converges to some finite number, or it is infinite. We want to generate a guess as to which it is.

When x is small, $\sqrt{x} > x^2$, so we might guess that our integral behaves like the integral of $\frac{1}{\sqrt{x}}$ when x is near to 0. On the other hand, when x is large, $\sqrt{x} < x^2$, so we might guess that our integral behaves like the integral of $\frac{1}{x^2}$ as x goes to infinity. This is the hunch that drives the following work:

$$0 \leqslant \frac{1}{x^2+\sqrt{x}} \leqslant \frac{1}{\sqrt{x}} \quad \text{and the integral } \int_0^1 \frac{dx}{\sqrt{x}} \text{ converges by Example 3.10.9 in the text, and}$$
$$0 \leqslant \frac{1}{x^2+\sqrt{x}} \leqslant \frac{1}{x^2} \quad \text{and the integral } \int_1^\infty \frac{dx}{x^2} \text{ converges by Example 3.10.8 in the text}$$

Note $\frac{dx}{x^2+\sqrt{x}}$ is defined and continuous everywhere, $\frac{1}{\sqrt{x}}$ is defined and continuous for $x > 0$, and $\frac{1}{x^2}$ is defined and continuous for $x \geqslant 1$. So, the integral converges by the comparison test, Theorem 3.10.17 in the text, together with Remark 3.10.16.

S-10: There are two “sources of impropriety:” the two (infinite) limits of integration. So, we break our integral into two pieces.

$$\begin{aligned}\int_{-\infty}^{\infty}\cos x\,dx &= \int_{-\infty}^{0}\cos x\,dx + \int_0^\infty \cos x\,dx\\ &= \lim_{a\to\infty}\left[\int_{-a}^{0}\cos x\,dx\right] + \lim_{b\to\infty}\left[\int_0^b \cos x\,dx\right]\end{aligned}$$

These are easy enough to antdifferentiate.

$$\begin{aligned}&= \lim_{a\to\infty}\left[\sin 0 - \sin(-a)\right] + \lim_{b\to\infty}\left[\sin b - \sin 0\right]\\ &= \text{DNE}\end{aligned}$$

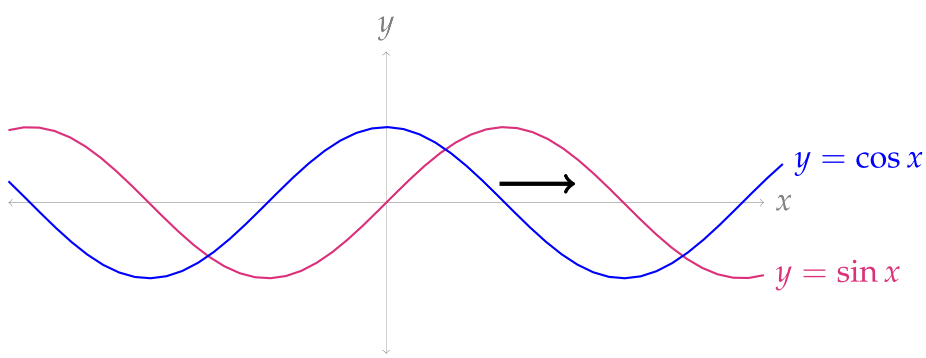
Since the limits don't exist, the integral diverges. (It happens that both limits don't exist; even if only one failed to exist, the integral would still diverge.)

S-11: There are two "sources of impropriety:" the two bounds. So, we break our integral into two pieces.

$$\begin{aligned}\int_{-\infty}^{\infty} \sin x \, dx &= \int_{-\infty}^{0} \sin x \, dx + \int_{0}^{\infty} \sin x \, dx \\ &= \lim_{a\to\infty} \left[\int_{-a}^{0} \sin x \, dx\right] + \lim_{b\to\infty} \left[\int_{0}^{b} \sin x \, dx\right] \\ &= \lim_{a\to\infty} [-\cos 0 + \cos(-a)] + \lim_{b\to\infty} [-\cos b + \cos 0] \\ &= \text{DNE}\end{aligned}$$

Since the limits don't exist, the integral diverges. (It happens that both limits don't exist; even if only one failed to exist, the integral would diverge.)

Remark: it's very tempting to think that this integral should converge, because as an odd function the area to the right of the x-axis "cancels out" the area to the left when the limits of integration are symmetric. One justification for not using this intuition is given in Example 3.10.11 in the text. Here's another: In Question 10 we saw that $\int_{-\infty}^{\infty} \cos x \, dx$ diverges. Since $\sin x = \cos(x - \pi/2)$, the area bounded by sine and the area bounded by cosine over an infinite region seem to be the same–only shifted by $\pi/2$. So if $\int_{-\infty}^{\infty} \sin x \, dx = 0$, then we ought to also have $\int_{-\infty}^{\infty} \cos x \, dx = 0$, but we saw in Question 10 this is not the case.



S-12: First, we check that the integrand has no singularities. The denominator is always positive when $x \geqslant 10$, so our only "source of impropriety" is the infinite limit of integration.

We further note that, for large values of x, the integrand resembles $\dfrac{x^4}{x^5} = \dfrac{1}{x}$. So, we have a two-part hunch: that the integral diverges, and that we can show it diverges by comparing it to $\displaystyle\int_{10}^{\infty} \frac{1}{x} \, dx$.

In order to use the comparison test, we'd need to show that $\dfrac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8} \geqslant \dfrac{1}{x}$. If this is true, it will be difficult to prove–and it's not at all clear that it's true. So, we will use the limiting comparison test instead, Theorem 3.10.22 in the text, with $g(x) = \frac{1}{x}$, $f(x) = \dfrac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8}$, and $a = 10$.

- Both $f(x)$ and $g(x)$ are defined and continuous for all $x > 0$, so in particular they are defined and continuous for $x \geqslant 10$.
- $g(x) \geqslant 0$ for all $x \geqslant 10$
- $\displaystyle\int_{10}^{\infty} g(x)\, dx$ diverges.
- Using l'Hôpital's rule (5 times!), or simply dividing both the numerator and denominator by x^5 (the common leading term), tells us:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{x^4-5x^3+2x-7}{x^5+3x+8}}{\frac{1}{x}} = \lim_{x\to\infty} x \cdot \frac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8}$$
$$= \lim_{x\to\infty} \frac{x^5 - 5x^4 + 2x^2 - 7x}{x^5 + 3x + 8} = 1$$

  That is, the limit exists and is nonzero.

By the limiting comparison test, we conclude $\displaystyle\int_{10}^{\infty} f(x)\, dx$ diverges.

S-13: Our domain of integration is finite, so the only potential "sources of impropriety" are infinite discontinuities in the integrand. To find these, we factor.

$$\int_0^{10} \frac{x-1}{x^2 - 11x + 10}\, dx = \int_0^{10} \frac{x-1}{(x-1)(x-10)}\, dx$$

A removable discontinuity doesn't affect the integral.

$$= \int_0^{10} \frac{1}{x-10}\, dx$$

Use the substitution $u = x - 10$, $du = dx$. When $x = 0$, $u = -10$, and when $x = 10$, $u = 0$.

$$= \int_{-10}^{0} \frac{1}{u}\, du$$

This is a p-integral with $p = 1$. From Example 3.10.9 and Theorem 3.10.20 in the text, we know it diverges.

S-14: You might think that, because the integrand is odd, the integral converges to 0. This is a common mistake– see Example 3.10.11 in the text, or Question 11 in this section. In the absence of such a shortcut, we use our standard procedure: identifying problem spots over the domain of integration, and replacing them with limits.

There are two "sources of impropriety," namely $x \to +\infty$ and $x \to -\infty$. So, we split the integral in two, and treat the two halves separately. The integrals below can be evaluated with the substitution $u = x^2+1$, $\frac{1}{2}\mathrm{d}u = x\mathrm{d}x$.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x &= \int_{-\infty}^{0} \frac{x}{x^2+1}\mathrm{d}x + \int_{0}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x \\
\int_{-\infty}^{0} \frac{x}{x^2+1}\mathrm{d}x &= \lim_{R\to\infty}\int_{-R}^{0} \frac{x}{x^2+1}\mathrm{d}x = \lim_{R\to\infty}\frac{1}{2}\ln(x^2+1)\Big|_{-R}^{0} \\
&= \lim_{R\to\infty}\frac{1}{2}\Big[\ln 1 - \ln(R^2+1)\Big] = \lim_{R\to\infty} -\frac{1}{2}\ln(R^2+1) = -\infty \\
\int_{0}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x &= \lim_{R\to\infty}\int_{0}^{R} \frac{x}{x^2+1}\mathrm{d}x = \lim_{R\to\infty}\frac{1}{2}\ln(x^2+1)\Big|_{0}^{R} \\
&= \lim_{R\to\infty}\frac{1}{2}\Big[\ln(R^2+1) - \ln 1\Big] = \lim_{R\to\infty}\frac{1}{2}\ln(R^2+1) = +\infty
\end{aligned}$$

Both halves diverge, so the whole integral diverges.

Once again: after we found that one of the limits diverged, we could have stopped and concluded that the original integrand diverges. Don't make the mistake of thinking that $\infty - \infty = 0$. That can get you into *big* trouble. ∞ is not a normal number. For example $2\infty = \infty$. So if ∞ were a normal number we would have both $\infty - \infty = 0$ and $\infty - \infty = 2\infty - \infty = \infty$.

S-15: We don't want to antidifferentiate this integrand, so let's use a comparison. Note the integrand is positive when $x > 0$.

For any x, $|\sin x| \leqslant 1$, so $\dfrac{|\sin x|}{x^{3/2}+x^{1/2}} \leqslant \dfrac{1}{x^{3/2}+x^{1/2}}$.

Since $x = 0$ and $x \to \infty$ both cause the integral to be improper, we need to break it into two pieces. Since both terms in the denominator give positive numbers when x is positive, $\dfrac{1}{x^{3/2}+x^{1/2}} \leqslant \dfrac{1}{x^{3/2}}$ and $\dfrac{1}{x^{3/2}+x^{1/2}} \leqslant \dfrac{1}{x^{1/2}}$. That gives us two options for comparison.

When x is positive and close to zero, $x^{1/2} \geqslant x^{3/2}$, so we guess that we should compare our integrand to $\frac{1}{x^{1/2}}$ near the limit $x = 0$. In contrast, when x is very large, $x^{1/2} \leqslant x^{3/2}$, so we guess that we should compare our integrand to $\frac{1}{x^{3/2}}$ as x goes to infinity.

$$\begin{aligned}
&\frac{|\sin x|}{x^{3/2}+x^{1/2}} \leqslant \frac{1}{x^{1/2}} \quad \text{and the integral } \int_0^1 \frac{\mathrm{d}x}{x^{1/2}} \text{ converges by the } p\text{-test, Example 3.10.9 in the text} \\
&\frac{|\sin x|}{x^{3/2}+x^{1/2}} \leqslant \frac{1}{x^{3/2}} \quad \text{and the integral } \int_1^\infty \frac{\mathrm{d}x}{x^{3/2}} \text{ converges by the } p\text{-test, Example 3.10.8 in the text}
\end{aligned}$$

Now we have all the data we need to apply the comparison test, Theorem 3.10.17 in the text.

- $\dfrac{|\sin x|}{x^{3/2}+x^{1/2}}$, $\dfrac{1}{x^{1/2}}$, and $\dfrac{1}{x^{3/2}}$ are defined and continuous for $x > 0$
- $\dfrac{1}{x^{1/2}}$ and $\dfrac{1}{x^{3/2}}$ are nonnegative for $x \geqslant 0$
- $\dfrac{|\sin x|}{x^{3/2}+x^{1/2}} \leqslant \dfrac{1}{x^{1/2}}$ for all $x > 0$ and $\displaystyle\int_0^1 \frac{1}{x^{1/2}}\,dx$ converges, so (using Remark 3.10.16 in the text) $\displaystyle\int_0^1 \frac{|\sin x|}{x^{3/2}+x^{1/2}}\,dx$ converges.
- $\dfrac{|\sin x|}{x^{3/2}+x^{1/2}} \leqslant \dfrac{1}{x^{3/2}}$ for all $x \geqslant 1$ and $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges, so $\displaystyle\int_1^\infty \frac{|\sin x|}{x^{3/2}+x^{1/2}}\,dx$ converges.

Therefore, our integral $\displaystyle\int_0^\infty \frac{|\sin x|}{x^{3/2}+x^{1/2}}\,dx$ converges.

S-16: Our goal is to decide when this integral diverges, and where it converges. We will leave q as a variable, and antidifferentiate. In order to antidifferentiate without knowing q, we'll need different cases. The integrand is x^{-5q}, so when $-5q \neq -1$, we use the power rule (that is, $\int x^n\,dx = \frac{x^{n+1}}{n+1}$) to antidifferentiate. Note $x^{(-5q)+1} = x^{1-5q} = \dfrac{1}{x^{5q-1}}$.

$$\int_1^t \frac{1}{x^{5q}}\,dx = \begin{cases} \left[\frac{x^{1-5q}}{1-5q}\right]_1^t \text{ with } 1-5q>0 & \text{if } q<\frac{1}{5} \\ \Big[\ln x\Big]_1^t & \text{if } q=\frac{1}{5} \\ \left[\frac{1}{(1-5q)x^{5q-1}}\right]_1^t \text{ with } 5q-1>0 & \text{if } q>\frac{1}{5} \end{cases}$$
$$= \begin{cases} \frac{1}{1-5q}(t^{1-5q}-1) \text{ with } 1-5q>0 & \text{if } q<\frac{1}{5} \\ \ln t & \text{if } q=\frac{1}{5} \\ \frac{1}{5q-1}\left(1-\frac{1}{t^{5q-1}}\right) \text{ with } 5q-1>0 & \text{if } q>\frac{1}{5}. \end{cases}$$

Therefore,

$$\int_1^\infty \frac{1}{x^{5q}}\,dx = \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^{5q}}\,dx\right) = \begin{cases} \frac{1}{1-5q}\left(\lim\limits_{t\to\infty} t^{1-5q}-1\right) = \infty & \text{if } q<\frac{1}{5} \\ \lim\limits_{t\to\infty} \ln t = \infty & \text{if } q=\frac{1}{5} \\ \frac{1}{5q-1}\left(1-\lim\limits_{t\to\infty}\frac{1}{t^{5q-1}}\right) = \frac{1}{5q-1} & \text{if } q>\frac{1}{5}. \end{cases}$$

The first two cases are divergent, and so the largest such value is $q = \frac{1}{5}$. (Alternatively, we might recognize this as a "p-integral" with $p = 5q$, and recall that the p-integral diverges precisely when $p \leqslant 1$.)

S-17: This integrand is a nice candidate for the substitution $u = x^2+1$, $\frac{1}{2}du = x\,dx$. Remember when we use substitution on a definite integral, we also need to adjust the limits of integration.

$$\begin{aligned}
\int_0^\infty \frac{x}{(x^2+1)^p}\,\mathrm{d}x &= \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+1)^p}\mathrm{d}x\\
&= \lim_{t\to\infty}\frac{1}{2}\int_1^{t^2+1}\frac{1}{u^p}\,\mathrm{d}u\\
&= \lim_{t\to\infty}\frac{1}{2}\int_1^{t^2+1} u^{-p}\,\mathrm{d}u\\
&= \begin{cases} \frac{1}{2}\lim\limits_{t\to\infty}\left[\dfrac{u^{1-p}}{1-p}\right]_1^{t^2+1} & \text{if } p\neq 1\\ \frac{1}{2}\lim\limits_{t\to\infty}\Big[\ln|u|\Big]_1^{t^2+1} & \text{if } p=1\end{cases}\\
&= \begin{cases} \frac{1}{2}\lim\limits_{t\to\infty}\dfrac{1}{1-p}\left[(t^2+1)^{1-p}-1\right] & \text{if } p\neq 1\\ \frac{1}{2}\lim\limits_{t\to\infty}\left[\ln(t^2+1)\right]=\infty & \text{if } p=1\end{cases}
\end{aligned}$$

At this point, we can see that the integral diverges when $p=1$. When $p\neq 1$, we have the limit

$$\lim_{t\to\infty}\frac{1/2}{1-p}\left[(t^2+1)^{1-p}-1\right] = \frac{1/2}{1-p}\left[\lim_{t\to\infty}(t^2+1)^{1-p}\right]-\frac{1/2}{1-p}$$

Since $t^2+1\to\infty$, this limit converges exactly when the exponent $1-p$ is negative; that is, it converges when $p>1$, and diverges when $p<1$.

So, the integral in the question converges when $p>1$.

S-18: There are three singularities in the integrand: $x=0$, $x=1$, and $x=2$. We'll need to break up the integral at each of these places.

$$\begin{aligned}
&\int_{-5}^{5}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x\\
&=\int_{-5}^{0}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x+\int_{0}^{1}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x\\
&+\int_{1}^{2}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x+\int_{2}^{5}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x
\end{aligned}$$

This looks rather unfortunate. Let's think again. If all of the integrals below converge, then we can write:

$$\int_{-5}^{5}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x=\int_{-5}^{5}\frac{1}{\sqrt{|x|}}\mathrm{d}x+\int_{-5}^{5}\frac{1}{\sqrt{|x-1|}}\mathrm{d}x+\int_{-5}^{5}\frac{1}{\sqrt{|x-2|}}\mathrm{d}x$$

That looks a lot better. Also, we have a good reason to guess these integrals converge–they look like p-integrals with $p = \frac{1}{2}$. Let's take a closer look at each one.

$$\begin{aligned}\int_{-5}^{5} \frac{1}{\sqrt{|x|}}\mathrm{d}x &= \int_{-5}^{0} \frac{1}{\sqrt{|x|}}\mathrm{d}x + \int_{0}^{5} \frac{1}{\sqrt{|x|}}\mathrm{d}x \\ &= 2\int_{0}^{5} \frac{1}{\sqrt{|x|}}\mathrm{d}x \qquad \text{(even function)} \\ &= 2\int_{0}^{5} \frac{1}{\sqrt{x}}\mathrm{d}x\end{aligned}$$

This is a p-integral, with $p = \frac{1}{2}$. By Example 3.10.9 in the text (and Theorem 3.10.20, since the upper limit of integration is not 1), it converges. The other two pieces behave similarly.

$$\int_{-5}^{5} \frac{1}{\sqrt{|x-1|}}\mathrm{d}x = \int_{-5}^{1} \frac{1}{\sqrt{|x-1|}}\mathrm{d}x + \int_{1}^{5} \frac{1}{\sqrt{|x-1|}}\mathrm{d}x$$

Use $u = x - 1$, $\mathrm{d}u = \mathrm{d}x$

$$\begin{aligned}&= \int_{-6}^{0} \frac{1}{\sqrt{|u|}}\mathrm{d}u + \int_{0}^{4} \frac{1}{\sqrt{|u|}}\mathrm{d}x \\ &= \int_{0}^{6} \frac{1}{\sqrt{u}}\mathrm{d}u + \int_{0}^{4} \frac{1}{\sqrt{u}}\mathrm{d}x\end{aligned}$$

Since our function is even, we use the reasoning of Example 3.2.9 in the text to consider the area under the curve when $x \geqslant 0$, rather than when $x \leqslant 0$. Again, these are p-integrals with $p = \frac{1}{2}$, so they both converge. Finally:

$$\int_{-5}^{5} \frac{1}{\sqrt{|x-2|}}\mathrm{d}x = \int_{-5}^{2} \frac{1}{\sqrt{|x-2|}}\mathrm{d}x + \int_{2}^{5} \frac{1}{\sqrt{|x-2|}}\mathrm{d}x$$

Use $u = x - 2$, $\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}&= \int_{-7}^{0} \frac{1}{\sqrt{|u|}}\mathrm{d}u + \int_{0}^{3} \frac{1}{\sqrt{|u|}}\mathrm{d}u \\ &= \int_{0}^{7} \frac{1}{\sqrt{u}}\mathrm{d}u + \int_{0}^{3} \frac{1}{\sqrt{u}}\mathrm{d}u\end{aligned}$$

Since $p = \frac{1}{2}$, so they both converge.

We conclude our original integral, as the sum of convergent integrals, converges.

S-19: We can use integration by parts twice to find the antiderivative of $e^{-x}\sin x$, as in Example 3.5.10 of the text. To keep our work a little simpler, we'll find the antiderivative first, then take the limit.

Let $u = e^{-x}$, $dv = \sin x \, dx$, so $du = -e^{-x} \, dx$ and $v = -\cos x$.

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

Now let $u = e^{-x}$, $dv = \cos x \, dx$, so $du = -e^{-x} \, dx$ and $v = \sin x$.

$$\begin{aligned} &= -e^{-x} \cos x - \left[e^{-x} \sin x + \int e^{-x} \sin x \, dx \right] \\ &= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \, dx \end{aligned}$$

All together, we found

$$\begin{aligned} \int e^{-x} \sin x \, dx &= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \, dx + C \\ 2 \int e^{-x} \sin x \, dx &= -e^{-x} \cos x - e^{-x} \sin x + C \\ \int e^{-x} \sin x \, dx &= -\frac{1}{2e^x} (\cos x + \sin x) + C \end{aligned}$$

(Remember, since C is an arbitrary constant, we can rename $\frac{C}{2}$ to simply C.) Now we can evaluate our improper integral.

$$\begin{aligned} \int_0^\infty e^{-x} \sin x \, dx &= \lim_{b \to \infty} \int_0^b e^{-x} \sin x \, dx \\ &= \lim_{b \to \infty} \left[-\frac{1}{2e^x} (\cos x + \sin x) \right]_0^b \\ &= \lim_{b \to \infty} \left(\frac{1}{2} - \frac{1}{2e^b} (\cos b + \sin b) \right) \end{aligned}$$

To find the limit, we use the Squeeze Theorem. Since $|\sin b|, |\cos b| \leqslant 1$ for any b, we can use the fact that $-2 \leqslant \cos b + \sin x \leqslant 2$ for any b.

$$\begin{aligned} \frac{-2}{2e^b} \leqslant \frac{1}{2e^b} (\cos b + \sin b) \leqslant \frac{2}{2e^b} \\ \lim_{b \to \infty} \frac{-2}{2e^b} = 0 = \frac{2}{2e^b} \\ \text{So,} \qquad \lim_{b \to \infty} \left[\frac{1}{2e^b} (\cos b + \sin b) \right] = 0 \\ \text{Therefore,} \qquad \frac{1}{2} = \lim_{b \to \infty} \left(\frac{1}{2} - \frac{1}{2e^b} (\cos b + \sin b) \right) \end{aligned}$$

That is, $\displaystyle\int_0^\infty e^{-x} \sin x \, dx = \frac{1}{2}$.

S-20: The integrand is positive everywhere. So either the integral converges to some finite number or it is infinite. There are two potential "sources of impropriety" — a possible singularity at $x = 0$ and the fact that the domain of integration extends to ∞. So, we split up the integral.

$$\int_0^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x = \int_0^1 \frac{\sin^4 x}{x^2}\,\mathrm{d}x + \int_1^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$$

Let's consider the first integral. By l'Hôpital's rule

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

Consequently,

$$\lim_{x\to 0} \frac{\sin^4 x}{x^2} = \Big(\lim_{x\to 0} \sin^2 x\Big)\Big(\lim_{x\to 0} \frac{\sin x}{x}\Big)\Big(\lim_{x\to 0} \frac{\sin x}{x}\Big) = 0 \times 1 \times 1 = 0$$

and the first integral is not even improper.

Now for the second integral. Since $|\sin x| \leqslant 1$, we'll compare it to $\int_1^\infty \frac{1}{x^2}$.

- $\frac{\sin^4 x}{x^2}$ and $\frac{1}{x^2}$ are defined and continuous for every $x \geqslant 1$
- $0 \leqslant \frac{\sin^4 x}{x^2} \leqslant \frac{1^4}{x^2} = \frac{1}{x^2}$ for every $x \geqslant 1$
- $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges by Example 3.10.8 in the text (it's a p-type integral with $p > 1$)

By the comparison test, Theorem 3.10.17 in the text, $\displaystyle\int_1^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ converges.

Since $\displaystyle\int_0^1 \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ and $\displaystyle\int_1^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ both converge, we conclude $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ converges as well.

S-21: Since the denominator is positive for all $x \geqslant 0$, the integrand is continuous over $[0, \infty)$. So, the only "source of impropriety" is the infinite domain of integration.

Solution 1: Let's try to use a direct comparison. Note $\dfrac{x}{e^x + \sqrt{x}} \geqslant 0$ whenever $x \geqslant 0$. Also note that, for large values of x, e^x is much larger than $\sqrt{x}$. That leads us to consider the following inequalty:

$$0 \leqslant \frac{x}{e^x + \sqrt{x}} \leqslant \frac{x}{e^x}$$

If $\int_0^\infty \frac{x}{e^x}\,\mathrm{d}x$ converges, we're in business. Let's figure it out. The integrand looks like a candidate for integration by parts: take $u = x$, $\mathrm{d}v = e^{-x}\,\mathrm{d}x$, so $\mathrm{d}u = \mathrm{d}x$ and

$v = -e^{-x}$.

$$\begin{aligned}\int_0^\infty \frac{x}{e^x}\,\mathrm{d}x &= \lim_{b\to\infty}\int_0^b \frac{x}{e^x}\,\mathrm{d}x = \lim_{b\to\infty}\left(\left[-\frac{x}{e^x}\right]_0^b + \int_0^b e^{-x}\,\mathrm{d}x\right)\\ &= \lim_{b\to\infty}\left(-\frac{b}{e^b} + \left[-e^{-x}\right]_0^b\right) = \lim_{b\to\infty}\left(-\frac{b}{e^b} - \frac{1}{e^b} + 1\right)\\ &= \lim_{b\to\infty}\left(1 - \underbrace{\frac{b+1}{e^b}}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}\right) = \lim_{b\to\infty}\left(1 - \frac{1}{e^b}\right) = 1\end{aligned}$$

Using l'Hôpital's rule, we see $\int_0^\infty \frac{x}{e^x}\,\mathrm{d}x$ converges. All together:

- $\frac{x}{e^x}$ and $\frac{x}{e^x+\sqrt{x}}$ are defined and continuous for all $x \geqslant 0$,
- $\left|\frac{x}{e^x+\sqrt{x}}\right| \leqslant \frac{x}{e^x}$, and
- $\int_0^\infty \frac{x}{e^x}\,\mathrm{d}x$ converges.

So, by Theorem 3.10.17 in the text, our integral $\displaystyle\int_0^\infty \frac{x}{e^x+\sqrt{x}}\,\mathrm{d}x$ converges.

Solution 2: Let's try to use a different direct comparison from Solution 1, and avoid integration by parts. We'd like to compare to something like $\dfrac{1}{e^x}$, but the inequality goes the wrong way. So, we make a slight modification: we consider $2e^{-x/2}$. To that end, we claim $x < 2e^{x/2}$ for all $x \geqslant 0$. We can prove this by noting the following two facts:

- $0 < 2 = 2e^{0/2}$, and
- $\frac{\mathrm{d}}{\mathrm{d}x}\{x\} = 1 \leqslant e^{x/2} = \frac{\mathrm{d}}{\mathrm{d}x}\{2e^{x/2}\}$.

So, when $x = 0$, $x < 2e^{x/2}$, and then as x increases, $2e^{x/2}$ grows faster than x.

Now we can make the following comparison:

$$0 \leqslant \frac{x}{e^x+\sqrt{x}} \leqslant \frac{x}{e^x} < \frac{2e^{x/2}}{e^x} = \frac{2}{e^{x/2}}$$

We have a hunch that $\int_0^\infty \frac{2}{e^{x/2}}\,\mathrm{d}x$ converges, just like $\int_0^\infty \frac{1}{e^x}\,\mathrm{d}x$. This is easy enough to prove. We can guess an antiderivative, or use the substitution $u = x/2$.

$$\begin{aligned}\int_0^\infty \frac{2}{e^{x/2}}\,\mathrm{d}x &= \lim_{R\to\infty}\int_0^R \frac{2}{e^{x/2}}\,\mathrm{d}x = \lim_{R\to\infty}\left[-\frac{4}{e^{x/2}}\right]_0^R\\ &= \lim_{R\to\infty}\left[\frac{4}{e^0} - \frac{4}{e^{R/2}}\right]_0^R = 4\end{aligned}$$

Now we know:

- $0 \leqslant \frac{x}{e^x+\sqrt{x}} \leqslant \frac{2}{e^{x/2}}$, and

- $\int_0^\infty \frac{2}{e^{x/2}}\,\mathrm{d}x$ converges.
- Furthermore, $\frac{x}{e^x+\sqrt{x}}$ and $\frac{2}{e^{x/2}}$ are defined and continuous for all $x \geqslant 0$.

By the comparison test (Theorem 3.10.17) in the text, we conclude the integral converges.

Solution 3: Let's use the limiting comparison test (Theorem 3.10.22 in the text). We have a hunch that our integral behaves similarly to $\int_0^\infty \frac{1}{e^x}\,\mathrm{d}x$, which converges (see Example 3.10.18 in the text). Unfortunately, if we choose $g(x) = \frac{1}{e^x}$ (and, of course, $f(x) = \frac{x}{e^x+\sqrt{x}}$), then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{e^x + \sqrt{x}} \cdot e^x = \lim_{x\to\infty} \frac{x}{1 + \underbrace{\frac{\sqrt{x}}{e^x}}_{\to 0}} = \infty$$

That is, the limit does not exist, so the limiting comparison test does not apply. (To find $\lim\limits_{x\to\infty} \frac{\sqrt{x}}{e^x}$, you can use l'Hôpital's rule.)

This setback encourages us to try a slightly different angle. If $g(x)$ gave larger values, then we could decrease $\frac{f(x)}{g(x)}$. So, let's try $g(x) = \frac{1}{e^{x/2}} = e^{-x/2}$. Now,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{e^x + \sqrt{x}} \div \frac{1}{e^{x/2}} = \lim_{x\to\infty} \frac{x}{e^{x/2} + \frac{\sqrt{x}}{e^{x/2}}}$$

Hmm... this looks hard. Instead of dealing with it directly, let's use the squeeze theorem.

$$0 \leqslant \frac{x}{e^{x/2} + \frac{\sqrt{x}}{e^{x/2}}} \leqslant \frac{x}{e^{x/2}}$$

Using l'Hôpital's rule,

$$\lim_{x\to\infty} \underbrace{\frac{x}{e^{x/2}}}_{\substack{\text{num}\to\infty \\ \text{den}\to\infty}} = \lim_{x\to\infty} \frac{1}{\frac{1}{2}e^{x/2}} = 0 = \lim_{x\to\infty} 0$$

So, by the squeeze theorem $\lim\limits_{x\to 0} \frac{\frac{x}{e^x+\sqrt{x}}}{\frac{1}{e^{x/2}}} = 0$. Since this limit exists, $\frac{1}{e^{x/2}}$ is a reasonable function to use in the limiting comparison test (provided its integral converges). So, we need to show that $\int_0^\infty \frac{1}{e^{x/2}}\,\mathrm{d}x$ converges. This can be done by simply evaluating it:

$$\int_0^\infty \frac{1}{e^{x/2}}\,\mathrm{d}x = \lim_{b\to\infty} \int_0^b e^{-x/2}\,\mathrm{d}x = \lim_{b\to\infty} -\frac{1}{2}\left[e^{-x/2}\right]_0^b = \lim_{b\to\infty} -\frac{1}{2}\left[\frac{1}{e^{b/2}} - 1\right] = \frac{1}{2}$$

So, all together:

- The functions $\frac{x}{e^x+\sqrt{x}}$ and $\frac{1}{e^{x/2}}$ are defined and continuous for all $x \geqslant 0$, and $\frac{1}{e^{x/2}} \geqslant 0$ for all $x \geqslant 0$.
- $\int_0^\infty \frac{1}{e^{x/2}}\,\mathrm{d}x$ converges.
- The limit $\lim\limits_{x\to\infty} \frac{\frac{x}{e^x}+\sqrt{x}}{\frac{1}{e^{x/2}}}$ exists (it's equal to 0).
- So, the limiting comparison test (Theorem 3.10.17 in the text) tells us that $\int_0^\infty \frac{x}{e^x+\sqrt{x}}\,\mathrm{d}x$ converges as well.

S-22: There are two sources of error: the upper bound is t, rather than infinity, and we're using an approximation with some finite number of intervals, n. Our plan is to first find a value of t that introduces an error of no more than $\frac{1}{2}10^{-4}$. That is, we'll find a value of t such that $\int_t^\infty \frac{e^{-x}}{1+x}\,\mathrm{d}x \leqslant \frac{1}{2}10^{-4}$. After that, we'll find a value of n that approximates $\int_0^t \frac{e^{-x}}{x+1}\,\mathrm{d}x$ to within $\frac{1}{2}10^{-4}$. Then, all together, our error will be at most $\frac{1}{2}10^{-4} + \frac{1}{2}10^{-4} = 10^{-4}$, as desired. (Note we could have broken up the error in another way—it didn't have to be $\frac{1}{2}10^{-4}$ and $\frac{1}{2}10^{-4}$. This will give us one of many possible answers.)

Let's find a t such that $\int_t^\infty \frac{e^{-x}}{1+x}\,\mathrm{d}x \leqslant \frac{1}{2}10^{-4}$. For all $x \geqslant 0$, $0 < \frac{e^{-x}}{1+x} \leqslant e^{-x}$, so

$$\int_t^\infty \frac{e^{-x}}{1+x}\,\mathrm{d}x \leqslant \int_t^\infty e^{-x}\,\mathrm{d}x = e^{-t} \overset{(*)}{\leqslant} \frac{1}{2}10^{-4}$$
$$\text{where } (*) \text{ is true if} \quad t \geqslant -\ln\left(\frac{1}{2}10^{-4}\right) \approx 9.90$$

Choose, for example, $t = 10$.

Now it's time to decide how many intervals we're going to use to approximate $\displaystyle\int_0^t \frac{e^{-x}}{x+1}\,\mathrm{d}x$. Again, we want our error to be less than $\frac{1}{2}10^{-4}$. To bound our error, we need to know the second derivative of $\frac{e^{-x}}{x+1}$.

$$f(x) = \frac{e^{-x}}{1+x} \implies f'(x) = -\frac{e^{-x}}{1+x} - \frac{e^{-x}}{(1+x)^2} \implies f''(x) = \frac{e^{-x}}{1+x} + 2\frac{e^{-x}}{(1+x)^2} + 2\frac{e^{-x}}{(1+x)^3}$$

Since $f''(x)$ is positive, and decreases as x increases,

$$|f''(x)| \leqslant f''(0) = 5 \implies |E_n| \leqslant \frac{5(10-0)^3}{24n^2} = \frac{5000}{24n^2} = \frac{625}{3n^2}$$

and $|E_n| \leqslant \frac{1}{2}10^{-4}$ if

$$\begin{aligned}
&& \frac{625}{3n^2} &\leqslant \frac{1}{2}10^{-4} \\
\iff && n^2 &\geqslant \frac{1250 \times 10^4}{3} \\
\iff && n &\geqslant \sqrt{\frac{1.25 \times 10^7}{3}} \approx 2041.2
\end{aligned}$$

So $t = 10$ and $n = 2042$ will do the job. There are many other correct answers.

S-23:

(a) Since $f(x)$ is odd, using the reasoning of Example 3.2.10 in the text,

$$\int_{-\infty}^{-1} f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_{-t}^{-1} f(x)\,\mathrm{d}x = \lim_{t\to\infty} -\int_1^t f(x)\,\mathrm{d}x = -\lim_{t\to\infty} \int_1^t f(x)\,\mathrm{d}x$$

Since $\int_1^\infty f(x)\,\mathrm{d}x$ converges, the last limit above converges. Therefore, $\int_{-\infty}^{-1} f(x)\,\mathrm{d}x$ converges.

(b) Since $f(x)$ is even, using the reasoning of Example 3.2.9 in the text,

$$\int_{-\infty}^{-1} f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_{-t}^{-1} f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t f(x)\,\mathrm{d}x$$

Since $\int_1^\infty f(x)\,\mathrm{d}x$ converges, the last limit above converges. Since $f(x)$ is continuous everywhere, by Theorem 3.10.20 in the text, $\int_{-1}^\infty f(x)\,\mathrm{d}x$ converges (note the adjusted lower limit). Then, since

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{-1} f(x)\,\mathrm{d}x + \int_{-1}^{\infty} f(x)\,\mathrm{d}x$$

and both summands converge, our original integral converges as well.

S-24: Define $F(x) = \int_0^x \frac{1}{e^t}\mathrm{d}t$.

$$F(x) = \int_0^x \frac{1}{e^t}\mathrm{d}t = \left[-\frac{1}{e^t}\right]_0^x = \frac{1}{e^0} - \frac{1}{e^x} < \frac{1}{e^0} = 1$$

So, the statement is false: there is no x such that $F(x) = 1$. For every real x, $F(x) < \frac{1}{e^0} = 1$.

We note here that $\lim_{x\to\infty} \int_0^x \frac{1}{e^t}\mathrm{d}t = 1$. So, as x grows larger, the gap between $F(x)$ and 1 grows infintesimally small. But there is no real value of x where $F(x)$ is exactly equal to 1.

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S-1: (A) Note $\int f'(x)f(x)\,\mathrm{d}x = \int u\,\mathrm{d}u$ if we substitute $u = f(x)$. This is the kind of integrand described in (I). It's quite possible that a u-substitution would work on the others, as well, but (I) is the most reliable kind of integrand for a u-substitution.
(B) A trigonometric substitution usually allows us to cancel out a square root containing a quadratic function, as in (IV).

(C) We can often antidifferentiate the product of a polynomial with an exponential function using integration by parts: see Examples 3.5.1, 3.5.6 in the text. If we let u be the polynomial function and $\mathrm{d}v$ be the exponential, as long as we can antidifferentiate $\mathrm{d}v$, we can repeatedly apply integration by parts until the polynomial function goes away. So, we go with (II)
(D) We apply partial fractions to rational functions, (III).

Note: without knowing more about the functions, there's no guarantee that the methods we chose will be the best methods, or even that they will work (with the exception of (I)). With practice, you gain intuition about likely methods for different integrals. Luckily for you, there's lots of practice below.

S-2:

The integrand is a product of powers of sine and cosine. Since cosine has an odd power, we want to substitute $u = \sin x$, $\mathrm{d}u = \cos x\mathrm{d}x$. Therefore, we should:

- reserve one cosine for the derivative of sine in our substitution, and
- change the rest of the cosines to sines using the identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^4 x \cos^5 x\mathrm{d}x &= \int_0^{\pi/2} \sin^4 x(\cos^2 x)^2 \cos x\mathrm{d}x \\
&= \int_0^{\pi/2} \sin^4 x(1-\sin^2 x)^2 \underbrace{\cos x\mathrm{d}x}_{du} \\
&= \int_{\sin(0)}^{\sin(\pi/2)} u^4(1-u^2)^2 du \\
&= \int_0^1 u^4(1-2u^2+u^4)du \\
&= \int_0^1 (u^4-2u^6+u^8)du \\
&= \left[\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9\right]_{u=0}^{u=1} \\
&= \left(\frac{1}{5}-\frac{2}{7}+\frac{1}{9}\right) - 0 \\
&= \frac{8}{315}
\end{aligned}$$

S-3:

We notice that there is a quadratic equation under the square root. If that equation were a perfect square, we could get rid of the square root: so we'll mould it into a perfect square using a trig substitution.

Our candidates will use one of the following identities:

$$1 - \sin^2\theta = \cos^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

We'll be substituting $x =$(something), so we notice that $3 - 5x^2$ has the general form of (constant)−(function), as does $1 - \sin^2\theta$. In order to get the constant right, we multiply through by three:

$$3 - 3\sin^2\theta = 3\cos^2\theta$$

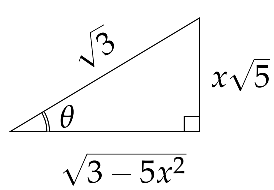
Our goal is to get $3 - 5x^2 = 3 - 3\sin^2\theta$; so we solve this equation for x and decide on the substitution

$$x = \sqrt{\frac{3}{5}}\sin\theta, \qquad \mathrm{d}x = \sqrt{\frac{3}{5}}\cos\theta \mathrm{d}\theta$$

Now we evaluate our integral.

$$\begin{aligned}
\int \sqrt{3-5x^2}\mathrm{d}x &= \int \sqrt{3 - 5\left(\sqrt{\frac{3}{5}}\sin\theta\right)^2}\sqrt{\frac{3}{5}}\cos\theta d\theta \\
&= \int \sqrt{3 - 3\sin^2\theta}\sqrt{3/5}\cos\theta\mathrm{d}\theta \\
&= \int \sqrt{3\cos^2\theta}\sqrt{3/5}\cos\theta\mathrm{d}\theta \\
&= \int \sqrt{3}\cos\theta\sqrt{3/5}\cos\theta\mathrm{d}\theta \\
&= \frac{3}{\sqrt{5}}\int \cos^2\theta\mathrm{d}\theta \\
&= \frac{3}{\sqrt{5}}\int \frac{1+\cos 2\theta}{2}\mathrm{d}\theta \\
&= \frac{3}{2\sqrt{5}}\int (1+\cos 2\theta)\mathrm{d}\theta \\
&= \frac{3}{2\sqrt{5}}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{3}{2\sqrt{5}}\left[\theta + \sin\theta\cos\theta\right] + C
\end{aligned}$$

From our substitution $x = \sqrt{3/5}\sin\theta$, we glean $\sin\theta = x\sqrt{5/3}$, and $\theta = \arcsin\left(x\sqrt{5/3}\right)$. To figure out $\cos\theta$, we draw a right triangle. Let θ be one angle, and since $\sin\theta = \dfrac{x\sqrt{5}}{\sqrt{3}}$, we let the hypotenuse be $\sqrt{3}$ and the side opposite θ be $x\sqrt{5}$. By Pythagorus, the missing side (adjacent to θ) has length $\sqrt{3-5x^2}$.



Therefore, $\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{3-5x^2}}{\sqrt{3}}$. So our integral evaluates to:

$$\begin{aligned}
\frac{3}{2\sqrt{5}}\left[\theta + \sin\theta\cos\theta\right] + C &= \frac{3}{2\sqrt{5}}\left[\arcsin(x\sqrt{5/3}) + x\sqrt{5/3}\cdot\frac{\sqrt{3-5x^2}}{\sqrt{3}}\right] + C \\
&= \frac{3}{2\sqrt{5}}\arcsin(x\sqrt{5/3}) + \frac{x}{2}\cdot\sqrt{3-5x^2} + C
\end{aligned}$$

S-4: First, we note the integral is improper. So, we'll need to replace the top bound with a variable, and take a limit. Second, we're going to have to antidifferentiate. The integrand is the product of an exponential function, e^{-x}, with a polynomial function, $x-1$, so we use integration by parts with $u = x-1$, $\mathrm{d}v = e^{-x}\mathrm{d}u$, $\mathrm{d}u = \mathrm{d}x$, and $v = -e^{-x}$.

$$\begin{aligned}
\int \frac{x-1}{e^x}\,\mathrm{d}x &= -(x-1)e^{-x} + \int e^{-x}\,\mathrm{d}x \\
&= -(x-1)e^{-x} - e^{-x} + C = -xe^{-x} + C \\
\text{So,}\qquad \int_0^\infty \frac{x-1}{e^x}\,\mathrm{d}x &= \lim_{b\to\infty}\int_0^b \frac{x-1}{e^x}\,\mathrm{d}x \\
&= \lim_{b\to\infty}\left[-\frac{x}{e^x}\right]_0^b = \lim_{b\to\infty}\left[-\underbrace{\frac{b}{e^b}}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}\right] \\
&\stackrel{(*)}{=} \lim_{b\to\infty}\frac{1}{e^b} = 0
\end{aligned}$$

(In the equality marked $(*)$, we used l'Hôpital's rule.)

So, $\displaystyle\int_0^\infty \frac{x-1}{e^x}\,\mathrm{d}x = 0$.

Remark: this shows that, interestingly, $\displaystyle\int_0^\infty \frac{x}{e^x}\,\mathrm{d}x = \int_0^\infty \frac{1}{e^x}\,\mathrm{d}x$.

S-5:

Solution 1: Notice the denominator factors as $(x+1)(3x+1)$. Since the integrand is a rational function (the quotient of two polynomials), we can use partial fraction decomposition.

$$\begin{aligned}
\frac{-2}{3x^2+4x+1} &= \frac{-2}{(x+1)(3x+1)} \\
&= \frac{A}{x+1} + \frac{B}{3x+1} \\
&= \frac{A(3x+1)+B(x+1)}{(x+1)(3x+1)} \\
&= \frac{(3A+B)x+(A+B)}{(x+1)(3x+1)}
\end{aligned}$$

So:

$$\begin{aligned}
-2 &= (3A+B)x+(A+B) \\
0 &= 3A+B \text{ and } -2 = A+B \\
B &= -3A \text{ and hence } -2 = A+(-3A) \\
A &= 1 \text{ so then } B = -3
\end{aligned}$$

So now:

$$\begin{aligned}
\frac{-2}{3x^2+4x+1} &= \frac{1}{x+1} - \frac{3}{3x+1} \\
\int \frac{-2}{3x^2+4x+1}\mathrm{d}x &= \int \left(\frac{1}{x+1} - \frac{3}{3x+1}\right)\mathrm{d}x \\
&= \ln|x+1| - \ln|3x+1| + C \\
&= \ln\left|\frac{x+1}{3x+1}\right| + C
\end{aligned}$$

Solution 2: The previous solution is probably the nicest. However, for the foolhardy or the brave, this integral can also be evaluated using trigonometric substitution.

We start by completing the square on the denominator.

$$\begin{aligned}
3x^2+4x+1 &= 3\left(x^2+\frac{4}{3}x+\frac{1}{3}\right) \\
&= 3\left(x^2+2\cdot\frac{2}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{1}{3}\right) \\
&= 3\left(\left(x+\frac{2}{3}\right)^2-\frac{4}{9}+\frac{3}{9}\right) \\
&= 3\left(\left(x+\frac{2}{3}\right)^2-\frac{1}{9}\right) \\
&= 3\left(x+\frac{2}{3}\right)^2-\frac{1}{3}
\end{aligned}$$

This has the form of a function minus a constant, which matches the trigonometric identity $\sec^2\theta - 1 = \tan^2\theta$. Multiplying through by $\frac{1}{3}$, we see we can use the identity $\frac{1}{3}\sec^2\theta - \frac{1}{3} = \frac{1}{3}\tan^2\theta$. So, to get the substitution right, we want to choose a substitution that makes the following true:

$$\begin{aligned}
3\left(x+\frac{2}{3}\right)^2 - \frac{1}{3} &= \frac{1}{3}\sec^2\theta - \frac{1}{3}\\
3\left(x+\frac{2}{3}\right)^2 &= \frac{1}{3}\sec^2\theta\\
9\left(x+\frac{2}{3}\right)^2 &= \sec^2\theta\\
{\color{magenta}3x+2} &= {\color{blue}\sec\theta}
\end{aligned}$$

And, accordingly:

$$ {\color{magenta}3\mathrm{d}x} = {\color{blue}\sec\theta\tan\theta\mathrm{d}\theta}$$

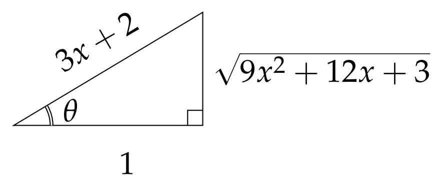
Now, let's simplify a little and use this substitution on our integral:

$$\begin{aligned}
\int \frac{-2}{3x^2+4x+1}\mathrm{d}x &= \int \frac{-2}{3\left(x+\frac{2}{3}\right)^2 - \frac{1}{3}}\mathrm{d}x\\
&= \int \frac{-2}{9\left(x+\frac{2}{3}\right)^2 - 1}3\mathrm{d}x\\
&= \int \frac{-2}{{\color{magenta}(3x+2)}^2 - 1}{\color{magenta}3\mathrm{d}x}\\
&= \int \frac{-2}{{\color{blue}(\sec\theta)}^2 - 1}{\color{blue}\sec\theta\tan\theta\mathrm{d}\theta}\\
&= \int \frac{-2}{\tan^2\theta}\sec\theta\tan\theta\mathrm{d}\theta\\
&= \int -2\frac{\sec\theta}{\tan\theta}\mathrm{d}\theta\\
&= \int -2\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\mathrm{d}\theta\\
&= \int -2\frac{1}{\sin\theta}\mathrm{d}\theta\\
&= \int -2\csc\theta\mathrm{d}\theta
\end{aligned}$$

Using the result of Example A.8.2 in the text, or a table of integrals:

$$= 2\ln|\csc\theta + \cot\theta| + C$$

Our final task is to translate this back from θ to x. Recall we used the substitution $3x+2 = \sec\theta$. Using this information, and $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$, we can fill in two sides of a right triangle with angle θ. The Pythagorean theorem tells us the third side (opposite to θ) has measure $\sqrt{(3x+2)^2-1} = \sqrt{9x^2-12x+3}$.



$$
\begin{aligned}
2\ln|\csc\theta+\cot\theta|+C &= 2\ln\left|\frac{3x+2}{\sqrt{9x^2+12x+3}}+\frac{1}{\sqrt{9x^2+12x+3}}\right|+C\\
&= 2\ln\left|\frac{3x+3}{\sqrt{9x^2+12x+3}}\right|+C\\
&= \ln\left|\frac{(3x+3)^2}{\sqrt{9x^2+12x+3}^2}\right|+C\\
&= \ln\left|\frac{(3x+3)^2}{9x^2+12x+3}\right|+C\\
&= \ln\left|\frac{9(x+1)^2}{3(3x+1)(x+1)}\right|+C\\
&= \ln\left|\frac{3(x+1)^2}{(3x+1)(x+1)}\right|+C\\
&= \ln\left|\frac{3(x+1)}{3x+1}\right|+C\\
&= \ln\left|\frac{x+1}{3x+1}\right|+\ln 3+C
\end{aligned}
$$

Since C is an arbitrary constant, we can write our final answer as

$$\ln\left|\frac{x+1}{3x+1}\right|+C$$

S-6:

We see that we have two functions multiplied, but they don't simplify nicely with each other. However, if we differentiate logarithm, and integrate x^2, we'll get a polynomial. So, let's use integration by parts.

$$u=\ln x \qquad \mathrm{d}v=x^2\mathrm{d}x$$

$$\mathrm{d}u=(1/x)\mathrm{d}x \qquad v=x^3/3$$

First, let's antidifferentiate. We'll deal with the limits of integration later.

$$\begin{aligned}\int x^2 \ln x \mathrm{d}x &= \underbrace{(\ln x)}_{u}\underbrace{(x^3/3)}_{v}) - \int \underbrace{(x^3/3)}_{v}\underbrace{(1/x)\mathrm{d}x}_{\mathrm{d}u} \\ &= \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2 \mathrm{d}x \\ &= \frac{1}{3}x^3\ln x - \frac{1}{3}\cdot\frac{1}{3}x^3 + C \\ &= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C\end{aligned}$$

We use the Fundamental Theorem of Calculus Part 2 to evaluate the definite integral.

$$\begin{aligned}\int_1^2 x^3 \ln x \mathrm{d}x &= \left[\frac{1}{3}x^3\ln x - \frac{1}{9}x^3\right]_1^2 \\ &= \left[\frac{1}{3}2^3\ln 2 - \frac{1}{9}2^3\right] - \left[\frac{1}{3}1^3\ln 1 - \frac{1}{9}1^3\right] \\ &= \frac{8\ln 2}{3} - \frac{8}{9} - 0 + \frac{1}{9} \\ &= \frac{8}{3}\ln 2 - \frac{7}{9}\end{aligned}$$

S-7: The derivative of the denominator shows up in the numerator, only differing by a constant, so we perform a substitution. Specifically, substitute $u = x^2 - 3$, $\mathrm{d}u = 2x\,\mathrm{d}x$. This gives

$$\int \frac{x}{x^2-3}\mathrm{d}x = \int \frac{\mathrm{d}u/2}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2-3\right| + C$$

S-8: (a) Although a quadratic under a square root often suggests trigonometric substitution, in this case we have an easier substitution. Specifically, let $y = 9 + x^2$. Then $\mathrm{d}y = 2x\mathrm{d}x,\ x\mathrm{d}x = \frac{\mathrm{d}y}{2},\ y(0) = 9$, and $y(4) = 25$.

$$\int_0^4 \frac{x}{\sqrt{9+x^2}}\,\mathrm{d}x = \int_9^{25} \frac{1}{\sqrt{y}}\frac{\mathrm{d}y}{2} = \frac{1}{2}\cdot\left.\frac{\sqrt{y}}{1/2}\right|_9^{25} = 5 - 3 = 2$$

(b) The power of cosine is odd, so we can reserve one cosine for the differential and change the rest to sines. Substituting $y = \sin x$, $\mathrm{d}y = \cos x,\ \mathrm{d}x$, $y(0) = 0$, $y(\pi/2) = 1$, $\cos^2 x = 1 - y^2$:

$$\begin{aligned}\int_0^{\pi/2}\cos^3 x\ \sin^2 x\,\mathrm{d}x = \int_0^{\pi/2}\cos^2 x\ \sin^2 x\ \cos x\,\mathrm{d}x &= \int_0^1 (1-y^2)y^2\,\mathrm{d}y = \int_0^1 (y^2-y^4)\,\mathrm{d}y \\ &= \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{5} \\ &= \frac{2}{15}\end{aligned}$$

(c) The integrand is the product of two different kinds of functions, with no obvious substitution or simplification. If we differentiate $\ln x$, it will match better with the polynomial nature of the rest of the integrand. So, integrate by parts with $u(x) = \ln x$ and $\mathrm{d}v = x^3\,\mathrm{d}x$, then $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and $v = x^4/4$.

$$\begin{aligned}\int_1^e x^3 \ln x\,\mathrm{d}x &= \frac{x^4}{4}\ln x\bigg|_1^e - \int_1^e \frac{x^4}{4}\cdot\frac{1}{x}\,\mathrm{d}x = \frac{e^4}{4} - \int_1^e \frac{x^3}{4}\,\mathrm{d}x = \frac{e^4}{4} - \frac{x^4}{16}\bigg|_1^e \\ &= \frac{3e^4}{16} + \frac{1}{16}\end{aligned}$$

S-9: (a) Integrate by parts with $u = x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$ so that $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos x$.

$$\int x\sin x\,\mathrm{d}x = -x\cos x - \int(-\cos x)\,\mathrm{d}x = -x\cos x + \sin x + C$$

$$\text{So,}\quad \int_0^{\pi/2} x\sin x\,\mathrm{d}x = \Big[-x\cos x + \sin x\Big]_0^{\pi/2} = 1$$

(b) The power of cosine is odd, so we can reserve one cosine for $\mathrm{d}u$ and change the rest into sines. Make the substitution $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$.

$$\begin{aligned}\int_0^{\pi/2}\cos^5 x\,\mathrm{d}x &= \int_0^{\pi/2}\left(1-\sin^2 x\right)^2\cos x\,\mathrm{d}x = \int_0^1\left(1-u^2\right)^2\mathrm{d}u = \int_0^1\left(1-2u^2+u^4\right)\mathrm{d}u \\ &= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}\end{aligned}$$

S-10: (a) This is a classic integration-by-parts example. If we integrate e^x, it doesn't change, and if we differentiate x it becomes a constant. So, let $u = x$ and $\mathrm{d}v = e^x\,\mathrm{d}x$, so that $\mathrm{d}u = \mathrm{d}x$ and $v = e^x$.

$$\int_0^2 xe^x\,\mathrm{d}x = \Big[xe^x\Big]_0^2 - \int_0^2 e^x\,\mathrm{d}x = 2e^2 - \Big[e^x\Big]_0^2 = e^2 + 1$$

(b) We have a quadratic function underneath a square root. In the absence of an easier substitution, we can get rid of the square root with a trigonometric substitution. Substitute $x = \tan y$, $\mathrm{d}x = \sec^2 y\,\mathrm{d}y$. When $x = 0$, $\tan y = 0$ so $y = 0$. When $x = 1$, $\tan y = 1$ so $y = \frac{\pi}{4}$. Also $\sqrt{1+x^2} = \sqrt{1+\tan^2 y} = \sqrt{\sec^2 y} = \sec y$, since $\sec y \geqslant 0$ for all $0 \leqslant y \leqslant \frac{\pi}{4}$.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x &= \int_0^{\pi/4}\frac{\sec^2 y\,\mathrm{d}y}{\sec y} = \int_0^{\pi/4}\sec y\,\mathrm{d}y = \Big[\ln|\sec y + \tan y|\Big]_0^{\pi/4} \\ &= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| \\ &= \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \ln(\sqrt{2}+1)\end{aligned}$$

$$\text{So,}\quad \int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \ln(\sqrt{2}+1)$$

(c) The integral is a rational function. In the absence of an obvious substitution, we use partial fractions.

$$\frac{4x}{(x^2-1)(x^2+1)} = \frac{4x}{(x-1)(x+1)(x^2+1)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}$$

Multiplying by the denominator,

$$4x = a(x+1)(x^2+1) + b(x-1)(x^2+1) + (cx+d)(x-1)(x+1) \qquad (*)$$

Setting $x = 1$ gives $4a = 4$, so $a = 1$. Setting $x = -1$ gives $-4b = -4$, so $b = 1$. Substituting in $a = b = 1$ in $(*)$ gives:

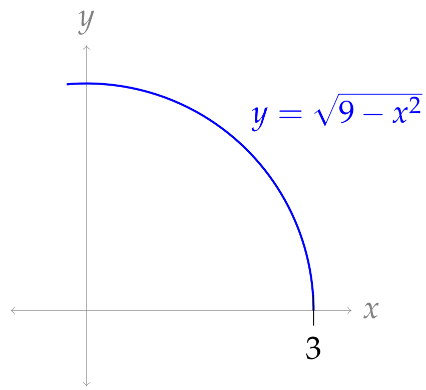
$$\begin{aligned}
4x &= (x+1)(x^2+1) + (x-1)(x^2+1) + (cx+d)(x-1)(x+1) \\
4x &= 2x(x^2+1) + (cx+d)(x-1)(x+1) \\
4x - 2x(x^2+1) &= (cx+d)(x-1)(x+1) \\
-2x(x^2-1) &= (cx+d)(x^2-1) \\
-2x &= cx+d \\
c &= -2,\ d = 0
\end{aligned}$$

So,

$$\begin{aligned}
\int_3^5 \frac{4x}{(x^2-1)(x^2+1)}\,\mathrm{d}x &= \int_3^5 \left(\frac{1}{x-1} + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)\mathrm{d}x \\
&= \Big[\ln|x-1| + \ln|x+1| - \ln(x^2+1)\Big]_3^5 \\
&= \ln 4 + \ln 6 - \ln 26 - \ln 2 - \ln 4 + \ln 10 \\
&= \ln\frac{6\times 10}{26\times 2} = \ln\frac{15}{13} \approx 0.1431
\end{aligned}$$

S-11: (a) $\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$ is the area of the portion of the disk $x^2+y^2 \leqslant 9$ that lies in the first quadrant. It is $\frac{1}{4}\pi 3^3 = \frac{9}{4}\pi$. Alternatively, you could also evaluate this integral using the substitution $x = 3\sin y$, $\mathrm{d}x = 3\cos y\,\mathrm{d}y$.

$$\begin{aligned}
\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x &= \int_0^{\pi/2} \sqrt{9-9\sin^2 y}\,(3\cos y)\,\mathrm{d}y = 9\int_0^{\pi/2}\cos^2 y\,\mathrm{d}y \\
&= \frac{9}{2}\int_0^{\pi/2}[1+\cos(2y)]\,\mathrm{d}y = \frac{9}{2}\Big[y + \frac{\sin(2y)}{2}\Big]_0^{\pi/2} \\
&= \frac{9}{4}\pi
\end{aligned}$$



(b) It's not immediately obvious what to do with this one, but remember we found $\int \ln x \mathrm{d}x$ using integration by parts with $u = \ln x$ and $\mathrm{d}v = \mathrm{d}x$. Let's hope a similar trick works here. Integrate by parts, using $u = \ln(1 + x^2)$ and $\mathrm{d}v = \mathrm{d}x$, so that $\mathrm{d}u = \frac{2x}{1+x^2}\,\mathrm{d}x$, $v = x$.

$$\begin{aligned}\int_0^1 \ln(1+x^2)\,\mathrm{d}x &= \Big[x\ln(1+x^2)\Big]_0^1 - \int_0^1 x\frac{2x}{1+x^2}\,\mathrm{d}x = \ln 2 - 2\int_0^1 \frac{x^2}{1+x^2}\,\mathrm{d}x \\ &= \ln 2 - 2\int_0^1 \left(1 - \frac{1}{1+x^2}\right)\mathrm{d}x = \ln 2 - 2\big[x - \arctan x\big]_0^1 \\ &= \ln 2 - 2 + \frac{\pi}{2} \approx 0.264\end{aligned}$$

(c) The integrand is a rational function with no obvious substitution, so we use partial fractions.

$$\frac{x}{(x-1)^2(x-2)} = \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c}{x-2} = \frac{a(x-2) + b(x-1)(x-2) + c(x-1)^2}{(x-1)^2(x-2)}$$

Multiply by the denominator.

$$x = a(x-2) + b(x-1)(x-2) + c(x-1)^2$$

Setting $x = 1$ gives $a = -1$. Setting $x = 2$ gives $c = 2$. Substituting in $a = -1$ and $c = 2$ gives

$$\begin{aligned}b(x-1)(x-2) = x + (x-2) - 2(x-1)^2 = -2x^2 + 6x - 4 = -2(x-1)(x-2) \\ \implies b = -2\end{aligned}$$

Hence

$$\begin{aligned}\int_3^\infty \frac{x}{(x-1)^2(x-2)}\,\mathrm{d}x &= \lim_{M\to\infty}\int_3^M \left(-\frac{1}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{x-2}\right)\mathrm{d}x \\ &= \lim_{M\to\infty}\left[\frac{1}{x-1} - 2\ln|x-1| + 2\ln|x-2|\right]_3^M \\ &= \lim_{M\to\infty}\left[\frac{1}{x-1} + 2\ln\left|\frac{x-2}{x-1}\right|\right]_3^M \\ &= \lim_{M\to\infty}\left[\frac{1}{M-1} + 2\ln\left|\frac{M-2}{M-1}\right|\right] - \left[\frac{1}{3-1} + 2\ln\left|\frac{3-2}{3-1}\right|\right] \\ &= 2\ln 2 - \frac{1}{2} \approx 0.886\end{aligned}$$

since

$$\lim_{M\to\infty} \ln\frac{M-2}{M-1} = \lim_{M\to\infty} \ln\frac{1-2/M}{1-1/M} = \ln 1 = 0$$

$$\text{and} \qquad \ln\frac{1}{2} = -\ln 2$$

S-12: This looks quite a lot like a rational function, but with variable $\sin\theta$ instead of x. So, we use the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$.

$$\int \frac{\sin^4\theta - 5\sin^3\theta + 4\sin^2\theta + 10\sin\theta}{\sin^2\theta - 5\sin\theta + 6}\cos\theta\,\mathrm{d}\theta = \int \frac{x^4-5x^3+4x^2+10x}{x^2-5x+6}\,\mathrm{d}x$$

Since the numerator does not have smaller degree than the denominator, we need to do some long division before we can set up our partial fractions decomposition.

$$\begin{array}{r@{}l}
 & \;x^2 \qquad\qquad -2 \\ \cline{2-2}
x^2-5x+6 & \big)\;x^4-5x^3+4x^2+10x \\
 & \;-x^4+5x^3-6x^2 \\ \cline{2-2}
 & \qquad\qquad -2x^2+10x \\
 & \qquad\qquad\;\; 2x^2-10x+12 \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad\quad 12
\end{array}$$

That is,

$$\frac{x^4-5x^3+4x^2+10x}{x^2-5x+6} = x^2-2+\frac{12}{x^2-5x+6} = x^2-2+\frac{12}{(x-2)(x-3)}$$

We use partial fractions decomposition on the rightmost term.

$$\frac{12}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3}$$
$$12 = A(x-3)+B(x-2)$$

Setting $x=3$ and $x=2$ gives us

$$B = 12, \quad A = -12$$

Now we can evaluate our integral.

$$\begin{aligned}
\int \frac{\sin^4\theta - 5\sin^3\theta + 4\sin^2\theta + 10\sin\theta}{\sin^2\theta - 5\sin\theta + 6}\cos\theta\mathrm{d}\theta &= \int \frac{x^4-5x^3+4x^2+10x}{x^2-5x+6}\,\mathrm{d}x \\
&= \int \left(x^2-2+\frac{12}{(x-2)(x-3)}\right)\mathrm{d}x \\
&= \int \left(x^2-2-\frac{12}{x-2}+\frac{12}{x-3}\right)\mathrm{d}x \\
&= \frac{1}{3}x^3-2x-12\ln|x-2|+12\ln|x-3|+C \\
&= \frac{1}{3}x^3-2x+12\ln\left|\frac{x-3}{x-2}\right|+C \\
&= \frac{1}{3}\sin^3\theta-2\sin\theta+12\ln\left|\frac{\sin\theta-3}{\sin\theta-2}\right|+C
\end{aligned}$$

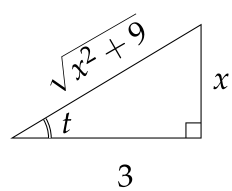
S-13: (a) It doesn't matter to us right now that the arguments of sine and cosine are $2x$ rather than x. This is still the integral of powers of products of sines and cosines. Since cosine has an odd power, we make the substitution $u = \sin(2x)$, $\mathrm{d}u = 2\cos(2x)\,\mathrm{d}x$.

$$\int_0^{\frac{\pi}{4}} \sin^2(2x)\cos^3(2x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \sin^2(2x)\left[1-\sin^2(2x)\right]\cos(2x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 u^2\left[1-u^2\right]\,\mathrm{d}u$$
$$= \frac{1}{2}\int_0^1 \left(u^2-u^4\right)\,\mathrm{d}u = \frac{1}{2}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 = \frac{1}{15}$$

(b) Make the substitution $x = 3\tan t$, $\mathrm{d}x = 3\sec^2 t\,\mathrm{d}t$ and use the trig identity $9 + 9\tan^2 t = 9\sec^2 t$.

$$\int \left(9+x^2\right)^{-\frac{3}{2}}\,\mathrm{d}x = \int \left(9+9\tan^2 t\right)^{-\frac{3}{2}} 3\sec^2 t\,\mathrm{d}t = \int \left(3\sec t\right)^{-3} 3\sec^2 t\,\mathrm{d}t$$
$$= \frac{1}{9}\int \cos t\,\mathrm{d}t = \frac{1}{9}\sin t + C = \frac{1}{9}\frac{x}{\sqrt{x^2+9}} + C$$

To convert back to x, in the last step, we used the triangle below, which is rigged to have $\tan t = \frac{x}{3}$.



(c) Seeing a rational function with no obvious substitutions, we use partial fractions.

$$\frac{1}{(x-1)(x^2+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+1} = \frac{a(x^2+1)+(bx+c)(x-1)}{(x-1)(x^2+1)}$$

Multiply by the original denominator.

$$1 = a(x^2+1) + (bx+c)(x-1) \tag{$*$}$$

Setting $x = 1$ gives $2a = 1$ or $a = \frac{1}{2}$. Substituting in $a = \frac{1}{2}$ in $(*)$ gives

$$\begin{aligned}
& \frac{1}{2}(x^2+1) + (bx+c)(x-1) = 1 \\
\iff \quad & (bx+c)(x-1) = \frac{1}{2}(1-x^2) = -\frac{1}{2}(x-1)(x+1) \\
\iff \quad & (bx+c) = -\frac{1}{2}(x+1) \\
\iff \quad & b = c = -\frac{1}{2}
\end{aligned}$$

So,

$$\int \frac{\mathrm{d}x}{(x-1)(x^2+1)} = \int \left[\frac{1/2}{x-1} - \frac{\frac{1}{2}(x+1)}{x^2+1}\right] \mathrm{d}x$$

To antidifferentiate the second piece, we split it into two integrals: one that can be handled with the substitution $u = x^2 + 1$, and another that looks like the derivative of arctangent.

$$\begin{aligned}
&= \int \left(\frac{1/2}{x-1} - \frac{x/2}{x^2+1} - \frac{1/2}{x^2+1}\right) \mathrm{d}x \\
&= \int \left(\frac{1/2}{x-1} - \frac{1}{4} \cdot \frac{2x}{x^2+1} - \frac{1/2}{x^2+1}\right) \mathrm{d}x \\
&= \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln(x^2+1) - \frac{1}{2} \arctan x + C
\end{aligned}$$

(d) We know the derivative of arctangent, and it would integrate nicely if multiplied to the antiderivative of x. So, we integrate by parts with $u = \arctan x$ and $\mathrm{d}v = x\,\mathrm{d}x$ so that $\mathrm{d}u = \frac{1}{1+x^2}\mathrm{d}x$ and $v = \frac{1}{2}x^2$. Then

$$\begin{aligned}
\int x \arctan x \,\mathrm{d}x &= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2}\,\mathrm{d}x \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2}\,\mathrm{d}x \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) \mathrm{d}x \\
&= \frac{1}{2}\left[x^2 \arctan x - x + \arctan x\right] + C
\end{aligned}$$

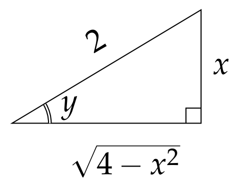
S-14: (a) We substitute $y = \sin(2x)$, $\mathrm{d}y = 2\cos(2x)\,\mathrm{d}x$. Note $\sin(2 \cdot 0) = 0$ and $\sin(2 \cdot \frac{\pi}{4}) = 1$.

$$\int_0^{\pi/4} \sin^5(2x)\,\cos(2x)\,\mathrm{d}x = \int_0^1 y^5\,\frac{\mathrm{d}y}{2} = \frac{1}{12}\left[y^6\right]_0^1 = \frac{1}{12}$$

(b) We can get rid of the square root with a trig substitution. Substituting $x = 2\sin y$, $\mathrm{d}x = 2\cos y\,\mathrm{d}y$,

$$\begin{aligned}
\int \sqrt{4-x^2}\,\mathrm{d}x &= \int \sqrt{4-4\sin^2 y}\,2\cos y\,\mathrm{d}y = 4\int \cos^2 y\,\mathrm{d}y = 2\int \left[1+\cos(2y)\right]\,\mathrm{d}y \\
&= 2y + \sin(2y) + C = 2y + 2\sin y \cos y + C \\
&= 2\sin^{-1}\frac{x}{2} + x\sqrt{1-\frac{x^2}{4}} + C
\end{aligned}$$

since $\sin y = \frac{x}{2}$ and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{x^2}{4}}$. Alternately, we can draw a triangle with $\sin y = \frac{x}{2}$, and use the Pythagorean Theorem to find the adjacent side.



(c) Seeing a rational function with no obvious substitution, we use the method of partial fractions. The denominator is already completely factored.

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$
$$x+1 = Ax(x-1) + B(x-1) + Cx^2$$

Setting $x = 1$ gives us $C = 2$. Setting $x = 0$ gives us $B = -1$. Furthermore, the coefficient of x^2 on the left hand side (after collecting like terms), namely $A + C$, must be the same as the coefficient of x^2 on the right hand side, namely 0. So $A + C = 0$ and $A = -2$. Checking,

$$-2x(x-1) - (x-1) + 2x^2 = -2x^2 + 2x - x + 1 + 2x^2 = x + 1$$

as desired. Thus,

$$\int \frac{x+1}{x^2(x-1)}\,\mathrm{d}x = \int \left[-\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right] \mathrm{d}x = -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$$

S-15: (a) Define

$$I_1 = \int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x \qquad I_2 = \int_0^\infty e^{-x}\cos(2x)\,\mathrm{d}x$$

We integrate by parts, with $u = \sin(2x)$ or $\cos(2x)$ and $\mathrm{d}v = e^{-x}\,\mathrm{d}x$. That is, $v = -e^{-x}$.

$$\begin{aligned} I_1 = \int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x &= \lim_{R\to\infty}\int_0^R e^{-x}\sin(2x)\,\mathrm{d}x \\ &= \lim_{R\to\infty}\left(\Big[-e^{-x}\sin(2x)\Big]_0^R + 2\int_0^R e^{-x}\cos(2x)\,\mathrm{d}x\right) = 2I_2 \\ I_2 = \int_0^\infty e^{-x}\cos(2x)\,\mathrm{d}x &= \lim_{R\to\infty}\int_0^R e^{-x}\cos(2x)\,\mathrm{d}x \\ &= \lim_{R\to\infty}\left(\Big[-e^{-x}\cos(2x)\Big]_0^R - 2\int_0^R e^{-x}\sin(2x)\,\mathrm{d}x\right) = 1 - 2I_1 \end{aligned}$$

Substituting $I_2 = \frac{1}{2}I_1$ into $I_2 = 1 - 2I_1$ gives $\frac{5}{2}I_1 = 1$, or $\int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x = \frac{2}{5}$.

(b) We can cancel out the square root if we use a trig substitution. Substitute $x = \sqrt{2}\tan y$, $\mathrm{d}x = \sqrt{2}\sec^2 y\,\mathrm{d}y$.

$$\int_0^{\sqrt{2}} \frac{1}{(2+x^2)^{3/2}}\,\mathrm{d}x = \sqrt{2}\int_0^{\pi/4}\frac{\sec^2 y}{(2+2\tan^2 y)^{3/2}}\,\mathrm{d}y = \frac{1}{2}\int_0^{\pi/4}\cos y\,\mathrm{d}y = \frac{1}{2}\Big[\sin y\Big]_0^{\pi/4} = \frac{1}{2\sqrt{2}}$$

(c)

Solution 1: Integrate by parts, using $u = \ln(1+x^2)$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{2x}{1+x^2}$, $v = \frac{x^2}{2}$.

$$\int_0^1 x\ln(1+x^2)\,\mathrm{d}x = \left[\frac{1}{2}x^2\ln(1+x^2)\right]_0^1 - \int_0^1 \frac{x^3}{1+x^2}\,\mathrm{d}y = \frac{1}{2}\ln 2 - \int_0^1 \left[x - \frac{x}{1+x^2}\right]\mathrm{d}x$$
$$= \frac{1}{2}\ln 2 - \left[\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)\right]_0^1 = \ln 2 - \frac{1}{2} \approx 0.193$$

Solution 2: First substitute $y = 1+x^2$, $\mathrm{d}y = 2x\,\mathrm{d}x$.

$$\int_0^1 x\ln(1+x^2)\,\mathrm{d}x = \frac{1}{2}\int_1^2 \ln y\;\mathrm{d}y$$

Then integrate by parts, using $u = \ln y$ and $\mathrm{d}v = \mathrm{d}y$, so that $\mathrm{d}u = \frac{1}{y}$, $v = y$.

$$\int_0^1 x\ln(1+x^2)\,\mathrm{d}x = \frac{1}{2}\int_1^2 \ln y\;\mathrm{d}y = \left[\frac{1}{2}y\ln y\right]_1^2 - \frac{1}{2}\int_1^2 y\frac{1}{y}\,\mathrm{d}y = \ln 2 - \frac{1}{2} \approx 0.193$$

(d) Seeing a rational function with no obvious substitution, we use partial fractions.

$$\frac{1}{(x-1)^2(x-2)} = \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c}{x-2}$$
$$1 = a(x-2) + b(x-1)(x-2) + c(x-1)^2 \qquad (*)$$

Setting $x = 1$ gives $a = -1$. Setting $x = 2$ gives $c = 1$. Substituting in $a = -1$ and $c = 1$ to $(*)$ gives

$$\begin{aligned} b(x-1)(x-2) &= 1 + (x-2) - (x-1)^2 \\ &= -x^2 + 3x - 2 \\ &= -(x-1)(x-2) \\ \implies \qquad b &= -1 \end{aligned}$$

Hence:

$$\begin{aligned} \int_3^\infty \frac{x}{(x-1)^2(x-2)}\,\mathrm{d}x &= \lim_{M\to\infty}\int_3^M \left(-\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x-2}\right)\mathrm{d}x \\ &= \lim_{M\to\infty}\left[\frac{1}{x-1} - \ln(x-1) + \ln(x-2)\right]_3^M \\ &= \lim_{M\to\infty}\left[\frac{1}{M-1} + \ln\frac{M-2}{M-1}\right] - \left[\frac{1}{3-1} + \ln\frac{3-2}{3-1}\right] \\ &= \ln 2 - \frac{1}{2} \approx 0.193 \end{aligned}$$

since

$$\lim_{M\to\infty}\ln\frac{M-2}{M-1} = \lim_{M\to\infty}\ln\frac{1-2/M}{1-1/M} = \ln 1 = 0$$

S-16: (a) Integrate by parts with $u = \ln x$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = \frac{1}{2}x^2$.

$$\int x \ln x \,\mathrm{d}x = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \cdot \frac{1}{x}\,\mathrm{d}x = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

(b) The denominator is an irreducible quadratic, so partial fractions can't get us any further. To integrate a function whose denominator is quadratic, we split the numerator up so that one piece can be evaluated with a u-substitution, and the other piece looks like arctangent.

$$\begin{aligned}
\int \frac{(x-1)\mathrm{d}x}{x^2+4x+5} &= \int \frac{x+2-3}{x^2+4x+5}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\mathrm{d}x - \int \frac{3}{x^2+4x+5}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\mathrm{d}x - 3\int \frac{1}{(x+2)^2+1}\mathrm{d}x \\
&= \frac{1}{2}\ln[x^2+4x+5] - 3\arctan(x+2) + C
\end{aligned}$$

For the last step, you can guess the antiderivative, or use the substitutions $u_1 = x^2 + 4x + 5$ and $u_2 = x + 2$, respectively, for the two integrals.

(c) We use partial fractions.

$$\begin{aligned}
\frac{1}{x^2-4x+3} = \frac{1}{(x-3)(x-1)} &= \frac{a}{x-3} + \frac{b}{x-1} \\
1 &= a(x-1) + b(x-3)
\end{aligned}$$

Setting $x = 3$ gives $a = \frac{1}{2}$. Setting $x = 1$ gives $b = -\frac{1}{2}$. So,

$$\int \frac{\mathrm{d}x}{x^2-4x+3} = \int \left(\frac{1/2}{x-3} - \frac{1/2}{x-1}\right)\mathrm{d}x = \frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + C$$

(d) Substitute $y = x^3$, $\mathrm{d}y = 3x^2\,\mathrm{d}x$.

$$\int \frac{x^2\,\mathrm{d}x}{1+x^6} = \frac{1}{3}\int \frac{\mathrm{d}y}{1+y^2} = \frac{1}{3}\arctan y + C = \frac{1}{3}\arctan x^3 + C$$

S-17: (a) Integrate by parts with $u = \arctan x$, $\mathrm{d}v = \mathrm{d}x$, $\mathrm{d}u = \frac{\mathrm{d}x}{1+x^2}$ and $v = x$. This gives

$$\int_0^1 \arctan x \,\mathrm{d}x = \big[x\arctan x\big]_0^1 - \int_0^1 \frac{x}{1+x^2}\mathrm{d}x = \arctan 1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

(b) Note that the derivative of the denominator is $2x - 2$, which differs from the numerator only by 1.

$$\begin{aligned}
\int \frac{2x-1}{x^2-2x+5}\,\mathrm{d}x &= \int \frac{2x-2}{x^2-2x+5}\,\mathrm{d}x + \int \frac{1}{x^2-2x+5}\,\mathrm{d}x \\
&= \int \frac{2x-2}{x^2-2x+5}\,\mathrm{d}x + \int \frac{1}{(x-1)^2+4}\,\mathrm{d}x \\
&= \ln|x^2-2x+5| + \frac{1}{2}\arctan\frac{x-1}{2} + C
\end{aligned}$$

In the last step, you can guess the antiderivative, or use the substitutions $u_1 = x^2 - 2x + 5$ and $u_2 = (x-1)/2$, respectively.

S-18: (a) Substituting $u = x^3 + 1$, $\mathrm{d}u = 3x^2\,\mathrm{d}x$

$$\int \frac{x^2}{(x^3+1)^{101}}\mathrm{d}x = \int \frac{1}{u^{101}} \cdot \frac{\mathrm{d}u}{3} = \frac{u^{-100}}{-100} \cdot \frac{1}{3} + C = -\frac{1}{300(x^3+1)^{100}} + C$$

(b) Substituting $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$, $\cos^2 x = 1 - \sin^2 x = 1 - u^2$,

$$\begin{aligned}\int \cos^3 x\ \sin^4 x\ \mathrm{d}x &= \int \cos^2 x\ \sin^4 x\ \cos x\ \mathrm{d}x = \int (1-u^2)u^4\ \mathrm{d}u\\ &= \int (u^4 - u^6)\ \mathrm{d}u = \frac{u^5}{5} - \frac{u^7}{7} + C\\ &= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C\end{aligned}$$

S-19: First, we note that the integral is improper, because $\sin \pi = 0$. So, we'll have to use a limit.

Second, we need to antidifferentiate. The substitution $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$ fits just right.

$$\begin{aligned}\int_{\pi/2}^{\pi} \frac{\cos x}{\sqrt{\sin x}}\,\mathrm{d}x &= \lim_{b\to\pi^-} \int_{\pi/2}^{b} \frac{\cos x}{\sqrt{\sin x}}\,\mathrm{d}x = \lim_{b\to\pi^-} \int_1^{\sin b} \frac{1}{\sqrt{u}}\,\mathrm{d}u\\ &= \lim_{b\to\pi^-} \Big[2\sqrt{u}\Big]_1^{\sin b} = 2\sqrt{0} - 2\sqrt{1} = -2\end{aligned}$$

S-20: (a) If the integrand had x's instead of e^x's it would be a rational function, ripe for the application of partial fractions. So let's start by making the substitution $u = e^x$, $\mathrm{d}u = e^x\,\mathrm{d}x$:

$$\int \frac{e^x}{(e^x+1)(e^x-3)}\mathrm{d}x = \int \frac{\mathrm{d}u}{(u+1)(u-3)}$$

Now, we follow the partial fractions protocol, starting with expressing

$$\frac{1}{(u+1)(u-3)} = \frac{A}{u+1} + \frac{B}{u-3}$$

To find A and B, the sneaky way, we cross multiply by the denominator

$$1 = A(u-3) + B(u+1)$$

and find A and B by evaluating at $u = -1$ and $u = 3$, respectively.

$$1 = A(-1-3) + B(-1+1) \iff A = -\frac{1}{4}$$
$$1 = A(3-3) + B(3+1) \iff B = \frac{1}{4}$$

Finally, we can do the integral:

$$\begin{aligned}\int \frac{e^x}{(e^x+1)(e^x-3)}\mathrm{d}x &= \int \frac{\mathrm{d}u}{(u+1)(u-3)} = \int \Big(\frac{-1/4}{u+1} + \frac{1/4}{u-3}\Big)\mathrm{d}u \\ &= -\frac{1}{4}\ln|u+1| + \frac{1}{4}\ln|u-3| + C \\ &= -\frac{1}{4}\ln|e^x+1| + \frac{1}{4}\ln|e^x-3| + C\end{aligned}$$

(b) The argument of the square root is

$$12 + 4x - x^2 = 12 - (x-2)^2 + 4 = 16 - (x-2)^2$$

Hmmm. The numerator is $x^2 - 4x + 4 = (x-2)^2$. So let's make the integral look somewhat simpler by substituting $u = x - 2$, $\mathrm{d}u = \mathrm{d}x$. When $x = 2$ we have $u = 0$, and when $x = 4$ we have $u = 2$, so:

$$\int_{x=2}^{x=4} \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\mathrm{d}x = \int_{u=0}^{u=2} \frac{u^2}{\sqrt{16-u^2}}\mathrm{d}u$$

This is perfect for the trig substitution $u = 4\sin\theta$, $\mathrm{d}u = 4\cos(\theta)\,\mathrm{d}\theta$. When $u = 0$ we have $4\sin\theta = 0$ and hence $\theta = 0$. When $u = 2$ we have $4\sin\theta = 2$ and hence $\theta = \frac{\pi}{6}$. So

$$\begin{aligned}\int_{u=0}^{u=2} \frac{u^2}{\sqrt{16-u^2}}\mathrm{d}u &= \int_{\theta=0}^{\theta=\pi/6} \frac{16\sin^2\theta}{\sqrt{16-16\sin^2\theta}} 4\cos\theta\mathrm{d}\theta \\ &= 16\int_0^{\pi/6} \sin^2\theta\,\mathrm{d}\theta \\ &= 8\int_0^{\pi/6} \big(1-\cos(2\theta)\big)\,\mathrm{d}\theta \\ &= 8\Big[\theta - \frac{1}{2}\sin(2\theta)\Big]_0^{\pi/6} = 8\Big[\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\Big] \\ &= \frac{4\pi}{3} - 2\sqrt{3}\end{aligned}$$

S-21: (a) Substituting $y = \cos x$, $\mathrm{d}y = -\sin x\,\mathrm{d}x$, $\sin^2 x = 1 - \cos^2 x = 1 - y^2$

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^3 x}\,\mathrm{d}x &= \int \frac{\sin^2 x}{\cos^3 x}\,\sin x\,\mathrm{d}x = \int \frac{1-y^2}{y^3}\,(-\mathrm{d}y) = -\int \big(y^{-3} - y^{-1}\big)\,\mathrm{d}y \\ &= -\frac{y^{-2}}{-2} + \ln|y| + C = \frac{1}{2}\sec^2 x + \ln|\cos x| + C\end{aligned}$$

(b) The integrand is an even function, and the limits of integration are symmetric. So, we can slightly simplify the integral by replacing the lower limit with 0, and doubling the integral.

We'd rather not use partial fractions here, because it would be pretty complicated. Instead, notice that the numerator is only off by a constant from the derivative of x^5. Substituting $x^5 = 4y$, $5x^4\,\mathrm{d}x = 4\,\mathrm{d}y$, and using that $x = 2 \implies 2^5 = 4y \implies y = 8$,

$$\int_{-2}^{2} \frac{x^4}{x^{10}+16}\,\mathrm{d}x = 2\int_0^2 \frac{x^4}{x^{10}+16}\,\mathrm{d}x = 2\cdot\frac{4}{5}\int_0^8 \frac{1}{16y^2+16}\,\mathrm{d}y = \frac{1}{10}\int_0^8 \frac{1}{y^2+1}\,\mathrm{d}y$$
$$= \frac{1}{10}\arctan 8 \approx 0.1446$$

S-22:

Solution 1: Let's use the substitution $u = x - 1$, $\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}\int x\sqrt{x-1}\,\mathrm{d}x &= \int (u+1)\sqrt{u}\,\mathrm{d}u \\ &= \int \left(u^{3/2}+u^{1/2}\right)\,\mathrm{d}u \\ &= \frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C \\ &= \frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C\end{aligned}$$

Solution 2: We have an integrand with x multiplied by something integrable. So, if we use integration by parts with $u = x$ and $\mathrm{d}v = \sqrt{x-1}\,\mathrm{d}x$, then $\mathrm{d}u = \mathrm{d}x$ (that is, the x goes away) and $v = \frac{2}{3}(x-1)^{3/2}$.

$$\begin{aligned}\int x\sqrt{x-1}\,\mathrm{d}x &= \frac{2}{3}x\sqrt{x-1}^3 - \frac{2}{3}\int (x-1)^{3/2}\mathrm{d}x \\ &= \frac{2}{3}x\sqrt{x-1}^3 - \frac{2}{3}\left(\frac{2}{5}(x-1)^{5/2}\right)+C \\ &= \frac{2}{3}\sqrt{x-1}\left(x(x-1)-\frac{2}{5}(x-1)^2\right)+C \\ &= \frac{2}{15}\sqrt{x-1}\cdot(3x^2-x-2)+C \\ &= \frac{2}{15}\sqrt{x-1}\cdot(3(x^2-2x+1)+5x-5)+C \\ &= \frac{2}{15}\sqrt{x-1}\cdot(3(x-1)^2+5(x-1))+C \\ &= \frac{2}{15}\cdot 3\sqrt{x-1}^5+\frac{2}{15}\cdot 5\sqrt{x-1}^3+C \\ &= \frac{2}{5}\sqrt{x-1}^5+\frac{2}{3}\sqrt{x-1}^3+C\end{aligned}$$

S-23:

$$\int \frac{\sqrt{x^2-2}}{x^2}\mathrm{d}x$$

We notice that there is a quadratic function under the square root. If that equation were a perfect square, we could get rid of the square root: so we'll mould it into a perfect square using a trig substitution.

Our candidates will use one of the following identities:

$$1-\sin^2\theta = \cos^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

We'll be substituting $x =$(something), so we notice that x^2-2 has the general form of (function)$-$(constant), as does $\sec^2\theta - 1$. In order to get the constant right, we multiply through by two:

$$2\sec^2\theta - 2 = 2\tan^2\theta$$

or:

$$(\sqrt{2}\sec\theta)^2 - 2 = 2\tan^2\theta$$

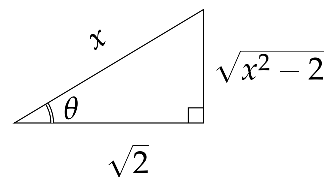
so we decide to use the substitution

$$x = \sqrt{2}\sec\theta \qquad \mathrm{d}x = \sqrt{2}\sec\theta\tan\theta\mathrm{d}\theta$$

Now that we've chosen the substitution, we evaluate the integral.

$$\begin{aligned}
\int \frac{\sqrt{x^2-2}}{x^2}\mathrm{d}x &= \int \frac{\sqrt{2\sec^2\theta-2}}{2\sec^2\theta}\sqrt{2}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \int \frac{\sqrt{2\tan^2\theta}}{2\sec^2\theta}\sqrt{2}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \int \frac{\sqrt{2}\tan\theta}{2\sec^2\theta}\sqrt{2}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \int \frac{\tan^2\theta}{\sec\theta}\mathrm{d}\theta \\
&= \int \frac{\sec^2\theta-1}{\sec\theta}\mathrm{d}\theta \\
&= \int \big(\sec\theta - \cos\theta\big)\,\mathrm{d}\theta \\
&= \ln|\sec\theta + \tan\theta| - \sin\theta + C
\end{aligned}$$

Now we need everything back in terms of x. We need a triangle. Since $x = \sqrt{2}\sec\theta$, that means if we label an angle θ, its secant (hypotenuse over adjacent side) is $\frac{x}{\sqrt{2}}$. By Pythagoras, the opposite side is $\sqrt{x^2-2}$.



So $\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sqrt{x^2-2}}{\sqrt{2}}$, and $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\sqrt{x^2-2}}{x}$. Then the value of the integral is:

$$\begin{aligned}
\ln|\sec\theta + \tan\theta| - \sin\theta + C &= \ln\left|\frac{x}{\sqrt{2}} + \frac{\sqrt{x^2-2}}{\sqrt{2}}\right| - \frac{\sqrt{x^2-2}}{x} + C \\
&= \ln\left|x + \sqrt{x^2-2}\right| - \ln\sqrt{2} - \frac{\sqrt{x^2-2}}{x} + C \\
&= \ln\left|x + \sqrt{x^2-2}\right| - \frac{\sqrt{x^2-2}}{x} + C
\end{aligned}$$

Note the simplification in the last step is due to our convention that C is an arbitrary constant. So, $C - \ln\sqrt{2}$ can be re-written as simply C.

S-24:

This is the product of secants and tangents, as in Section 3.6.2 of the text. If $u = \tan x$, then $\mathrm{d}u = \sec^2 x\mathrm{d}x$. We can get the remaining two secants to turn into tangents with the identity $\sec^2 x = 1 + \tan^2 x$, so we'll use this substitution.

$$\begin{aligned}
\int_0^{\pi/4} \sec^4 x \tan^5 x\mathrm{d}x &= \int_0^{\pi/4} \sec^2 x \tan^5 x \sec^2 x\mathrm{d}x \\
&= \int_0^{\pi/4} (1 + \tan^2 x)\tan^5 x \underbrace{\sec^2 x\mathrm{d}x}_{\mathrm{d}u} \\
&= \int_{\tan(0)}^{\tan(\pi/4)} (1 + u^2)u^5\mathrm{d}u \\
&= \int_0^1 (u^5 + u^7)\mathrm{d}u \\
&= \left[\frac{1}{6}u^6 + \frac{1}{8}u^8\right]_0^1 \\
&= \frac{1}{6} + \frac{1}{8} - 0 = \frac{7}{24}
\end{aligned}$$

S-25: We can use partial fraction decomposition to break this into chunks that we can deal with. The denominator has a repeated linear factor, so it can be decomposed as the sum of constants divided by powers of that factor.

$$\begin{aligned}
\frac{3x^2+4x+6}{(x+1)^3} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \\
&= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} \\
\Rightarrow \quad 3x^2+4x+6 &= A(x+1)^2 + B(x+1) + C \\
&= Ax^2 + (2A+B)x + (A+B+C)
\end{aligned}$$

So, by matching coefficients:

$$\begin{gathered}
A = 3,\ 2A+B = 4,\ \text{and } A+B+C = 6 \\
A = 3,\ B = -2,\ C = 5
\end{gathered}$$

Therefore:

$$\frac{3x^2+4x+6}{(x+1)^3} = \frac{3}{x+1} + \frac{-2}{(x+1)^2} + \frac{5}{(x+1)^3}$$

Now, the integration is easy, with a substitution of $u = x+1$ and $\mathrm{d}u = \mathrm{d}x$:

$$\begin{aligned}
\int \frac{3x^2+4x+6}{(x+1)^3}\mathrm{d}x &= \int \left(\frac{3}{x+1} + \frac{-2}{(x+1)^2} + \frac{5}{(x+1)^3}\right)\mathrm{d}x \\
&= \int \left(3u^{-1} - 2u^{-2} + 5u^{-3}\right)\mathrm{d}u \\
&= 3\ln|u| + 2u^{-1} - \frac{5}{2}u^{-2} + C \\
&= 3\ln|x+1| + \frac{2}{x+1} - \frac{5}{2(x+1)^2} + C
\end{aligned}$$

S-26:

If the denominator were x^2+1, the antiderivative would be arctangent. So, by completing the square, let's aim for the fraction to look like $\frac{1}{u^2+1}$, for some u. This is a good strategy for integrating an irreducible quadratic under a constant.

First: complete the square

$$\int \frac{1}{x^2+x+1}\mathrm{d}x = \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}\mathrm{d}x = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x$$

Second: get the denominator in the form u^2+1. To do this, we need to fix the constant

$$\begin{aligned}
&= \int \left(\frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \right) \left(\frac{\frac{4}{3}}{\frac{4}{3}} \right) \mathrm{d}x \\
&= \frac{4}{3} \int \frac{1}{\frac{4}{3} \cdot \left(x+\frac{1}{2}\right)^2 + 1} \mathrm{d}x
\end{aligned}$$

Now a quick wiggle to make that first part of the denominator into something squared again:

$$= \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} \mathrm{d}x$$

Now we see that $u = \dfrac{2}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}$, $\mathrm{d}u = \dfrac{2}{\sqrt{3}}\mathrm{d}x$ will do the job

$$\begin{aligned}
&= \frac{4}{3} \int \frac{1}{u^2+1} \cdot \frac{\sqrt{3}}{2} \mathrm{d}u = \frac{2}{\sqrt{3}} \int \frac{1}{u^2+1} \mathrm{d}u \\
&= \frac{2}{\sqrt{3}} \arctan u + C \\
&= \frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \right) + C
\end{aligned}$$

S-27: Since $\tan x = \frac{\sin x}{\cos x}$,

$$\begin{aligned}
\int \sin x \cos x \tan x \, \mathrm{d}x = \int \sin^2 x \, \mathrm{d}x &= \int \frac{1}{2} (1 - \cos(2x)) \, \mathrm{d}x \\
&= \frac{1}{2} \left(x - \frac{1}{2} \sin(2x) \right) + C \\
&= \frac{1}{2} (x - \sin x \cos x) + C
\end{aligned}$$

S-28: We have the integral of a rational function with no obvious substitution, so we use partial fractions. That means we need to factor the denominator. We see that $x=-1$ is a root of the denominator, so $x+1$ is a factor. You might be able to figure out the rest of the factorization by inspection, or from having seen this common expression before; alternately, we can use long division.

$$
\begin{array}{r}
x^2 - x + 1 \\
x+1 \overline{\smash{\big)}\; x^3 \qquad\qquad + 1} \\
\underline{- x^3 - x^2 \qquad\qquad} \\
- x^2 \qquad\quad \\
\underline{x^2 + x \qquad} \\
x + 1 \\
\underline{- x - 1} \\
0
\end{array}
$$

Note $x^2 - x + 1$ is an irreducible quadratic.

$$
\begin{aligned}
\frac{1}{x^3+1} &= \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \\
1 &= A(x^2-x+1) + (Bx+C)(x+1) \qquad (*)
\end{aligned}
$$

When $x = -1$, we see $1 = 3A$, so $\frac{1}{3} = A$. We plug this into $(*)$.

$$
\begin{aligned}
1 &= \frac{1}{3}(x^2-x+1) + (Bx+C)(x+1) \\
-\frac{1}{3}x^2 + \frac{1}{3}x + \frac{2}{3} &= Bx^2 + (B+C)x + C
\end{aligned}
$$

Matching up coefficients of corresponding power of x, we see $B = -\frac{1}{3}$ and $C = \frac{2}{3}$.

$$
\int \frac{1}{x^3+1}\,\mathrm{d}x = \int \left(\frac{1/3}{x+1} - \frac{\frac{1}{3}x - \frac{2}{3}}{x^2-x+1} \right)\,\mathrm{d}x
$$

To integrate the second fraction, we break it up into two pieces: one we can integrate using the substitution $u = x^2 - x + 1$, the other will look like the derivative of arctangent.

$$
\begin{aligned}
&= \frac{1}{3}\ln|x+1| - \int \frac{\frac{1}{3}x - \frac{1}{6} - \frac{1}{2}}{x^2-x+1}\,\mathrm{d}x \\
&= \frac{1}{3}\ln|x+1| - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}\,\mathrm{d}x \\
&= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{\frac{3}{4}\left(\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1\right)}\,\mathrm{d}x \\
&= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{2}{3}\int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1}\,\mathrm{d}x
\end{aligned}
$$

Let $u = \frac{2x-1}{\sqrt{3}}$, $\mathrm{d}u = \frac{2}{\sqrt{3}}\,\mathrm{d}x$.

$$
\begin{aligned}
&= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\,\mathrm{d}x \\
&= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C
\end{aligned}
$$

S-29: By process of elimination, we decide to use integration by parts. We won't get anything better by antidifferentiating arcsine, so let's plan on differentiating it:

$$u = \arcsin x \qquad \mathrm{d}v = (3x)^2\mathrm{d}x$$
$$\mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \qquad v = 3x^3$$

$$\begin{aligned}\int (3x)^2 \arcsin x \mathrm{d}x &= \underbrace{\arcsin x}_{u} \cdot \underbrace{3x^3}_{v} - \int \underbrace{3x^3}_{v} \cdot \underbrace{\frac{1}{\sqrt{1-x^2}}\mathrm{d}x}_{\mathrm{d}u} \\ &= 3x^3 \arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}}\mathrm{d}x\end{aligned}$$

So: we've gotten rid of the ugly pairing of arcsine with a polynomial, but now we're in another pickle. From here, two options present themselves. We could use the substitution $u = 1 - x^2$, or we could use a trig substitution.

Option 1: Let $u = 1 - x^2$. Then $-\frac{1}{2}\mathrm{d}u = \mathrm{d}x$, and $x^2 = 1 - u$.

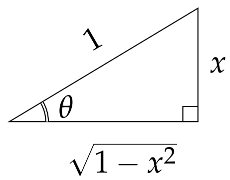
$$\begin{aligned}\int (3x)^2 \arcsin x \mathrm{d}x &= 3x^3 \arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}}\mathrm{d}x \\ &= 3x^3 \arcsin x - 3\int \frac{x^2}{\sqrt{1-x^2}} \cdot x \, \mathrm{d}x \\ &= 3x^3 \arcsin x + \frac{3}{2}\int \frac{1-u}{\sqrt{u}} \, \mathrm{d}u \\ &= 3x^3 \arcsin x + \frac{3}{2}\int \left(u^{-1/2} - u^{1/2}\right) \, \mathrm{d}u \\ &= 3x^3 \arcsin x + \frac{3}{2}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right) + C \\ &= 3x^3 \arcsin x + 3\sqrt{1-x^2} - \sqrt{1-x^2}^3 + C\end{aligned}$$

Option 2: If we let $x = \sin\theta$, then $\sqrt{1-x^2} = \sqrt{\cos^2\theta} = \cos\theta$. So let's use the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\mathrm{d}\theta$.

$$\begin{aligned}\int (3x)^2 \arcsin x \mathrm{d}x &= 3x^3 \arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}}\mathrm{d}x \\ &= 3x^3 \arcsin x - \int \frac{3\sin^3\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\mathrm{d}\theta \\ &= 3x^3 \arcsin x - \int 3\sin^3\theta\mathrm{d}\theta\end{aligned}$$

And now: a substitution from Section 3.6.1 of the text, $u = \cos x$ and $\mathrm{d}u = -\sin x \mathrm{d}x$

$$\begin{aligned}
3x^3 \arcsin x - \int 3\sin^3\theta \mathrm{d}\theta &= 3x^3 \arcsin x - 3\int \sin^2\theta \sin\theta \mathrm{d}\theta \\
&= 3x^3 \arcsin x - 3\int (1-\cos^2\theta)\sin\theta \mathrm{d}\theta \\
&= 3x^3 \arcsin x + 3\int (1-u^2)\mathrm{d}u \\
&= 3x^3 \arcsin x + 3\left(u - \frac{1}{3}u^3\right) + C \\
&= 3x^3 \arcsin x + 3u - u^3 + C \\
&= 3x^3 \arcsin x + 3\cos\theta - \cos^3\theta + C
\end{aligned}$$



Recall $x = \sin\theta$; so we draw a triangle with angle θ, opposite side x, hypotenuse 1. Then by Pythagoras, adjacent side is $\sqrt{1-x^2}$, so $\cos\theta = \sqrt{1-x^2}$.

$$= 3x^3 \arcsin x + 3\sqrt{1-x^2} - (1-x^2)^{3/2} + C$$

S-30: We would like to not have that square root there. Luckily, there's a way of turning cosine into cosine squared: the identity $\cos(2x) = 2\cos^2 x - 1$. If we take $2x = t$, then $\cos t = 2\cos^2(t/2) - 1$.

$$\int_0^{\pi/2} \sqrt{\cos t + 1}\, \mathrm{d}t = \int_0^{\pi/2} \sqrt{2\cos^2(t/2)}\, \mathrm{d}t = \sqrt{2}\int_0^{\pi/2} |\cos(t/2)|\, \mathrm{d}t$$

Over the interval $[0, \frac{\pi}{2}]$, $\cos(t/2) > 0$, so we can drop the absolute values.

$$\begin{aligned}
&= \sqrt{2}\int_0^{\pi/2} \cos(t/2)\, \mathrm{d}t = \sqrt{2}\left[2\sin\left(\frac{t}{2}\right)\right]_0^{\pi/2} \\
&= 2\sqrt{2}\sin\left(\frac{\pi}{4}\right) = 2
\end{aligned}$$

S-31:

Solution 1: Using logarithm rules, $\ln\sqrt{x} = \ln\left(x^{1/2}\right) = \frac{1}{2}\ln x$, so we can simplify:

$$\int_1^e \frac{\ln\sqrt{x}}{x}\mathrm{d}x = \int_1^e \frac{\ln x}{2x}\mathrm{d}x$$

We use the substitution $u = \ln x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$:

$$\begin{aligned}
\int_1^e \frac{\ln x}{2x}\mathrm{d}x &= \frac{1}{2}\int_1^e \underbrace{\ln(x)}_{u} \cdot \underbrace{\frac{1}{x}\,\mathrm{d}x}_{\mathrm{d}u} \\
&= \frac{1}{2}\int_{\ln(1)}^{\ln(e)} u\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 u\,\mathrm{d}u \\
&= \frac{1}{2}\left[\frac{1}{2}u^2\right]_0^1 \\
&= \frac{1}{2}\left[\frac{1}{2} - 0\right] = \frac{1}{4}
\end{aligned}$$

Solution 2: We use the substitution $u = \ln\sqrt{x}$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$, hence $2\mathrm{d}u = \frac{1}{x}\mathrm{d}x$. This fits our integral nicely!

$$\begin{aligned}
\int_1^e \frac{\ln\sqrt{x}}{x}\mathrm{d}x &= \int_{\ln\sqrt{1}}^{\ln\sqrt{e}} u \cdot 2\mathrm{d}u \\
&= \left[u^2\right]_0^{1/2} \\
&= \left(\frac{1}{2}\right)^2 - 0^2 = \frac{1}{4}
\end{aligned}$$

S-32:

$$\int_{0.1}^{0.2} \frac{\tan x}{\ln(\cos x)}\mathrm{d}x$$

It might not be immediately obvious how to proceed on this one, so this is another example of an integral where you should not be discouraged by finding methods that don't work. One thing that's worked for us in the past is to use a u-substitution with the denominator. With that in mind, let's find the derivative of the denominator.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\ln(\cos x)\right\} = \frac{1}{\cos x} \cdot (-\sin x) = \frac{-\sin x}{\cos x} = -\tan x$$

So, if we let $u = \ln(\cos x)$, we see $-\mathrm{d}u = \tan x\mathrm{d}x$, which will work for a substitution.

$$\begin{aligned}
\int_{0.1}^{0.2} \frac{\tan x}{\ln(\cos x)} \mathrm{d}x &= \int_{\ln(\cos(0.1))}^{\ln(\cos(0.2))} \frac{-\mathrm{d}u}{u} \\
&= \Big[-\ln|u| \Big]_{\ln(\cos(0.1))}^{\ln(\cos(0.2))} \\
&= -\ln|\ln(\cos 0.2)| + \ln|\ln(\cos 0.1)| \\
&= \ln\left|\frac{\ln(\cos(0.1))}{\ln(\cos(0.2))}\right| \\
&= \ln\left(\frac{\ln(\cos(0.1))}{\ln(\cos(0.2))}\right)
\end{aligned}$$

Things to notice: the integrand is only defined when $\ln(\cos x)$ exists AND is nonzero. So, for instance, it is not defined when $x = 0$, because then $\ln\cos x = \ln 1 = 0$, and we can't divide by zero.

In the final simplification, since 0.1 and 0.2 are between 0 and $\pi/2$, the cosine term is positive but less than one, so $\ln(\cos 0.1)$ and $\ln(\cos 0.2)$ are both negative; then their quotient is positive, so we can drop the absolute value signs.

Using the base change formula, we can also write the final answer as $\ln\Big(\ln_{\cos(0.2)}\cos(0.1)\Big)$.

S-33: (a) Without any other ideas, we see we have a compound function–a function of a function. We often find it useful to substitute for the "inside" function. So, we substitute $u = \ln x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. Then $\mathrm{d}x = x\,\mathrm{d}u = e^u\,\mathrm{d}u$.

$$\int \sin(\ln x)\ \mathrm{d}x = \int \sin(u)\ e^u\ \mathrm{d}u$$

We have already seen, in Example 3.5.11 of the text, that

$$\int \sin(u)\ e^u\ \mathrm{d}u = \frac{1}{2}e^u\big(\sin u - \cos u\big) + C$$

So,

$$\int \sin(\ln x)\ \mathrm{d}x = \frac{1}{2}x\big[\sin(\ln x) - \cos(\ln x)\big] + C$$

(b) The integrand is of the form $N(x)/D(x)$ with $N(x)$ of lower degree than $D(x)$. So we factor $D(x) = (x-2)(x-3)$ and look for a partial fractions decomposition:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Multiplying through by the denominator yields

$$1 = A(x-3) + B(x-2)$$

Setting $x = 2$ we find:

$$1 = A(2-3) + 0 \implies A = -1$$

Setting $x = 3$ we find:

$$1 = 0 + B(3-2) \implies B = 1$$

So we have found that $A = -1$ and $B = 1$. Therefore

$$\begin{aligned}\int \frac{1}{(x-2)(x-3)}\,\mathrm{d}x &= \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right)\mathrm{d}x \\ &= \ln|x-3| - \ln|x-2| + C\end{aligned}$$

and the definite integral

$$\begin{aligned}\int_0^1 \frac{1}{(x-2)(x-3)}\,\mathrm{d}x &= \Big[\ln|x-3| - \ln|x-2|\Big]_0^1 \\ &= \big[\ln 2 - \ln 1\big] - \big[\ln 3 - \ln 2\big] \\ &= 2\ln 2 - \ln 3 = \ln\frac{4}{3}\end{aligned}$$

S-34: (a) If we expand the integrand, one part of it is quite familiar–a portion of a circle. So, we split the specified integral in two.

$$\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x = \int_0^3 \sqrt{9-x^2}\,\mathrm{d}x + \int_0^3 x\sqrt{9-x^2}\,\mathrm{d}x$$

The first piece represents the area above the x–axis and below the curve $y = \sqrt{9-x^2}$, i.e. $x^2 + y^2 = 9$, with $0 \leqslant x \leqslant 3$. That's the area of one quadrant of a disk of radius 3. So

$$\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x = \frac{1}{4}(\pi \cdot 3^2) = \frac{9}{4}\pi$$

For the second part, we substitute $u = 9 - x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$. Note $u(0) = 9$ and $u(3) = 0$. So,

$$\int_0^3 x\sqrt{9-x^2}\,\mathrm{d}x = \int_9^0 \sqrt{u}\,\frac{\mathrm{d}u}{-2} = -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_9^0 = -\frac{1}{2}\left[-\frac{27}{3/2}\right] = 9$$

All together,

$$\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x = \frac{9}{4}\pi + 9$$

(b) The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. We immediately look for a partial fractions decomposition:

$$\frac{4x+8}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}.$$

Multiplying through by the denominator yields

$$4x + 8 = A(x^2 + 4) + (Bx + C)(x - 2) \tag{$*$}$$

Setting $x = 2$ we find:

$$8 + 8 = A(4 + 4) + 0 \implies 16 = 8A \implies A = 2$$

Substituting $A = 2$ in $(*)$ gives

$$\begin{aligned} & & 4x + 8 &= A(x^2 + 4) + (Bx + C)(x - 2) \\ &\implies & -2x^2 + 4x &= (x - 2)(Bx + C) \\ &\implies & (-2x)(x - 2) &= (Bx + C)(x - 2) \\ &\implies & B &= -2,\ C = 0 \end{aligned}$$

So we have found that $A = 2$, $B = -2$, and $C = 0$. Therefore

$$\begin{aligned} \int \frac{4x + 8}{(x - 2)(x^2 + 4)}\, dx &= \int \left(\frac{2}{x - 2} - \frac{2x}{x^2 + 4} \right) dx \\ &= 2 \ln|x - 2| - \ln(x^2 + 4) + C \end{aligned}$$

Here the second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2 + 4$, $du = 2x\, dx$.

(c) The given integral is improper, but only because of its infinite limits of integration. (The integrand is continuous for all real numbers.) So, we'll have to take two limits. Before we do that, though, let's find the antiderivative. We would like to use the substitution $u = e^x$, $du = e^x\, dx$. That is, $\frac{1}{u}\, du = dx$.

$$\begin{aligned} \int \frac{1}{e^x + e^{-x}}\, dx &= \int \frac{1}{u(u + \frac{1}{u})}\, du = \int \frac{1}{u^2 + 1} = \arctan u + C \\ &= \arctan(e^x) + C \end{aligned}$$

Now we can deal with the limits of integration.

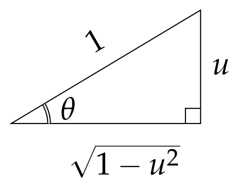
$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx &= \int_{-\infty}^{0} \frac{1}{e^x + e^{-x}}\, dx + \int_{0}^{\infty} \frac{1}{e^x + e^{-x}}\, dx \\ &= \lim_{a \to -\infty} \left[\int_a^0 \frac{1}{e^x + e^{-x}}\, dx \right] + \lim_{b \to \infty} \left[\int_0^b \frac{1}{e^x + e^{-x}}\, dx \right] \\ &= \lim_{a \to -\infty} \big[\arctan(e^x) \big]_a^0 + \lim_{b \to \infty} \big[\arctan(e^x) \big]_0^b \\ &= \lim_{a \to -\infty} \big[\arctan(e^0) - \arctan(e^a) \big] + \lim_{b \to \infty} \big[\arctan(e^b) - \arctan(e^0) \big] \\ &= \lim_{a \to -\infty} \big[-\arctan(e^a) \big] + \lim_{b \to \infty} \big[\arctan(e^b) \big] \\ &= -\arctan(0) + \frac{\pi}{2} = \frac{\pi}{2} \end{aligned}$$

S-35: It's not immediately clear where to start, but a common method we've seen is to use the denominator in a u-substitution, especially when square roots are involved.

Let $u = \sqrt{1-x}$, $\mathrm{d}u = -\frac{1}{2\sqrt{1-x}}\mathrm{d}x$. Then $u^2 = 1-x$, so $x = 1-u^2$.

$$\int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = 2\int \frac{\sqrt{x}}{2\sqrt{1-x}}\,\mathrm{d}x = -2\int \sqrt{1-u^2}\,\mathrm{d}u$$

Now we're back in familiar territory. Let $u = \sin\theta$, $\mathrm{d}u = \cos\theta\,\mathrm{d}\theta$.



$$\begin{aligned}
&= -2\int \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta \\
&= -2\int \cos^2\theta\,\mathrm{d}\theta \\
&= -\int \big(1+\cos(2\theta)\big)\,\mathrm{d}\theta \\
&= -\theta - \frac{1}{2}\sin(2\theta) + C \\
&= -\theta - \sin\theta\cos\theta + C \\
&= -\arcsin u - u\sqrt{1-u^2} + C \qquad (*) \\
&= -\arcsin(\sqrt{1-x}) - \sqrt{1-x}\sqrt{x} + C
\end{aligned}$$

In $(*)$, to convert from θ to u, our substitution $u = \sin\theta$ tells us $\theta = \arcsin u$. To find $\cos\theta$, we can either trace our work backwards to see that we already simplified $\sqrt{1-u^2}$ into $\cos\theta$, or we can draw a right triangle with angle θ and $\sin\theta = u$, then use the Pythagorean theorem to find the length of the adjacent side of the triangle and $\cos\theta$.

S-36: Let's use the substitution $u = e^x$. There are a few reasons to think this is a good choice. It's an "inside function," in that if we let $f(x) = e^x$, then $f(e^x) = e^{e^x}$, which is a piece of our integrand. Also its derivative, e^x, is multiplied by the rest of the integrand, since $e^{2x} = e^x \cdot e^x$.

Let $u = e^x$, $\mathrm{d}u = e^x\mathrm{d}x$. When $x = 0$, $u = 1$, and when $x = 1$, $u = e$.

$$\int_0^1 e^{2x}e^{e^x}\mathrm{d}x = \int_0^1 e^x e^{e^x} e^x\mathrm{d}x = \int_1^e ue^u\,\mathrm{d}u$$

This is more familiar. We use integration by parts with $\mathrm{d}v = e^u\,\mathrm{d}u$, $v = e^u$. Conveniently, the "u" we brought in with the substitution is what we want to use for the "u" in integration by parts, so we don't have to change the names of our variables.

$$\begin{aligned}
&= \big[ue^u\big]_1^e - \int_1^e e^u\,\mathrm{d}u \\
&= e\cdot e^e - e - e^e + e = e^e(e-1)
\end{aligned}$$

S-37: The substitution $u = x+1$ looks promising at first, but doesn't result in something easily integrable. We can't use partial fractions because our integration isn't rational. This doesn't look like something from the trig-substitution family. So, let's think about integration by parts. There's a lot of different ways we could break up the integrand into two parts. For example, we could view it as $\left(\frac{x}{(x+1)^2}\right)\left(e^x\right)$, or we could view it as $\left(\frac{x}{x+1}\right)\left(\frac{e^x}{x+1}\right)$. After some trial and error, we settle on $u = xe^x$ and $\mathrm{d}v = (x+1)^{-2}\,\mathrm{d}x$. Then $\mathrm{d}u = e^x(x+1)$ and $v = \frac{-1}{x+1}$.

$$\begin{aligned}
\int \frac{xe^x}{(x+1)^2}\,\mathrm{d}x &= -\frac{xe^x}{x+1} + \int \frac{e^x(x+1)}{x+1}\mathrm{d}x \\
&= -\frac{xe^x}{x+1} + \int e^x \mathrm{d}x \\
&= -\frac{xe^x}{x+1} + e^x + C \\
&= \frac{e^x}{x+1} + C
\end{aligned}$$

S-38: It would be nice to use integration by parts with $u = x$, because then we would integrate $\int v\,\mathrm{d}u$, and $\mathrm{d}u = \mathrm{d}x$. That is, the x would go away, and we'd be left with a pure trig integral. If we use $u = x$, then $\mathrm{d}v = \frac{\sin x}{\cos^2 x}$. We need to find v:

$$v = \int \frac{\sin x}{\cos^2 x}\,\mathrm{d}x = \int \tan x \sec x\,\mathrm{d}x = \sec x$$

Now we use integration by parts.

$$\int \frac{x\sin x}{\cos^2 x}\mathrm{d}x = x\sec x - \int \sec x\,\mathrm{d}x = x\sec x - \ln|\sec x + \tan x| + C$$

S-39: If the unknown exponent gives you the jitters, think about what this looks like in easier cases. If n is a whole number, the integrand is a polynomial. Not so scary, right? However, it's a little complicated to expand. (You can do it using the very handy binomial theorem.) Let's think of an easier way.

If we had simply the variable x raised to the power n, rather than the binomial $x+a$, that might be nicer. So, let's use the substitution $u = x+a$, $\mathrm{d}u = \mathrm{d}x$. Note $x = u - a$.

$$\int x(x+a)^n\,\mathrm{d}x = \int (u-a)u^n\,\mathrm{d}x = \int \left(u^{n+1} - au^n\right)\mathrm{d}u$$

Now, if $n \neq -1$ and $n \neq -2$, we can just use the power rule:

$$\begin{aligned}
&= \frac{u^{(n+2)}}{n+2} - a\frac{u^{n+1}}{n+1} + C \\
&= \frac{(x+a)^{(n+2)}}{n+2} - a\frac{(x+a)^{n+1}}{n+1} + C
\end{aligned}$$

If $n = -1$, then

$$\int x(x+a)^n \, \mathrm{d}x = \int \left(u^{n+1} - au^n\right) \mathrm{d}u = \int \left(1 - \frac{a}{u}\right) \mathrm{d}u$$
$$= u - a\ln|u| + C = (x+a) - a\ln|x+a| + C$$

If $n = -2$, then

$$\int x(x+a)^n \, \mathrm{d}x = \int \left(u^{n+1} - au^n\right) \mathrm{d}u = \int \left(\frac{1}{u} - au^{-2}\right) \mathrm{d}u$$
$$= \ln|u| + \frac{a}{u} + C = \ln|x+a| + \frac{a}{x+a} + C$$

All together,

$$\int x(x+a)^n \, \mathrm{d}x = \begin{cases} \frac{(x+a)^{(n+2)}}{n+2} - a\frac{(x+a)^{n+1}}{n+1} + C & \text{if } n \neq -1, -2 \\ (x+a) - a\ln|a+x| + C & \text{if } n = -1 \\ \ln|x+a| + \frac{a}{x+a} + C & \text{if } n = -2 \end{cases}$$

S-40: We've seen how to antidifferentiate $\arctan x$: integration by parts. Let's hope the same thing will work here.

Step 1: integration by parts.
Let $u = \arctan(x^2)$ and $\mathrm{d}v = \mathrm{d}x$. Then $\mathrm{d}u = \frac{2x}{x^4+1}\,\mathrm{d}u$ and $v = x$.

$$\int \arctan(x^2)\, \mathrm{d}x = x\arctan(x^2) - \int \frac{2x^2}{x^4+1}\, \mathrm{d}x$$

Now we have a rational function. There's no obvious substitution, but we can use partial fractions. The degree of the numerator is strictly less than the degree of the denominator, so we don't need to long divide first. We do, however, need to factor the denominator. It's a common function, so you might already know the factorization, or you might be able to guess it. Below, we show another way to find the factorization, similar to the method of partial fractions.

Step 2: factor $\mathbf{x^4 + 1}$.
For any real x, note $x^4 + 1 > 0$. Since it has no roots, it has no linear factors. That means it factors as the product of two irreducible quadratics. That is,

$$x^4 + 1 = (ax^2 + bx + c)(dx^2 + ex + f)$$

Since the coefficient of x^4 on the left-hand is 1, we may assume $a = d = 1$.

$$x^4 + 1 = (x^2 + bx + c)(x^2 + ex + f)$$

Since the constant term is 1, $cf = 1$. That is, $f = \frac{1}{c}$.

$$x^4 + 1 = (x^2 + bx + c)(x^2 + ex + 1/c)$$
$$= x^4 + \underbrace{(b+e)}_{(1)} x^3 + \underbrace{\left(\frac{1}{c} + be + c\right)}_{(3)} x^2 + \underbrace{\left(\frac{b}{c} + ec\right)}_{(2)} x + 1$$

(1) The coefficient of x^3 tells us $e = -b$.

(2) Then the coefficient of x tells us $0 = \frac{b}{c} + ec = \frac{b}{c} - bc$. So, $c = \frac{1}{c}$, hence $c = \pm 1$.

(3) Finally, the coefficient of x^2 tells us $0 = \frac{1}{c} + be + c = \frac{1}{c} - b^2 + c$. Since $-b^2$ is negative (or zero), $\frac{1}{c} + c$ is positive, so $c = 1$. That is, $0 = 1 - b^2 + 1$. So, $b = \sqrt{2}$.

All together,

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

Step 3: partial fraction decomposition.
Now that we have the denominator factored into irreducible quadratics, we can find the partial fraction decomposition of the integrand.

$$\begin{aligned}
\frac{2x^2}{x^4+1} &= \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \\
2x^2 &= (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1) \\
&= (A+C)x^3 + (B+D-\sqrt{2}A+\sqrt{2}C)x^2 + (A+C-\sqrt{2}B+\sqrt{2}D)x + (B+D)
\end{aligned}$$

From the coefficient of x^3, we see $C = -A$.

$$2x^2 = (B+D-2\sqrt{2}A)x^2 + (-\sqrt{2}B+\sqrt{2}D)x + (B+D)$$

From the constant term, we see $D = -B$.

$$2x^2 = (-2\sqrt{2}A)x^2 + (-2\sqrt{2}B)x$$

From the coefficient of x^2, we see $-2\sqrt{2}A = 2$, so $A = -1/\sqrt{2}$. Since $C = -A$, then $C = 1/\sqrt{2}$.

From the coefficient of x, we see $B = 0$. Since $D = -B$, also $D = 0$.

Step 4: integration.

$$\begin{aligned}
\int \frac{2x^2}{x^4+1}\,dx &= \int \left(\frac{(-1/\sqrt{2})x}{x^2+\sqrt{2}x+1} + \frac{(1/\sqrt{2})x}{x^2-\sqrt{2}x+1} \right) dx \\
&= \frac{1}{\sqrt{2}} \int \left(\frac{-x}{x^2+\sqrt{2}x+1} + \frac{x}{x^2-\sqrt{2}x+1} \right) dx
\end{aligned}$$

To integrate, we want to break the fractions into two pieces each: one we can integrate with a substitution $u = x^2 \pm \sqrt{2}x + 1$, $du = \left(2x \pm \sqrt{2}\right)dx$ (shown in blue), and one that looks like the derivative of arctangent (shown in red).

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \int \left(\frac{-x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{x^2+\sqrt{2}x+1} + \frac{x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{x^2-\sqrt{2}x+1} \right) dx \\
&= \frac{1}{\sqrt{2}} \int \left(\frac{-\frac{1}{2}(2x+\sqrt{2})}{x^2+\sqrt{2}x+1} + \frac{\frac{\sqrt{2}}{2}}{x^2+\sqrt{2}x+1} + \frac{\frac{1}{2}(2x-\sqrt{2})}{x^2-\sqrt{2}x+1} + \frac{\frac{\sqrt{2}}{2}}{x^2-\sqrt{2}x+1} \right) dx \\
&= \frac{1}{\sqrt{2}} \left(-\frac{1}{2}\ln\left|x^2+\sqrt{2}x+1\right| + \int \frac{\frac{\sqrt{2}}{2}}{x^2+\sqrt{2}x+1}\,dx \right. \\
&\qquad \left. + \frac{1}{2}\ln|x^2-\sqrt{2}x+1| + \int \frac{\frac{\sqrt{2}}{2}}{x^2-\sqrt{2}x+1}\,dx \right)
\end{aligned}$$

We use logarithm rules to compress our work. In order to evaluate the remaining integrals, we complete the squares of the denominators.

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\ln\left|\tfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + \int \frac{\frac{\sqrt{2}}{2}}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\,\mathrm{d}x + \int \frac{\frac{\sqrt{2}}{2}}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\mathrm{d}x\right)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\ln\left|\tfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + \int \frac{\sqrt{2}}{(\sqrt{2}x+1)^2+1}\,\mathrm{d}x + \int \frac{\sqrt{2}}{(\sqrt{2}x-1)^2+1}\mathrm{d}x\right)$$

Now, we can either guess the antiderivatives of the remaining integrals, or use the substitutions $u = (\sqrt{2}x \pm 1)$.

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\ln\left|\tfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + \arctan\left(\sqrt{2}x+1\right) + \arctan\left(\sqrt{2}x-1\right)\right) + C$$

Step 5: finishing touches.
Finally, we can put our work together. (Remember way back in Step 1, we used integration by parts.)

$$\int \arctan(x^2)\,\mathrm{d}x = x\arctan(x^2) - \int \frac{2x^2}{x^4-1}\,\mathrm{d}x$$
$$= x\arctan(x^2) - \frac{1}{\sqrt{2}}\left(\frac{1}{2}\ln\left|\tfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + \arctan\left(\sqrt{2}x+1\right) + \arctan\left(\sqrt{2}x-1\right)\right) + C$$

Remark: although this integral calculation was longer than average, it didn't use any new ideas (except for the factoring of x^4+1 mentioned in the hint). It's good exercise to apply familiar techniques in challenging situations, to deepen your mastery.

Solutions to Exercises 3.12 — Jump to TABLE OF CONTENTS

S-1:

(a) If $y = 5(e^x - 3x^2 - 6x - 6)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 5(e^x - 6x - 6)$. Let's see whether this is equal to $y + 15x^2$:

$$\begin{aligned} y + 15x^2 &= 5(e^x - 3x^2 - 6x - 6) + 15x^2 \\ &= 5(e^x - 3x^2 - 6x - 6 + 3x^2) \\ &= 5(e^x - 6x - 6) \\ &= \frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

So, $y = 5(e^x - 3x^2 - 6x - 6)$ is indeed a solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y + 15x^2$.

(b) If $y = \dfrac{-2}{x^2+1}$, then $\frac{dy}{dx} = \dfrac{4x}{(x+1)^2}$. Let's see whether this is equal to xy^2:

$$\begin{aligned} xy^2 &= x\left(\frac{-2}{x^2+1}\right)^2 \\ &= \frac{4x}{(x^2+1)^2} \\ &= \frac{dy}{dx} \end{aligned}$$

So, $y = \dfrac{-2}{x^2+1}$ is indeed a solution to the differential equation $\frac{dy}{dx} = yx^2$.

(c) If $y = x^{3/2} + x$, then $\frac{dy}{dx} = \frac{3}{2}\sqrt{x} + 1$.

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} &= \left(\frac{3}{2}\sqrt{x}+1\right)^2 + \frac{3}{2}\sqrt{x}+1 \\ &= \frac{9}{4}x + \frac{9}{2}\sqrt{x} + 2 \\ &\neq \frac{dy}{dx} \end{aligned}$$

So, $y = x^{3/2} + x$ is not a solution to the differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = y$.

S-2:

(a) $3y\frac{dy}{dx} = x\sin y$ can be written as $\frac{3y}{\sin y}\frac{dy}{dx} = x$, which fits the form of a separable equation with $f(x) = x$, $g(y) = \frac{3y}{\sin y}$.

(b) $\frac{dy}{dx} = e^{x+y} = e^x e^y$, so $e^{-y}\frac{dy}{dx} = e^x$ which fits the form of a separable equation using $f(x) = e^x$, $g(y) = e^{-y}$.

(c) $\frac{dy}{dx} + 1 = x$ can be written as $\frac{dy}{dx} = (x-1)$, which fits the form of a separable equation using $f(x) = x-1$, $g(y) = 1$. (We can solve it by simply antidifferentiating.)

(d) Notice the left side of the equation $\left(\frac{dy}{dx}\right)^2 - 2x\frac{dy}{dx} + x^2 = 0$ is a perfect square. So, this equation is equivalent to $\left(\frac{dy}{dx} - x\right)^2 = 0$, that is, $\frac{dy}{dx} = x$. This has the form of a separable equation with $f(x) = x$, $g(y) = 1$.

S-3: The mnemonic allows us to skip from the separable differential equation we want to solve (very first line) to the equation

$$\int g(y)\,dy = \int f(x)\,dx$$

So, the mnemonic is just a shortcut for the substitution we performed to get this point.

We also generally skip the explanation about C_1 and C_2 being replaced with C.

S-4: To say $y = f(x) + C$ is a solution to the differential equation means:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) + C\} = x(f(x) + C)$$

Since $y = f(x)$ is a solution, we know $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)\} = xf(x)$. Also, $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) + C\} = \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)\}$. So, $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) + C\} = xf(x)$.

$$\begin{aligned} xf(x) &= x(f(x) + C) \\ 0 &= xC \end{aligned}$$

Our equation should hold for all x in our domain, and for the derivative to y with respect to x to make sense, our domain should not be a single point. So, there is some x in our domain such that $x \neq 0$. Therefore, the C *must be zero*. So, $f(x) + C$ is *not* a solution to the differential equation for any constant C.

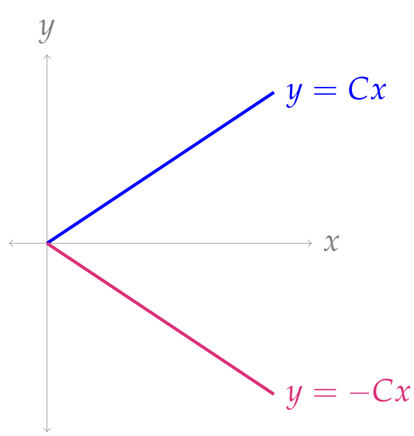
When we're finding a general antiderivative, we add "$+C$" at the end. When we're finding a general solution to a differential equation, the "$+C$" gets added when we antidifferentiate–we don't add another one at the end of our work.

S-5:

(a) Since $|y| \geqslant 0$ no matter what y is, we see $Cx \geqslant 0$ for all x in the domain of $f(x)$. Since C is positive, that means the domain of $f(x)$ only includes nonnegative numbers. So, the largest possible domain of $f(x)$ is $[0, \infty)$.

(b) None exists.

The graph of Cx is given below for some positive constant C, also with the graph of $-Cx$. If $y = f(x)$ were sometimes the top function, and other times the bottom function, then there would be a jump discontinuity where it switched. Then the derivative of $f(x)$ would not exist, violating the second property.



A tiny technical note is that it's possible that $f(x) = Cx$ when $x = 0$ and $f(x) = -Cx$ when $x > 0$ (or vice-versa). This would not introduce a jump discontinuity, but it also does not satisfy that $f(x) > 0$ for some values of x.

Remark: in several instances below, solving a differential equation will lead us to conclude something like $|y| = g(x)$. In these cases, we choose either $y = g(x)$, or

$y = -g(x)$, but not $y = \pm g(x)$ (which is not a function) or that y is sometimes $g(x)$, and other times $-g(x)$. The reasoning above somewhat explains this choice: if y were sometimes positive and sometimes negative, then $\frac{\mathrm{d}y}{\mathrm{d}x}$ would not exist at the values of x where the sign of y switches, unless that switch occurrs at a root of $g(x)$. Since that's a pretty specific occurrence, we usually feel safe ignoring it to avoid getting bogged down in technical details.

S-6: Let $Q(t)$ be the quantity of morphine in a patient's bloodstream at time t, where t is measured in minutes.

Using the definition of a derivative,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \lim_{h \to 0} \frac{Q(t+h) - Q(t)}{h} \approx \frac{Q(t+1) - Q(t)}{1}$$

So, $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is roughly the change in the amount of morphine in one minute, from t to $t+1$.

The sentence tells us that the change in the amount of morphine in one minute is about $-0.003Q$, where Q is the quantity in the bloodstream. That is:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -0.003Q(t)$$

S-7: If $p(t)$ is the proportion of times speakers use the new form, measured between 0 and 1, then $1 - p(t)$ is the proportion of times speakers use the old form.

The law, then, states that $\frac{\mathrm{d}p}{\mathrm{d}t}$ is proportional to $p(t) \times \big(1 - p(t)\big)$. When we say two quantities are proportional, we mean that one is a constant multiple of the other. So, the law says

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \alpha p(t)\big(1 - p(t)\big)$$

for some constant α.

Remark: it follows from this model that, when a new form is either very rare or entirely ubiquitous, the rate of change of its adoption is small. This makes sense: if the new form is used all the time ($p(t) \approx 1$), there's nobody left to convert; if the new form is almost never used ($p(t) \approx 0$) then people don't know about it, so they won't pick it up.

S-8:

(a) When $y = 0$, $y' = \frac{0}{2} - 1 = -1$.

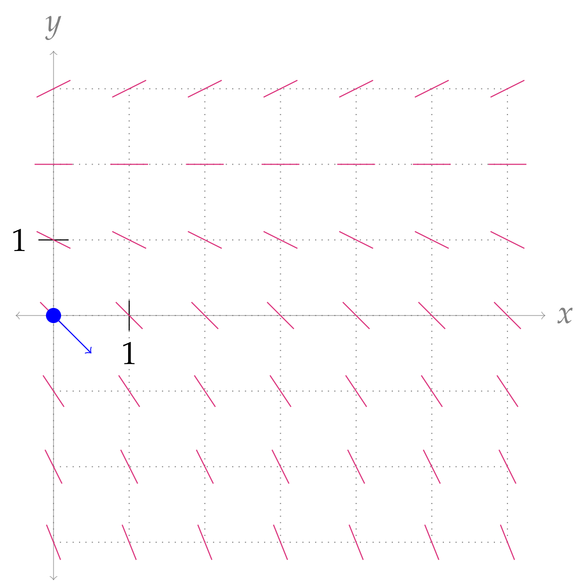
(b) When $y = 2$, $y' = \frac{2}{2} - 1 = 0$.

(c) When $y = 3$, $y' = \frac{3}{2} - 1 = 0.5$.

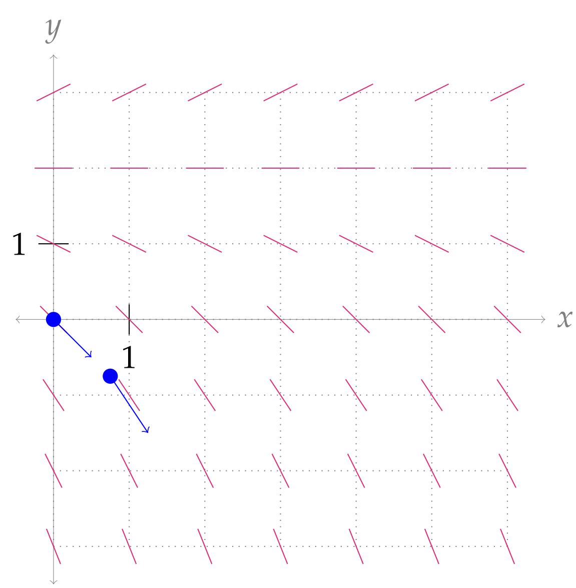
(d) The small red lines have varying slopes. The red lines on points with y-coordinate 2 have slopes of 0; this matches y' when $y = 0$, as we saw above. The red lines on points with y-coordinate 0 have slopes of approximately -1; again, this matches what we found for y' when $y = 0$.

The red lines correspond to a tiny section of $y(x)$, *if* $y(x)$ passes through that point. So, we can sketch a possible curve $y(x)$ satisfying the equation by starting somewhere, then following the slopes.

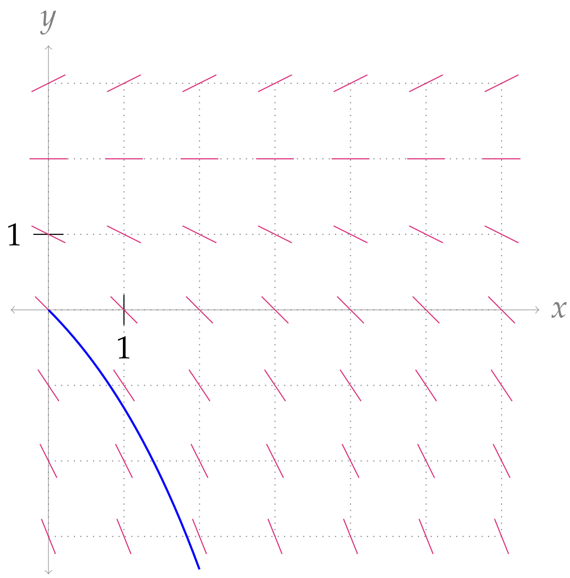
For example, suppose we start at the origin.



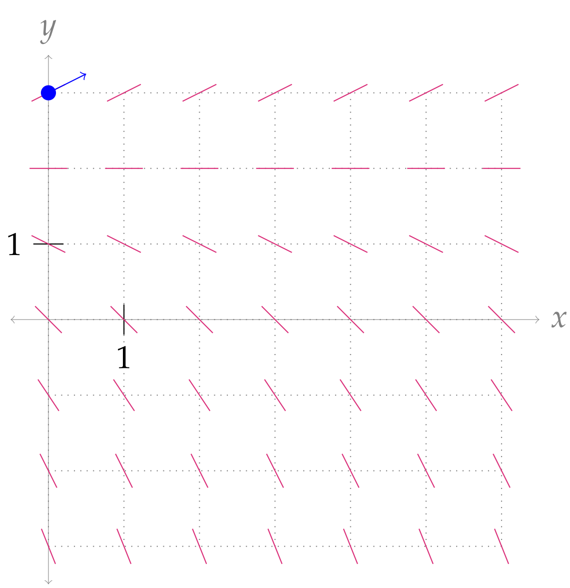
Then our function is decreasing at that point, which leads us to a coordinate where (as we see from the red marks) the function is decreasing slightly faster.



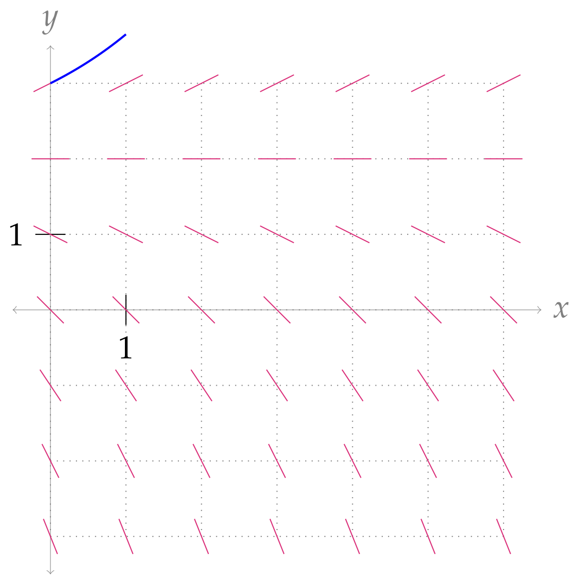
Following the red marks leads us down even further, so our function $y(x)$ might look something like this:



However, we didn't have to start at the origin. Suppose $y(0) = 3$. Then at $x = 0$, y is increasing, with slope $\frac{1}{2}$.



Our red marks run out that high up, but we now $y' = \frac{1}{2}y - 1$, so y' increases as y increases. That means our function keeps getting steeper and steeper, possibly something like this:



If $y(0) = 2$, we see another possible curve is the constant function $y(x) = 2$.

Remark: from Theorem 3.12.10 in the text, we see the solutions to the equation $y' = \frac{1}{2}y - 1 = \frac{1}{2}(y-2)$ are of the form $y(x) = Ce^{x/2} + 2$ for some constant C. Check that the curves you're sketching look exponential.

S-9: By Theorem 3.12.10 in the text, the function

$$y(t) = (y(0) - b)e^{at} + b$$

is the only function that satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(y-b)\ .$$

In our case, $\frac{\mathrm{d}y}{\mathrm{d}t} = 5y - 7 = 5\left(y - \frac{7}{5}\right)$, so $a = 5$ and $b = \frac{7}{5}$. Then

$$\begin{aligned} y(t) &= (y(0) - b)e^{at} + b \\ &= \left(-3 - \frac{7}{5}\right)e^{5t} + \frac{7}{5} \\ &= \frac{7}{5} - \frac{22}{5}e^{5t} \end{aligned}$$

S-10: By Theorem 3.12.10 in the text, the function

$$y(t) = (y(0) - b)e^{at} + b$$

is the only function that satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(y-b)\ .$$

In our case, $\frac{dy}{dt} = 1 + 2y = 2\left(y - \left(-\frac{1}{2}\right)\right)$, so $a = 2$ and $b = -\frac{1}{2}$. Then

$$\begin{aligned} y(t) &= (y(0) - b)e^{at} + b \\ &= \left(0 + \frac{1}{2}\right)e^{2t} - \frac{1}{2} \\ &= \frac{1}{2}\left(e^{2t} - 1\right) \end{aligned}$$

S-11: By Theorem 3.12.10 in the text, the function

$$y(t) = (y(0) - b)e^{at} + b$$

is the only function that satisfies the differential equation

$$\frac{dy}{dt} = a(y - b)\ .$$

In our case, $\frac{dy}{dt} = 2y + 3 = 2\left(y + \frac{3}{2}\right)$, so $a = 2$ and $b = -\frac{3}{2}$. Then

$$\begin{aligned} y(t) &= (y(0) - b)e^{at} + b \\ &= \left(y(0) + \frac{3}{2}\right)e^{2t} - \frac{3}{2} \end{aligned}$$

We're given $y(1) = 1$.

$$\begin{aligned} 1 = y(1) &= \left(y(0) + \frac{3}{2}\right)e^{2\cdot 1} - \frac{3}{2} \\ \frac{5}{2} &= \left(y(0) + \frac{3}{2}\right)e^{2} \\ \frac{5}{2e^2} &= y(0) + \frac{3}{2} \\ \frac{5}{2e^2} - \frac{3}{2} &= y(0) \end{aligned}$$

So,

$$\begin{aligned} y(t) &= \left(\frac{5}{2e^2} - \frac{3}{2} + \frac{3}{2}\right)e^{2t} - \frac{3}{2} \\ &= \frac{5}{2e^2}e^{2t} - \frac{3}{2} \end{aligned}$$

Or, equivalently,

$$y(t) = \frac{5}{2}e^{2(t-1)} - \frac{3}{2}$$

S-12: The steady-state solution is the constant solution $y = b$. This has derivative zero, so

$$0 = 3y - 7$$

tells us $y = \frac{7}{3}$.

S-13: A steady-state solution is a constant solution $y = b$. This has derivative zero, so

$$0 = y(y^2 - 1)$$

Factoring the right-hand side, we get

$$0 = y(y+1)(y-1)$$

So the steady-state solutions are $y = 0$, $y = 1$, and $y = -1$.

S-14: Rearranging, we have:

$$e^y \, dy = 2x \, dx.$$

Integrating both sides:

$$\begin{aligned} \int e^y \, dy &= \int 2x \, dx \\ e^y &= x^2 + C \end{aligned}$$

Since $y = \ln 2$ when $x = 0$, we have

$$\begin{aligned} e^{\ln 2} &= 0^2 + C \\ 2 &= C, \end{aligned}$$

and therefore

$$\begin{aligned} e^y &= x^2 + 2 \\ y &= \ln(x^2 + 2) \end{aligned}$$

S-15: Using separation of variables:

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy}{x^2+1} \\ \frac{dy}{y} &= \frac{x}{x^2+1} dx \\ \int \frac{dy}{y} &= \int \frac{x}{x^2+1} dx \\ \ln|y| &= \frac{1}{2}\ln(1+x^2) + C \end{aligned}$$

To satisfy $y(0) = 3$, we need $\ln 3 = \frac{1}{2}\ln(1+0) + C$, so $C = \ln 3$. Thus:

$$\begin{aligned}
\ln|y| &= \frac{1}{2}\ln(1+x^2) + \ln 3 \\
&= \ln\sqrt{1+x^2} + \ln 3 \\
&= \ln 3\sqrt{1+x^2}
\end{aligned}$$

So,

$$|y| = 3\sqrt{1+x^2}$$

We are told to find a function $y(x)$. So far, we have two possible functions from the work above: maybe $y = 3\sqrt{1+x^2}$, and maybe $y = -3\sqrt{1+x^2}$. It's important to note that $y = \pm 3\sqrt{1+x^2}$ is not a function: for an equation to represent a function, for every input in the domain, there must only be one output. That is, functions pass the vertical line test. So, we need to decide whether our function is $y = 3\sqrt{1+x^2}$ or $y = -3\sqrt{1+x^2}$. Since $y(0) = 3$, we conclude

$$y(x) = 3\sqrt{1+x^2}$$

S-16: The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
y' &= e^{\frac{y}{3}}\cos t \\
e^{-y/3}\mathrm{d}y &= \cos t\,\mathrm{d}t \\
\int e^{-y/3}\mathrm{d}y &= \int \cos t\,\mathrm{d}t \\
-3e^{-y/3} &= \sin t + C \\
\frac{1}{e^{y/3}} &= \frac{\sin t + C}{-3} \\
e^{y/3} &= \frac{-3}{C + \sin t} \\
\frac{y}{3} &= \ln\left(\frac{-3}{C+\sin t}\right) \\
y(t) &= 3\ln\left(\frac{-3}{C+\sin t}\right)
\end{aligned}$$

for any constant C.

Since the domain of logarithm is $(0, \infty)$, the solution only exists when $C + \sin t < 0$.

S-17: The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= xe^{x^2 - \ln(y^2)} = \frac{xe^{x^2}}{y^2} \\
y^2\mathrm{d}y &= xe^{x^2}\,\mathrm{d}x \\
\int y^2\mathrm{d}y &= \int xe^{x^2}\,\mathrm{d}x
\end{aligned}$$

We can guess the antiderivative of xe^{x^2}, or use the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$.

$$\begin{aligned} \frac{y^3}{3} &= \frac{1}{2}e^{x^2} + C' \\ y^3 &= \frac{3}{2}e^{x^2} + 3C' \end{aligned}$$

Since C' can be any constant in $(-\infty, \infty)$, then also $3C'$ can be any constant in $(-\infty, \infty)$, so we replace $3C'$ with the arbitrary constant C.

$$\begin{aligned} y^3 &= \frac{3}{2}e^{x^2} + C \\ y &= \sqrt[3]{\frac{3}{2}e^{x^2} + C} \end{aligned}$$

for any constant C.

S-18: The given differential equation is separable and we solve it accordingly.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= xe^y \\ \frac{\mathrm{d}y}{e^y} &= x\,\mathrm{d}x \\ \int \frac{\mathrm{d}y}{e^y} &= \int x\,\mathrm{d}x \\ -e^{-y} &= \frac{1}{2}x^2 + C \\ e^{-y} &= -\frac{1}{2}x^2 - C \end{aligned}$$

Since C can be any constant in $(-\infty, \infty)$, then also $-C$ can be any constant in $(-\infty, \infty)$, so we write C instead of $-C$.

$$\begin{aligned} e^{-y} &= C - \frac{1}{2}x^2 \\ -y &= \ln\left(C - \frac{x^2}{2}\right) \\ y &= -\ln\left(C - \frac{x^2}{2}\right) \end{aligned}$$

for any constant C.

The solution only exists for $C - \frac{x^2}{2} > 0$. For this to happen, we need $C > 0$, and then the domain of the function is those values x for which $|x| < \sqrt{2C}$.

S-19: The given differential equation is separable and we solve it accordingly.
Cross–multiplying, we rewrite the equation as

$$\begin{aligned} y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= e^x - 2x \\ y^2\,\mathrm{d}y &= (e^x - 2x)\,\mathrm{d}x. \end{aligned}$$

Integrating both sides, we find

$$\int y^2\,\mathrm{d}y = \int (e^x - 2x)\,\mathrm{d}x$$
$$\frac{1}{3}y^3 = e^x - x^2 + C$$

Setting $x = 0$ and $y = 3$, we find $\frac{1}{3}3^3 = e^0 - 0^2 + C$ and hence $C = 8$.

$$\frac{1}{3}y^3 = e^x - x^2 + 8$$
$$y = (3e^x - 3x^2 + 24)^{1/3}$$

S-20: This is a separable differential equation that we solve in the usual way.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -xy^3 \\
-\frac{\mathrm{d}y}{y^3} &= x\,\mathrm{d}x \\
\int -\frac{\mathrm{d}y}{y^3} &= \int x\,\mathrm{d}x \\
-\frac{y^{-2}}{-2} &= \frac{x^2}{2} + C \\
y^{-2} &= x^2 + 2C. \qquad (*)
\end{aligned}$$

To have $y = -\frac{1}{4}$ when $x = 0$, we must choose C to obey

$$\begin{aligned}
\left(-\frac{1}{4}\right)^{-2} &= 0 + 2C \\
16 &= 2C
\end{aligned}$$

So, from $(*)$,

$$\begin{aligned}
y^{-2} &= x^2 + 2C = x^2 + 16 \\
y^2 &= \frac{1}{x^2+16}
\end{aligned}$$

Now, we have two potential candidates for $y(x)$:

$$y = \frac{1}{\sqrt{x^2+16}} \qquad \text{OR} \qquad y = -\frac{1}{\sqrt{x^2+16}}$$

We know $y = -\frac{1}{4}$ when $x = 0$. The only function above that fits this is

$$y = -\frac{1}{\sqrt{x^2+16}}$$

So, $f(x) = -\dfrac{1}{\sqrt{x^2+16}}$.

S-21: This is a separable differential equation that we solve in the usual way. Cross-multiplying and integrating,

$$\begin{aligned}
y\,\mathrm{d}y &= (15x^2+4x+3)\,\mathrm{d}x \\
\int y\,\mathrm{d}y &= \int (15x^2+4x+3)\,\mathrm{d}x \\
\frac{y^2}{2} &= 5x^3+2x^2+3x+C.
\end{aligned}$$

Plugging in $x=1$ and $y=4$ gives $\frac{4^2}{2}=5+2+3+C$, and so $C=-2$. Therefore

$$\begin{aligned}
\frac{y^2}{2} &= 5x^3+2x^2+3x-2 \\
y^2 &= 10x^3+4x^2+6x-4
\end{aligned}$$

This leaves us with two possible functions for y:

$$y=\sqrt{10x^3+4x^2+6x-4} \qquad \text{OR} \qquad y=-\sqrt{10x^3+4x^2+6x-4}$$

When $x=1$, $y=4$. This only fits the first equation, so

$$y=\sqrt{10x^3+4x^2+6x-4}$$

S-22: The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x^3y \\
\frac{\mathrm{d}y}{y} &= x^3\,\mathrm{d}x \\
\int \frac{\mathrm{d}y}{y} &= \int x^3\,\mathrm{d}x \\
\ln|y| &= \frac{x^4}{4}+C \\
|y| &= e^{x^4/4+C} = e^{x^4/4}e^C
\end{aligned}$$

We are told that $y=1$ when $x=0$. That is, $1=e^0e^C$, so $e^C=1$. That is, $C=0$.

$$|y|=e^{x^4/4}$$

This leaves us with two potential functions:

$$y=e^{x^4/4} \qquad \text{OR} \qquad y=-e^{x^4/4}$$

The first is always positive, and the second is always negative. Since $y=1$ (a positive number) when $x=0$, we see

$$y=e^{x^4/4}$$

S-23: This is a separable differential equation, even if it doesn't quite look like it. First move the y from the left hand side to the right hand side.

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} + y &= y^2 \\ x\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 - y = y(y-1) \\ \frac{\mathrm{d}y}{y(y-1)} &= \frac{\mathrm{d}x}{x} \end{aligned}$$

Using the method of partial fractions, we see $\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}$.

$$\begin{aligned} \left(\frac{1}{y-1} - \frac{1}{y}\right)\,\mathrm{d}y &= \frac{\mathrm{d}x}{x} \\ \int\left(\frac{1}{y-1} - \frac{1}{y}\right)\,\mathrm{d}y &= \int\frac{\mathrm{d}x}{x} \\ \ln|y-1| - \ln|y| &= \ln|x| + C \\ \ln\frac{|y-1|}{|y|} &= \ln|x| + C \qquad (*) \end{aligned}$$

To determine C we set $x = 1$ and $y = -1$.

$$\begin{aligned} \ln\frac{|-2|}{|-1|} &= \ln|1| + C \\ \ln 2 &= C \end{aligned}$$

Returning to $(*)$,

$$\begin{aligned} \ln\frac{|y-1|}{|y|} &= \ln|x| + \ln 2 \\ \ln\left|\frac{y-1}{y}\right| &= \ln|2x| \\ \left|\frac{y-1}{y}\right| &= |2x| \end{aligned}$$

As $y(1) = -1$ is an initial condition, we have that $x \geqslant 1$ and $|2x| = 2x$. For $x = 1$, we have $y = -1$. So at least for x near 1, we have y near -1, so that $\frac{y-1}{y}$ is positive and we may drop the absolute value signs. There remains the possibility that $\frac{y(x)-1}{y(x)}$ changes sign for some larger $x > 1$. For now, we will simply ignore that possibility. At the end, we will explicitly check that the $y(x)$ we come up with really does satisfy the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2$ and the initial condition $y(1) = -1$.

$$\begin{aligned} \frac{y-1}{y} &= 2x \\ y - 1 &= 2xy \\ y - 2xy &= 1 \\ y(1-2x) &= 1 \\ y &= \frac{1}{1-2x} \end{aligned}$$

As a check, we compute:

$$\begin{aligned}
x\frac{\mathrm{d}y}{\mathrm{d}x} + y &= x\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-2x}\right\} + y \\
&= x\frac{2}{(1-2x)^2} + \frac{1}{1-2x} \\
&= \frac{2x + (1-2x)}{(1-2x)^2} \\
&= \frac{1}{(1-2x)^2} \\
&= y^2
\end{aligned}$$

So, our differential equation is satisfied. Furthermore:

$$y(1) = \frac{1}{1-2\times 1} = -1$$

as desired. This confirms that our solution is correct.

S-24: The unknown function $f(x)$ satisfies an equation that involves the derivative of f. That means we're in differential equation territory. Specifically, we are told that $y = f(x)$ obeys the separable differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= xy \\
\frac{\mathrm{d}y}{y} &= x\,\mathrm{d}x \\
\int \frac{\mathrm{d}y}{y} &= \int x\,\mathrm{d}x \\
\ln|y| &= \frac{x^2}{2} + C
\end{aligned}$$

To determine C we set $x = 0$ and $y = e$.

$$\begin{aligned}
\ln e &= \frac{0^2}{2} + C \\
1 &= C
\end{aligned}$$

So, the solution is

$$\ln|y| = \frac{x^2}{2} + 1$$

We are told that $y = f(x) > 0$, so may drop the absolute value signs.

$$\begin{aligned}
\ln y &= \frac{x^2}{2} + 1 \\
y &= e^{1+\frac{1}{2}x^2} = e \cdot e^{x^2/2}
\end{aligned}$$

S-25: This is a separable differential equation.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{(x^2+x)y} \\ y\,\mathrm{d}y &= \frac{\mathrm{d}x}{x(x+1)}\end{aligned}$$

Using partial fractions decomposition, we find $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$.

$$\begin{aligned}y\,\mathrm{d}y &= \left(\frac{1}{x} - \frac{1}{x+1}\right)\,\mathrm{d}x \\ \int y\,\mathrm{d}y &= \int \left(\frac{1}{x} - \frac{1}{x+1}\right)\,\mathrm{d}x \\ \frac{y^2}{2} &= \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C\end{aligned}$$

To satisfy the initial condition $y(1) = 2$ we must choose C to obey

$$\begin{aligned}\frac{2^2}{2} &= \ln\left|\frac{1}{1+1}\right| + C \\ 2 &= \ln\frac{1}{2} + C \\ C &= 2 - \ln\frac{1}{2}\end{aligned}$$

So,

$$\begin{aligned}\frac{y^2}{2} &= \ln\left|\frac{x}{x+1}\right| + 2 - \ln\frac{1}{2} \\ y^2 &= 2\ln\left|\frac{x}{x+1}\right| + 4 - 2\ln\frac{1}{2}\end{aligned}$$

Note that the question specifies that $y(1) = 2$ is an initial condition. So we always have $x \geqslant 1$. Then $\frac{x}{x+1}$ is positive, and we can drop the absolute values.

$$y^2 = 2\ln\frac{x}{x+1} + 4 - 2\ln\frac{1}{2}$$

This leaves two options for $y(x)$: the positive or negative square root of the right hand side above. Since $y(1) = 2$, which is positive, we must choose the positive square root.

$$\begin{aligned}y(x) &= \sqrt{2\Big(\ln\frac{x}{x+1} - \ln\frac{1}{2} + 2\Big)} \\ &= \sqrt{4 + 2\ln\frac{2x}{x+1}}\end{aligned}$$

You might worry that $y(x)$ could pass through zero, changing sign, at some $x > 1$. But the differential equation says that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(x^2+x)y}$ is positive whenever $y > 0$ and $x \geqslant 1$. So $y(x)$ is an increasing function whenever $y > 0$ and $x \geqslant 1$. As $y(1) = 2$, we have $y(x) \geqslant 2$ for all $x \geqslant 1$.

S-26: This is a separable differential equation.

$$\begin{aligned}\frac{1+\sqrt{y^2-4}}{\tan x}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sec x}{y}\\ y\left[1+\sqrt{y^2-4}\right]\,\mathrm{d}y &= \sec x\ \tan x\,\mathrm{d}x\\ \int y\left[1+\sqrt{y^2-4}\right]\,\mathrm{d}y &= \int \sec x\ \tan x\,\mathrm{d}x\end{aligned}$$

For the integral on the left, we use the substitution $u = y^2 - 4$, $\frac{1}{2}\mathrm{d}u = y\,\mathrm{d}y$.

$$\begin{aligned}\frac{1}{2}\int (1+\sqrt{u})\,\mathrm{d}u &= \sec x + C\\ \frac{1}{2}\left(u+\frac{2}{3}u^{3/2}\right) &= \sec x + C\\ \frac{1}{2}\left(y^2-4+\frac{2}{3}(y^2-4)^{3/2}\right) &= \sec x + C\\ y^2+\frac{2}{3}(y^2-4)^{3/2} &= 2\sec x + 2C + 4\end{aligned}$$

To find C we set $x = 0$ and $y = 2$.

$$\begin{aligned}4+\frac{2}{3}\sqrt{4-4}^3 &= 2\sec(0)+2C+4\\ 4 &= 2+2C+4\\ 2 &= 2C+4\end{aligned}$$

So,

$$y^2+\frac{2}{3}(y^2-4)^{3/2} = 2\sec x + 2$$

S-27: The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= -k\sqrt{P}\\ \frac{\mathrm{d}P}{\sqrt{P}} &= -k\,\mathrm{d}t\\ \int\frac{\mathrm{d}P}{\sqrt{P}} &= \int -k\,\mathrm{d}t\\ 2\sqrt{P} &= -kt+C\end{aligned}$$

At $t = 0$, $P = 90,000$ so

$$\begin{aligned}2\sqrt{90,000} &= -k\times 0 + C\\ C &= 2\times 300 = 600\end{aligned}$$

Therefore,

$$2\sqrt{P} = -kt + 600 \tag{$*$}$$

Now, we find k. Let t be measured in weeks. Then when $t = 6$, $P = 40,000$.

$$\begin{aligned}
2\sqrt{40,000} &= -6k + 600 \\
2 \cdot 200 &= -6k + 600 \\
k &= \frac{200}{6} = \frac{100}{3}
\end{aligned}$$

Substituting our value of k into ($*$):

$$2\sqrt{P} = -\frac{100}{3}t + 600$$

To find when the population will be 10,000, we set $P = 10,000$ and solve for t.

$$\begin{aligned}
2\sqrt{10,000} &= -\frac{100}{3}t + 600 \\
2 \cdot 100 &= -\frac{100}{3}t + 600 \\
\frac{100}{3}t &= 400 \\
t &= 12
\end{aligned}$$

Since we measured t in weeks when we found k, we see that in 12 weeks the population will decrease to 10,000 individuals.

S-28: The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
m\frac{\mathrm{d}v}{\mathrm{d}t} &= -(mg + kv^2) \\
\frac{m}{mg + kv^2}\,\mathrm{d}v &= -\mathrm{d}t \\
\int \frac{m}{mg + kv^2}\,\mathrm{d}v &= \int -\mathrm{d}t
\end{aligned}$$

The left integral looks something like the antiderivative of arctangent. Let's factor out that mg from the denominator.

$$\begin{aligned}
\frac{1}{mg}\int \frac{m}{1 + \frac{k}{mg}v^2}\,\mathrm{d}v &= -t + C \\
\frac{1}{g}\int \frac{1}{1 + \left(\sqrt{\frac{k}{mg}}v\right)^2}\,\mathrm{d}v &= -t + C
\end{aligned}$$

Now it looks even more like the derivative of arctangent. We can guess the antiderivative from here, or use the substitution $u = \sqrt{\frac{k}{mg}}v$, $\mathrm{d}u = \frac{k}{mg}\,\mathrm{d}v$.

$$\begin{aligned}
\frac{1}{g}\sqrt{\frac{mg}{k}}\arctan\left(\sqrt{\frac{k}{mg}}v\right) &= -t + C \\
\sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v\right) &= -t + C \qquad (*)
\end{aligned}$$

At $t = 0$, $v = v_0$, so:

$$\sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right) = C$$

Plug C into $(*)$.

$$\sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v\right) = \sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right) - t$$

At its highest point, the object has velocity $v = 0$. This happens when t obeys:

$$\begin{aligned}
\sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}0\right) &= \sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right) - t \\
0 &= \sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right) - t \\
t &= \sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right)
\end{aligned}$$

S-29: (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= -k\,v^2 \\
-\frac{\mathrm{d}v}{v^2} &= k\,\mathrm{d}t \\
\int -\frac{\mathrm{d}v}{v^2} &= \int k\,\mathrm{d}t \\
\frac{1}{v} &= kt + C
\end{aligned}$$

At $t = 0$, $v = 40$ so

$$\begin{aligned}
\frac{1}{40} &= k \times 0 + C \\
C &= \frac{1}{40}
\end{aligned}$$

Therefore,

$$v(t) = \frac{1}{kt + C} = \frac{1}{kt + 1/40} = \frac{40}{40kt + 1} \tag{$*$}$$

The constant of proportionality k is determined by

$$\begin{aligned}
v(10) &= 20 \\
20 &= \frac{40}{40k \times 10 + 1} \\
\frac{1}{2} &= \frac{1}{400k + 1} \\
400k + 1 &= 2 \\
k &= \frac{1}{400}
\end{aligned}$$

(b) Subbing in the value of k to $(*)$,

$$v(t) = \frac{40}{40kt + 1} = \frac{40}{t/10 + 1}$$

We want to know the value of t that gives $v(t) = 5$.

$$\begin{aligned}
5 &= \frac{40}{t/10 + 1} \\
\frac{t}{10} + 1 &= 8 \\
t &= 70 \text{ sec}
\end{aligned}$$

S-30: (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= k(3 - x)(2 - x) \\
\frac{\mathrm{d}x}{(x-2)(x-3)} &= k\mathrm{d}t
\end{aligned}$$

Using the method of partial fractions, we find $\frac{1}{(x-2)(x-3)} = \frac{1}{x-3} - \frac{1}{x-2}$.

$$\begin{aligned}
\int \left[\frac{1}{x-3} - \frac{1}{x-2}\right] \mathrm{d}x &= \int k\mathrm{d}t \\
\ln|x-3| - \ln|x-2| &= kt + C \\
\ln\left|\frac{x-3}{x-2}\right| &= kt + C \\
\left|\frac{x-3}{x-2}\right| &= e^{kt+C} = e^{kt}e^{C} \\
\frac{x-3}{x-2} &= De^{kt}
\end{aligned}$$

where $D = \pm e^C$. When $t = 0$, $x = 1$, forcing

$$\begin{aligned} \frac{1-3}{1-2} &= De^0 \\ D &= 2 \end{aligned}$$

Hence

$$\begin{aligned} \frac{x-3}{x-2} &= 2e^{kt} \\ x - 3 &= 2e^{kt}(x-2) \\ x - 2e^{kt}x &= 3 - 4e^{kt} \\ x(t) &= \frac{3-4e^{kt}}{1-2e^{kt}} \end{aligned}$$

(b) To evaluate the limit, we could use l'Hôpital's rule, but we could also just multiply the numerator and denominator by e^{-kt}. Note $\lim\limits_{t\to\infty} e^{-tk} = 0$.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \underbrace{\frac{3-4e^{kt}}{1-2e^{kt}}}_{\substack{\text{num}\to-\infty \\ \text{den}\to-\infty}} = \lim_{t\to\infty} \frac{3-4e^{kt}}{1-2e^{kt}} \cdot \frac{e^{-kt}}{e^{-kt}} = \lim_{t\to\infty} \frac{3e^{-kt}-4}{e^{-kt}-2} = \frac{0-4}{0-2} = 2$$

S-31: (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= 4P - P^2 \\ \frac{\mathrm{d}P}{4P-P^2} &= \mathrm{d}t \\ \frac{\mathrm{d}P}{P(4-P)} &= \mathrm{d}t \end{aligned}$$

Using the method of partial fractions, we see $\frac{1}{P(4-P)} = \frac{1/4}{P} + \frac{1/4}{4-P}$.

$$\begin{aligned} \frac{1}{4}\left[\frac{1}{P} + \frac{1}{4-P}\right]\mathrm{d}P &= \mathrm{d}t \\ \int \frac{1}{4}\left[\frac{1}{P} + \frac{1}{4-P}\right]\mathrm{d}P &= \int \mathrm{d}t \\ \frac{1}{4}\Big[\ln|P| - \ln|4-P|\Big] &= t + C \end{aligned}$$

When $t = 0$, $P = 2$, so $\frac{1}{4}\big[\ln|2| - \ln|2|\big] = C \implies C = 0$. So,

$$\frac{1}{4}\ln\left|\frac{P}{4-P}\right| = t$$

At time $t = 0$, $\frac{P}{4-P} = 1 > 0$. The ratio may not change sign at any finite time, because this could only happen if at some finite time P took either the value 0 or the value 4. But at this time $t = \frac{1}{4} \ln \left| \frac{P}{4-P} \right|$ would have to be infinite. So $\frac{P}{4-P} > 0$ for all time and:

$$\begin{aligned}
\frac{1}{4} \ln \frac{P}{4-P} &= t \\
\ln \frac{P}{4-P} &= 4t \\
\frac{P}{4-P} &= e^{4t} \\
P &= (4-P)e^{4t} \\
P + Pe^{4t} &= 4e^{4t} \\
P &= \frac{4e^{4t}}{1+e^{4t}} = \frac{4}{1+e^{-4t}}
\end{aligned}$$

(b) At $t = \frac{1}{2}$, $P = \frac{4}{1+e^{-2}} \approx 3.523$.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{4}{1+e^{-4t}} = \frac{4}{1+0} = 4$$

S-32:

(a) The rate of change of speed at time t is $-kv(t)^2$ for some constant of proportionality k (to be determined–but we assume it is positive, since the speed is decreasing). So $v(t)$ obeys the differential equation $\frac{dv}{dt} = -kv^2$.

(b) The equation $\frac{dv}{dt} = -kv^2$ is a separable differential equation, which we can solve in the usual way.

$$\begin{aligned}
\frac{dv}{dt} &= -kv^2 \\
\frac{dv}{-v^2} &= k dt \\
\int -\frac{dv}{v^2} &= \int k dt \\
\frac{1}{v} &= kt + C
\end{aligned}$$

At time $t = 0$, $v = 400$, so $C = \frac{1}{400}$. Then:

$$\frac{1}{v} = kt + \frac{1}{400} \tag{$*$}$$

At time $t = 1$, $v = 200$, so

$$\begin{aligned}
\frac{1}{200} &= k + \frac{1}{400} \\
k &= \frac{1}{400}
\end{aligned}$$

Therefore, from $(*)$,

$$\begin{aligned}\frac{1}{v} &= \frac{t}{400} + \frac{1}{400} = \frac{t+1}{400}\\ v &= \frac{400}{t+1}\end{aligned}$$

(c) To find when the speed is 50, we set $v = 50$ in the equation from (b) and solve for t.

$$\begin{aligned}50 &= \frac{400}{t+1}\\ 50(t+1) &= 400\\ t+1 &= 8\\ t &= 7\end{aligned}$$

S-33: (a) The first order derivatives are

$$f_r(r,\theta) = mr^{m-1}\cos m\theta \qquad f_\theta(r,\theta) = -mr^m \sin m\theta$$

The second order derivatives are

$$f_{rr}(r,\theta) = m(m-1)r^{m-2}\cos m\theta \quad f_{r\theta}(r,\theta) = -m^2r^{m-1}\sin m\theta \quad f_{\theta\theta}(r,\theta) = -m^2r^m\cos m\theta$$

so that

$$f_{rr}(1,0) = m(m-1),\ f_{r\theta}(1,0) = 0,\ f_{\theta\theta}(1,0) = -m^2$$

(b) By part (a), the expression

$$f_{rr} + \frac{\lambda}{r}f_r + \frac{1}{r^2}f_{\theta\theta} = m(m-1)r^{m-2}\cos m\theta + \lambda mr^{m-2}\cos m\theta - m^2r^{m-2}\cos m\theta$$

vanishes for all r and θ if and only if

$$m(m-1) + \lambda m - m^2 = 0 \iff m(\lambda-1) = 0 \iff \lambda = 1$$

S-34: (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} &= (0.06 + 0.02\sin t)B\\ \frac{\mathrm{d}B}{B} &= (0.06 + 0.02\sin t)\,\mathrm{d}t\\ \int \frac{\mathrm{d}B}{B} &= \int (0.06 + 0.02\sin t)\,\mathrm{d}t\\ \ln|B(t)| &= 0.06t - 0.02\cos t + C'\end{aligned}$$

Since $B(t)$ is our bank account balance and we're not withdrawing money, $B(t)$ is positive, so we can drop the absolute value signs.

$$\begin{aligned}
\ln B(t) &= 0.06t - 0.02\cos t + C' \\
B(t) &= e^{0.06t-0.02\cos t}e^{C'} \\
B(t) &= Ce^{0.06t-0.02\cos t}
\end{aligned}$$

for arbitrary constants C' and $C = e^{C'} \geqslant 0$.

Remark: the function $B(t) = 0$ obeys the differential equation so that $C = 0$ is allowed, even though it is not of the form $C = e^{C'}$. This seeming discrepancy arose because, in our very first step of part (a), we divided both sides of the differential equation by B, which is only allowable if $B \neq 0$. So, in this step, we implicitly assumed B was nonzero.

(b) We are told that $B(0) = 1000$. This allows us to find C.

$$\begin{aligned}
1000 = B(0) &= Ce^{0-0.02\cos 0} = Ce^{-0.02} \\
C &= 1000e^{0.02}
\end{aligned}$$

So, when $t = 2$,

$$B(2) = \underbrace{1000e^{0.02}}_{C}\, e^{0.06\times 2-0.02\cos 2} = \$1159.89$$

rounded to the nearest cent.

Note that $\cos 2$ is the cosine of 2 radians, $\cos 2 \approx -0.416$.

S-35: What we're given is an equation relating y to the integral of a function of y. What we know how to solve is an equation relating the *derivative* of y to a function of y. We can create this by differentiating the given integral equation. By the Fundamental Theorem of Calculus, part 1:

$$y'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_0^x \big(y(t)^2 - 3y(t) + 2\big)\sin t\,\mathrm{d}t\right\} = \big(y(x)^2 - 3y(x) + 2\big)\sin x$$

So $y(x)$ satisfies the differential equation $y' = \big(y^2 - 3y + 2\big)\sin x = (y-2)(y-1)\sin x$ and the initial equation $y(0) = 3$ (just substitute $x = 0$ into $(*)$). For $y \neq 1, 2$:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= (y-2)(y-1)\sin x \\
\frac{\mathrm{d}y}{(y-2)(y-1)} &= \sin x\,\mathrm{d}x \\
\int \frac{\mathrm{d}y}{(y-2)(y-1)} &= \int \sin x\,\mathrm{d}x
\end{aligned}$$

Using the method of partial fractions, we see $\frac{1}{(y-2)(y-1)} = \frac{1}{y-2} - \frac{1}{y-1}$.

$$\begin{aligned}
\int \left[\frac{1}{y-2} - \frac{1}{y-1}\right] \mathrm{d}y &= \int \sin x \, \mathrm{d}x \\
\ln|y-2| - \ln|y-1| &= -\cos x + c \\
\ln\left|\frac{y-2}{y-1}\right| &= -\cos x + c \\
\left|\frac{y-2}{y-1}\right| &= e^{c-\cos x}
\end{aligned}$$

The condition $y(0) = 3$ forces $\left|\frac{3-2}{3-1}\right| = e^{c-1}$ or $e^c = \frac{1}{2}e$, hence

$$\left|\frac{y-2}{y-1}\right| = \frac{1}{2}e^{1-\cos x}$$

Observe that, when $x = 0$, $\frac{y-2}{y-1} = \frac{1}{2} > 0$. Furthermore $\frac{1}{2}e^{1-\cos x}$, and hence $\left|\frac{y-2}{y-1}\right|$, can never take the value zero. As $y(x)$ varies continuously with x, $y(x)$ must remain larger than 2. Consquently, $\frac{y-2}{y-1}$ remains positive and we may drop the absolute value signs. Hence

$$\frac{y-2}{y-1} = \frac{1}{2}e^{1-\cos x}$$

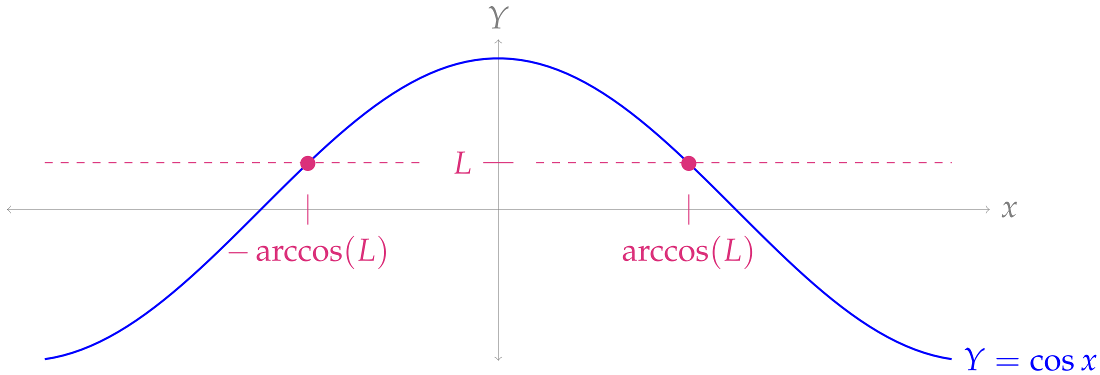
Solving for y,

$$\begin{aligned}
\frac{y-2}{y-1} &= \frac{1}{2}e^{1-\cos x} \\
2(y-2) &= e^{1-\cos x}(y-1) \\
2y-4 &= ye^{1-\cos x} - e^{1-\cos x} \\
y\big(2-e^{1-\cos x}\big) &= 4 - e^{1-\cos x} \\
y &= \frac{4-e^{1-\cos x}}{2-e^{1-\cos x}}
\end{aligned}$$

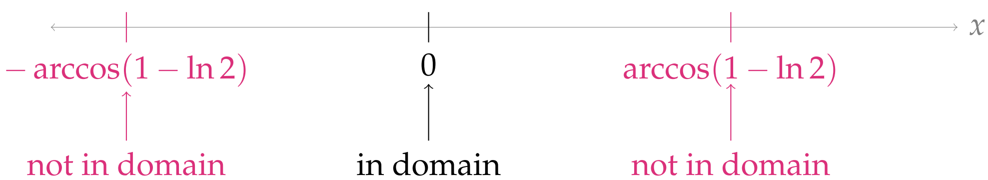
To avoid division by zero in the last step, we need

$$\begin{aligned}
e^{1-\cos x} &\neq 2 \\
1-\cos x &\neq \ln 2 \\
\cos x &\neq 1 - \ln 2
\end{aligned}$$

Let $L = 1 - \ln 2$, for brevity, and note that $L > 0$. (This can be seen by observing $2 < e$, so, $\ln 2 < \ln e = 1$, hence $1 - \ln 2 > 0$.)



We know $x = 0$ is in the domain of our function, but the points $x = \pm\arccos(L) = \pm\arccos(1 - \ln 2)$ are *not*.



Therefore, the largest interval for which our answer makes sense is

$$-\arccos(1 - \ln 2)) > x > \arccos(1 - \ln 2)$$

or approximately $-1.259 < x < 1.259$.

S-36: (a) Setting $x = 0$ gives

$$f(0) = 3 + \int_0^0 \big(f(t) - 1\big)\big(f(t) - 2\big)\, \mathrm{d}t = 3$$

(b) By the Fundamental Theorem of Calculus part 1,

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \big(f(t) - 1\big)\big(f(t) - 2\big)\, \mathrm{d}t = \big(f(x) - 1\big)\big(f(x) - 2\big)$$

Thus $y = f(x)$ obeys the differential equation $y' = (y - 1)(y - 2)$.

(c) If $y \neq 1, 2$,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (y-1)(y-2) \\ \frac{\mathrm{d}y}{(y-1)(y-2)} &= \mathrm{d}x \\ \int \frac{\mathrm{d}y}{(y-1)(y-2)} &= \int \mathrm{d}x \end{aligned}$$

Using the method of partial fractions,

$$\begin{aligned}
\int \left(\frac{1}{y-2} - \frac{1}{y-1}\right) \mathrm{d}y &= \int \mathrm{d}x \\
\ln|y-2| - \ln|y-1| &= x + C \\
\ln\left|\frac{y-2}{y-1}\right| &= x + C
\end{aligned}$$

Observe that $\frac{\mathrm{d}y}{\mathrm{d}x} = (y-1)(y-2) > 0$ for all $y \geqslant 2$. That is, $f(x)$ is increasing at all x for which $f(x) > 2$. As $f(0) = 3$, $f(x)$ increases for all $x \geqslant 0$, and $f(x) \geqslant 3$ for all $x \geqslant 0$. So we may drop the absolute value signs.

$$\begin{aligned}
\ln\frac{f(x)-2}{f(x)-1} &= x + C \\
\frac{f(x)-2}{f(x)-1} &= e^C e^x
\end{aligned}$$

At $x = 0$, $\frac{f(x)-2}{f(x)-1} = \frac{1}{2}$ so $e^C = \frac{1}{2}$.

$$\begin{aligned}
\frac{f(x)-2}{f(x)-1} &= \frac{1}{2}e^x \\
2f(x) - 4 &= [f(x)-1]e^x \\
[2-e^x]f(x) &= 4 - e^x \\
f(x) &= \frac{4-e^x}{2-e^x}
\end{aligned}$$

S-37:

(a) If we let $f(t) = 0$ for all t, then its average over any interval is 0, as is its root mean square.

(b) Let's start by simplifying the given equation.

$$\frac{1}{x-a}\int_a^x f(t)\,\mathrm{d}t = \sqrt{\frac{1}{x-a}\int_a^x f^2(t)\,\mathrm{d}t}$$

$$\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t = \sqrt{\int_a^x f^2(t)\,\mathrm{d}t} \tag{5.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}\right\} \tag{5.2}$$

For the derivative on the left, we use the product rule and the Fundamental Theorem of Calculus, part 1.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x-a}}\right\}\int_a^x f(t)\,\mathrm{d}t + \frac{1}{\sqrt{x-a}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_a^x f(t)\,\mathrm{d}t\right\} \\ &= -\frac{1}{2\sqrt{x-a}^3}\int_a^x f(t)\,\mathrm{d}t + \frac{f(x)}{\sqrt{x-a}} \\ &= \frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right]\end{aligned}$$

For the derivative on the right in Equation (5.2), we use the chain rule and the Fundamental Theorem of Calculus, part 1.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}\right\} &= \frac{1}{2}\left(\int_a^x f^2(t)\,\mathrm{d}t\right)^{-\frac{1}{2}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_a^x f^2(t)\,\mathrm{d}t\right\} \\ &= \frac{f^2(x)}{2\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}}\end{aligned}$$

So, Equation (5.2) yields the following:

$$\frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] = \frac{f^2(x)}{2\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}} \tag{5.3}$$

(c) From Equation (5.1), $\sqrt{\int_a^x f^2(t)\,\mathrm{d}t} = \frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t$.

$$\begin{aligned}\frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] &= \frac{f^2(x)}{2\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t} \\ \frac{2}{x-a}\int_a^x f(t)\,\mathrm{d}t\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] &= f^2(x)\end{aligned}$$

(d) Now what we have is a differential equation, although it might not look like it. Let $Y = \int_a^x f(t)\,\mathrm{d}t$. Then $\frac{\mathrm{d}Y}{\mathrm{d}x} = f(x)$.

$$\frac{2}{x-a}Y\left[\frac{\mathrm{d}Y}{\mathrm{d}x} - \frac{1}{2(x-a)}Y\right] = \left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)^2 \tag{5.4}$$

We're used to solving differential equations of the form $\frac{\mathrm{d}Y}{\mathrm{d}x}$ =(something). So, let's manipulate our equation until it has this form.

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)^2 - \left(\frac{2Y}{x-a}\right)\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right) + \left(\frac{Y}{x-a}\right)^2 = 0$$

This is a quadratic equation, with variable $\frac{\mathrm{d}Y}{\mathrm{d}x}$. Its solutions are:

$$\begin{aligned}
\frac{\mathrm{d}Y}{\mathrm{d}x} &= \frac{\left(\frac{2Y}{x-a}\right) \pm \sqrt{\left(\frac{2Y}{x-a}\right)^2 - 4\cdot\left(\frac{Y}{x-a}\right)^2}}{2} \\
&= \frac{\frac{2Y}{x-a} \pm 0}{2} \\
&= \frac{Y}{x-a}
\end{aligned}$$

This gives us the separable differential equation

$$\begin{aligned}
\frac{\mathrm{d}Y}{\mathrm{d}x} &= \frac{Y}{x-a} \\
\frac{\mathrm{d}Y}{Y} &= \frac{\mathrm{d}x}{x-a} \\
\int \frac{\mathrm{d}Y}{Y} &= \int \frac{\mathrm{d}x}{x-a} \\
\ln|Y| &= \ln|x-a| + C \\
|Y| &= e^{\ln(x-a)+C} = (x-a)e^C \\
Y &= D(x-a)
\end{aligned} \tag{5.5}$$

where D is some constant, e^C or $-e^C$. Note this covers all real constants except $D = 0$. If $D = 0$, then $Y(x) = 0$ for all x. This function also satisfies Equation (5.4), so indeed,

$$Y(x) = D(x-a) \tag{5.6}$$

for *any* constant D is the family of equations satisfying our differential equation.

Remark: the reason we "lost" the solution $Y(x) = 0$ is that in Equation (5.5), we divided by Y, thus tacitly assuming it was not identically 0.

(e) Remember $Y = \int_a^x f(t)\,\mathrm{d}t$. So, Equation (5.6) tells us:

$$\begin{aligned}
\int_a^x f(t)\,\mathrm{d}t &= D(x-a) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_a^x f(t)\,\mathrm{d}t\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{D(x-a)\} \\
f(x) &= D
\end{aligned}$$

We should check that this function works.

$$f_{\text{avg}} = \frac{1}{x-a}\int_a^x D\,\mathrm{d}t = \frac{1}{x-a}\Big[Dt\Big]_{t=a}^{t=x} = \frac{Dx - Da}{x-a} = D$$

$$f_{\text{RMS}} = \sqrt{\frac{1}{x-a}\int_a^x D^2\,\mathrm{d}t} = \sqrt{\frac{1}{x-a}\Big[D^2x\Big]_{t=a}^{t=x}} = \sqrt{\frac{D^2x - D^2a}{x-a}} = \sqrt{D^2} = |D|$$

So, $f(x) = D$ works only if D is nonnegative.

That is: the only functions whose average matches their root square mean over every interval are constant, nonnegative functions.

Remark: it was step (c) where we introduced the erroneous answer $f(x) = D$, $D < 0$ to our solution. In Equation (5.3), $f(x) = D$ is not a solution if $D < 0$:

$$\begin{aligned}
\frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] &= \frac{f^2(x)}{2\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}} \\
\frac{1}{\sqrt{x-a}}\left[D - \frac{1}{2(x-a)}\int_a^x D\,\mathrm{d}t\right] &= \frac{D^2}{2\sqrt{\int_a^x D^2\,\mathrm{d}t}} \\
\frac{1}{\sqrt{x-a}}\left[D - \frac{1}{2(x-a)}D(x-a)\right] &= \frac{D^2}{2\sqrt{D^2(x-a)}} \\
\frac{1}{\sqrt{x-a}}\left[\frac{1}{2}D\right] &= \frac{D^2}{2|D|\sqrt{x-a}} \\
D &= \frac{D^2}{|D|} = |D|
\end{aligned}$$

In (c), we replace $\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}$, which cannot be negative, with $\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t$, which could be negative if $f(t) = D < 0$. Indeed, if $f(t) = D$, then $\sqrt{\int_a^x f^2(t)\,\mathrm{d}t} = |D|\sqrt{x-a}$, while $\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t = D\sqrt{x-a}$. It is at this point that negative functions creep into our solution.

S-38: We start by antidifferentiating both sides with respect to x.

$$\int\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\mathrm{d}x = \int\left(\frac{2}{y^3}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x$$

The right integral is in exactly the form we would use for a change of variables (substitution) to y.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int\left(\frac{2}{y^3}\right)\mathrm{d}y = -\frac{1}{y^2} + C$$

When $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$.

$$\begin{aligned}
3 &= -\frac{1}{1} + C \\
C &= 4
\end{aligned}$$

So,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y^2} + 4$$

This is a separable differential equation.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4y^2-1}{y^2} \\
\frac{y^2}{4y^2-1}\,\mathrm{d}y &= \mathrm{d}x \\
\int \frac{y^2}{4y^2-1}\,\mathrm{d}y &= \int \mathrm{d}x \qquad (*)
\end{aligned}$$

We can evaluate the left integral with partial fractions, but because the numerator has the same degree as the denominator, we have to simplify first. We do this by inspection, but you can also use long division.

$$\begin{aligned}
\frac{y^2}{4y^2-1} &= \frac{\frac{1}{4}(4y^2-1)+\frac{1}{4}}{4y^2-1} \\
&= \frac{1}{4}\left(1+\frac{1}{4y^2-1}\right) \\
&= \frac{1}{4}\left(1+\frac{1}{(2y-1)(2y+1)}\right) \\
&= \frac{1}{4}\left(1+\frac{1/2}{2y-1}-\frac{1/2}{2y+1}\right)
\end{aligned}$$

Now, we return to $(*)$.

$$\begin{aligned}
\int \mathrm{d}x &= \int \frac{y^2}{4y^2-1}\,\mathrm{d}y \\
&= \int \frac{1}{4}\left(1+\frac{1/2}{2y-1}-\frac{1/2}{2y+1}\right)\,\mathrm{d}y \\
&= \frac{1}{4}\left(y+\frac{1}{4}\ln|2y-1|-\frac{1}{4}\ln|2y+1|\right) \\
&= \frac{1}{4}\left(y+\frac{1}{4}\ln\left|\frac{2y-1}{2y+1}\right|\right) \\
x+C &= \frac{1}{4}\left(y+\frac{1}{4}\ln\left|\frac{2y-1}{2y+1}\right|\right)
\end{aligned}$$

When $x=-\frac{1}{16}\ln 3$, $y=1$.

$$\begin{aligned}
-\frac{1}{16}\ln 3+C &= \frac{1}{4}\left(1+\frac{1}{4}\ln\left|\frac{2-1}{2+1}\right|\right) = \frac{1}{4}+\frac{1}{16}\ln\frac{1}{3} \\
C &= \frac{1}{4}
\end{aligned}$$

So,

$$\begin{aligned}
x+\frac{1}{4} &= \frac{1}{4}\left(y+\frac{1}{4}\ln\left|\frac{2y-1}{2y+1}\right|\right) \\
x &= \frac{1}{4}\left(y-1+\frac{1}{4}\ln\left|\frac{2y-1}{2y+1}\right|\right)
\end{aligned}$$

We can check our answer by differentiating with respect to x.

$$\begin{aligned}
x &= \frac{1}{4}\left(y-1+\frac{1}{4}\ln\left|\frac{2y-1}{2y+1}\right|\right)\\
4x &= y-1+\frac{1}{4}\ln|2y-1|-\frac{1}{4}\ln|2y+1|\\
\frac{\mathrm{d}}{\mathrm{d}x}\{4x\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{y-1+\frac{1}{4}\ln|2y-1|-\frac{1}{4}\ln|2y+1|\right\}\\
4 &= \frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{4}\cdot\frac{2\frac{\mathrm{d}y}{\mathrm{d}x}}{2y-1}-\frac{1}{4}\cdot\frac{2\frac{\mathrm{d}y}{\mathrm{d}x}}{2y+1}\\
4 &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(1+\frac{1/2}{2y-1}-\frac{1/2}{2y+1}\right)=\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{4y^2}{4y^2-1}\right)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4y^2-1}{y^2}=4-\frac{1}{y^2} \qquad (**)
\end{aligned}$$

Differentiating with respect to x again, using the chain rule,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{2}{y^3}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

This is exactly the differential equation we were meant to solve.

S-39:

(a) The functions are constant when their derivatives are zero.

$$\begin{aligned}
\frac{\mathrm{d}W}{\mathrm{d}t} &= rW+I(H)+a & \qquad \frac{\mathrm{d}H}{\mathrm{d}t} &= sH+I(W)+b\\
0 &= rW+0+a & 0 &= sH+0+b\\
W &= -\frac{a}{r} & H &= -\frac{b}{r}
\end{aligned}$$

(b) For this part, $\frac{\mathrm{d}W}{\mathrm{d}t}=rW+a$. If $\frac{\mathrm{d}W}{\mathrm{d}r}>0$, then:

$$\begin{aligned}
rW+a &> 0\\
rW &> -a
\end{aligned}$$

If $r>0$, then dividing both sides by r does not change the inequality. If $r<0$, then dividing both sides by r flips the inequality

$$\begin{cases} W & > -\frac{a}{r} \text{ if } r>0\\ W & < -\frac{a}{r} \text{ if } r<0\end{cases}$$

We want the second case, so $r<0$.

Similarly, if $\frac{\mathrm{d}W}{\mathrm{d}r}<0$, then:

$$\begin{aligned}
rW+a &< 0\\
rW &< -a
\end{aligned}$$

If $r > 0$, then dividing both sides by r does not change the inequality. If $r < 0$, then dividing both sides by r flips the inequality

$$\begin{cases} W & < -\frac{a}{r} \text{ if } r > 0 \\ W & > -\frac{a}{r} \text{ if } r < 0 \end{cases}$$

We want the second case, so again $r < 0$.

Note the inequality did not depend on the sign of a.

(c) If $W = H$, and $I(H) = cH$, then the first differential equation becomes:

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}t} &= rW + I(H) + a \\ &= rW + cH + a \\ &= rW + cW + a \\ &= (r+c)W + a \end{aligned}$$

We recognize this as a first-order linear differential equation, so we rearrange it a little to better fit the format of Theorem 3.12.10 in the text:

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}t} &= (r+c)\left(W - \frac{-a}{r+c}\right) \\ W(t) &= \left(W(0) + \frac{a}{r+c}\right)e^{(r+c)t} - \frac{a}{r+c} \end{aligned}$$

S-40:

(a) $\Delta x = \frac{300-0}{300} = 1$, and $x_0 = 0$, so $x_i = a + i\Delta x = i$. Then:

$$\begin{aligned} \int_0^{300} f(t)\mathrm{d}dt &\approx \frac{1}{2}f(0) + \sum_{i=1}^{299} f(i) + \frac{1}{2}f(300) \\ &= \frac{1}{2}f(0) + \sum_{i=1}^{300} f(i) - \frac{1}{2}f(300) \\ &= \frac{1}{2} - \frac{1}{2}e^{3/4} + \sum_{i=1}^{300} e^{i/400} \end{aligned}$$

(b) $f(t) = e^{t/400}$, so $f'(t) = \frac{1}{400}e^{t/400}$, and $f''(t) = \frac{1}{400^2}e^{t/400}$. This is a positive increasing function, so its max is at the right endpoint of the interval. That is, for x in $[0, 300]$, $|f''(x)| \leqslant f''(300) = \frac{e^{3/4}}{400^2}$. So, we choose $M = \frac{e^{3/4}}{400^2}$. Then the error is bounded by

$$\frac{M}{12}\frac{(b-a)^3}{n^2} = \frac{\frac{e^{3/4}}{400^2}}{12}\frac{300^3}{300^2} = \frac{e^{3/4}}{6400}$$

(c) $\frac{1}{2} - \frac{1}{2}e^{3/4} + \sum_{i=1}^{300} e^{i/400} - \sum_{t=1}^{300} e^{t/400} = \frac{1}{2} - \frac{1}{2}e^{3/4}$

(d) From (c), we know our error bound:

$$\left|\int_0^{300} e^{t/400}\mathrm{d}t - \left(\frac{1}{2}-\frac{1}{2}e^{3/4}+\sum_{i=1}^{300} e^{i/400}\right)\right| \leqslant \frac{e^{3/4}}{6400}$$

This gives us an interval:

$$-\frac{e^{3/4}}{6400} \leqslant \int_0^{300} e^{t/400}\mathrm{d}t - \left(\frac{1}{2}-\frac{1}{2}e^{3/4}+\sum_{i=1}^{300} e^{i/400}\right) \leqslant \frac{e^{3/4}}{6400}$$

$$-\frac{e^{3/4}}{6400} \leqslant \int_0^{300} e^{t/400}\mathrm{d}t - \frac{1}{2}+\frac{1}{2}e^{3/4}-\sum_{i=1}^{300} e^{i/400} \leqslant \frac{e^{3/4}}{6400}$$

Subtracting the coloured terms from all parts of the inequality,

$$-\frac{e^{3/4}}{6400}+\frac{1}{2}-\frac{1}{2}e^{3/4}-\int_0^{300} e^{t/400}\mathrm{d}t \leqslant -\sum_{i=1}^{300} e^{i/400} \leqslant \frac{e^{3/4}}{6400}+\frac{1}{2}-\frac{1}{2}e^{3/4}-\int_0^{300} e^{t/400}\mathrm{d}t$$

$$-\frac{e^{3/4}}{6400}+\frac{1}{2}-\frac{1}{2}e^{3/4}-400\left(e^{3/4}-1\right) \leqslant -\sum_{i=1}^{300} e^{i/400} \leqslant \frac{e^{3/4}}{6400}+\frac{1}{2}-\frac{1}{2}e^{3/4}-400\left(e^{3/4}-1\right)$$

$$400.5-e^{3/4}\left(400.5+\frac{1}{6400}\right) \leqslant -\sum_{i=1}^{300} e^{i/400} \leqslant 400.5-e^{3/4}\left(400.5-\frac{1}{6400}\right)$$

Now we'll work this into the form of our original expression.

$$400.5-e^{3/4}\left(400.5+\frac{1}{6400}\right) \leqslant -\sum_{t=1}^{300} e^{t/400} \leqslant 400.5-e^{3/4}\left(400.5-\frac{1}{6400}\right)$$

Adding $300e^{3/4}$ to all parts:

$$400.5-e^{3/4}\left(100.5+\frac{1}{6400}\right) \leqslant 300e^{3/4}-\sum_{t=1}^{300} e^{t/400} \leqslant 400.5-e^{3/4}\left(100.5-\frac{1}{6400}\right)$$

Multiplying all parts by the positive constant $\frac{1875}{e^{3/4}-1}$:

$$\frac{1875\left(400.5-e^{3/4}\left(100.5+\frac{1}{6400}\right)\right)}{e^{3/4}-1} \leqslant \frac{1875}{e^{3/4}-1}\left(300e^{3/4}-\sum_{t=1}^{300} e^{t/400}\right)$$
$$\leqslant \frac{1875\left(400.5-e^{3/4}\left(100.5-\frac{1}{6400}\right)\right)}{e^{3/4}-1}$$

Now we use a calculator to get numerical approximations.

$$\frac{1875\left(400.5-e^{3/4}\left(100.5+\frac{1}{6400}\right)\right)}{e^{3/4}-1} \approx 315142.957850791$$

$$\frac{1875\left(400.5-e^{3/4}\left(100.5-\frac{1}{6400}\right)\right)}{e^{3/4}-1} \approx 315144.068351846$$

So,

$$315342.95 \leqslant \frac{1875}{e^{3/4}-1}\left(300\cdot e^{3/4} - \sum_{t=1}^{300} e^{t/400}\right) \leqslant 315144.07$$

(e) The text gave the approximation $\frac{1875}{e^{3/4}-1}\left(300\cdot e^{3/4} - \sum_{t=1}^{300} e^{t/400}\right) \approx 316081.01$, which is an over-approximation, since the actual value is at most 315,144.07. The largest the absolute error could possibly be would occur if the real value were as small as possible. In that case, our absolute error would be $316081.01 - 315342.95$. So, the an absolute error is less than \$938.

The relative error is $\frac{\text{approx}-\text{exact}}{\text{exact}}$. We know the approximate value, but we do not know the exact value. However, this function decreases as the exact value increases. So, its maximum occurs in the case that the exact value is as small as possible, 315342.95. In that worst case, the relative error is $\frac{316081.01-315342.95}{315342.95} \approx 0.00234$. So, the relative error is less than 0.24%.

So, in the text, our approximation was actually pretty good! It was off by less than one-quarter of one percent.

S-41:

(a) The interest owed is r percent of $P(t)$, so it's $\frac{r}{100}P(t)$.

(b) $P'(t)$ is the rate of change of the amount owed, so $-P'(t)$ is the rate at which you're paying off the loan. In this model, it's approximately your monthly payment.

(c) You pay two things: interest and principal. So your payment C will satisfy

$$C = \frac{r}{100}P(t) - P'(t)$$

(d) The differential equation is linear, so we can use Theorem 3.12.10: The differentiable function $y(x)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$$

if and only if

$$y(x) = (y(0)-b)e^{ax} + b$$

We re-write our differential equation as $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{r}{100}\left(P - \frac{100}{r}C\right)$. Then we see $a = \frac{r}{100}$ and $b = \frac{100}{r}C$, so

$$P(t) = \left(P_0 - \frac{100}{r}C\right)e^{rt/100} + \frac{100}{r}C$$

(e) To find C, we use the as-yet-unused fact that $P(N) = 0$.

$$0 = P(N) = \left(P_0 - \frac{100}{r}C\right)e^{rN/100} + \frac{100}{r}C = P_0e^{rN/100} + \frac{100}{r}C\left(1 - e^{rN/100}\right)$$

$$\frac{100}{r}C\left(1 - e^{rN/100}\right) = -P_0e^{rN/100}$$

$$C = \frac{r}{100} \cdot \frac{P_0e^{rN/100}}{(e^{rN/100} - 1)}$$

(f) The setup is the same as in part (c), only C is replaced with $C_0 \cdot 1.001^t$:

$$\frac{r}{100}P(t) - P'(t) = C_0 \cdot 1.001^t$$

We note here that this is not a separable differential equation, so we haven't yet learned a technique to solve it.

Solutions to Exercises 4.1 — Jump to TABLE OF CONTENTS

S-1: $Pr(X = 5) = 0.1$

S-2: Yes: all the elements of its sample space can be listed (as opposed to belonging to a continuum).

S-3: The events are disjoint for X: if $X > 4.5$ and X is a whole number, then $X \geqslant 5$. That is, if B happens, then A does not.

The events are not disjoint for Y, because (for example) if $Y = 4.9$ then both A and B occur.

S-4: The outcomes $X \leqslant 5$ and $X > 5$ are disjoint, so

$$Pr(X \leqslant 5 \text{ or } X > 5) = Pr(X \leqslant 5) + Pr(X > 5)$$

Since "$X \leqslant 5$ or $X > 5$" describes all real numbers, $Pr(X \leqslant 5 \text{ or } X > 5) = 1$. So, it follows that

$$Pr(X > 5) = Pr(X \leqslant 5 \text{ or } X > 5) - Pr(X \leqslant 5) = 1 - \frac{99}{100} = \frac{1}{100}$$

S-5: The outcomes $X = x$ and $X \neq x$ are disjoint, so

$$Pr(X = x \text{ or } X \neq x) = Pr(X = x) + Pr(X \neq x)$$

Since "$X = x$ or $X \neq x$" describes all real numbers, $Pr(X = x \text{ or } X \neq x) = 1$. So, it follows that

$$Pr(X \neq 5) = 1 - \frac{78}{93} = \frac{15}{93}$$

S-6: Out of the six values in the sample space, the four values $\{2,3,4,6\}$ make "X even or $X = 3$" occur. Since all six values in the sample space are equally likely, $Pr(X \text{ even OR } X = 3) = \frac{4}{6} = \frac{2}{3}$.

S-7: There are 10 values in $\mathcal{S}$. Exactly 7 of these values, $\{10, 15, 20, 30, 40, 45, 50\}$, are divisible by 3 or 10. Since all values are equally likely, the probability that Y is divisible by 3 or 10 is $\frac{7}{10}$.

We have to be a little careful that we're considering disjoint events here. There are five values of $\mathcal{S}$ that are divisible by 10, and three that are divisible by 3, but 30 is divisible by both.

Solutions to Exercises 4.2 — Jump to TABLE OF CONTENTS

S-1: Probability Mass Function

S-2: Discrete

S-3:

(a) Values not in the sample space are not included in the table. We can see that the probabiliy values in the table add to 1, i.e. 100% of the time X is 5, 6, 7, or 8. So, $\mathcal{S} = \{5, 6, 7, 8\}$.

(b) No, all values are not equally likely. $X = 5$ is the most likely.

S-4: By Theorem 4.2.3,

$$\begin{aligned}1 &= Pr(X = 3.1) + Pr(X = 3.2) + Pr(X = 3.3) + Pr(X = 3.4) + Pr(X = 3.5)\\ &= 0.1 + 0.15 + 0.35 + 0.2 + Pr(X = 3.5)\end{aligned}$$

so $Pr(X = 3.5) = 0.2$

S-5: The largest probability in the list is $\frac{1}{3}$, so the most likely outcome is 8.2.

S-6: $f(x)$ is nonzero for all whole numbers 1,2,3,..., so $\mathcal{S} = \{1, 2, 3, \ldots\}$.

Since $\mathcal{S}$ consists of whole numbers,

$$Pr(X \leqslant 3) = Pr(X = 1) + Pr(X = 2) + Pr(X = 3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

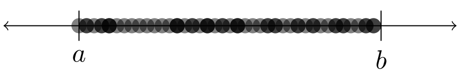
Solutions to Exercises 4.3 — Jump to TABLE OF CONTENTS

S-1: First, we note that $T > x$ and $T \leqslant x$ are disjoint events, and together they describe all possible outcomes, so $Pr(T \leqslant x) = 1 - Pr(T > x)$.

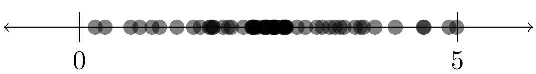
- $F(0) = Pr(T \leqslant 0) = 1 - Pr(T > 0) = 1 - \frac{9}{10} = \frac{1}{10}$. In words: if 9/10 days had a temperature of more than 0 degrees, then the remaining 1/10 days had a temperature of less than or equal to 0 degrees.
- $F(20) = Pr(T \leqslant 20) = \frac{4}{10} = \frac{2}{5}$, as stated in the problem.
- $F(30) = Pr(T \leqslant 30) = 1 - Pr(T > 30) = 1 - \frac{1}{10} = \frac{9}{10}$. In words: if 1/10 days had a temperature of more than 30 degrees, then the remaining 9/10 days had a temperature of less than or equal to 30 degrees.

Intuitively, these things sound about right: it's pretty usual to have a temperature less than or equal to 30 degrees; but it's pretty unusual to have temperature below freezing.

S-2: The dots should be fairly evenly distributed between a and b, and appear nowhere else.



Remark: the dot diagram above was NOT actually generated from pesudorandom numbers. Actual random numbers tend to have clusters and gaps – they don't behave the way you'd necessarily expect! If we used enough random numbers so that the dot diagram actually did look to have uniform density, then there would be so many dots, it would look just like one thick line.

S-3: 

S-4: By Definition 4.3.6, since $F(x)$ is continuous, W is a continuous random variable.

S-5:

(a) True, Corollary 4.3.10, part 3.

(b) False, for examples a variable X that takes the value $1,000,001$ with probability 1. Contrast this with the previous part: $F(x)$ approaches 1 eventually, but we don't have an absolute framework for what counts as "eventually."

(c) True, Corollary 4.3.10, part 2.

(d) True, Corollary 4.3.10, part 1. $F(x)$ is a probability no matter what real number x is, so $F(x) \geqslant 0$ (and also $F(x) \leqslant 1$)) for any x.

S-6: By Corollary 4.3.10, part 2, $F(11) \geqslant F(10)$. By part 1, $F(11) \leqslant 1$. So, $F(11) = 1$.

S-7:

- X never takes values less than -1. So, if $x < -1$, then $F(x) = Pr(X \leqslant x) = 0$.

- $F(-1) = Pr(X \leqslant -1) = Pr(X = -1) = \frac{1}{2}$

- X never takes values in the interval $(-1, 1)$. So, if $-1 < x < 1$, then $F(x) = Pr(X \leqslant x) = Pr(X = -1) = \frac{1}{2}$.

- $F(1) = Pr(X \leqslant 1) = 1$

- If $x > 1$, then also $F(x) = Pr(X \leqslant x) = 1$.

All together,

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leqslant x < 1 \\ 1 & 1 \leqslant x \end{cases}$$

S-8: This system is fairly similar in structure to the random variable of Example 4.3.4 in the text and Question 7 above.

- If x is 1, 2, 3, 4, 5, or 6, then $F(x) = Pr(D \leqslant x) = \frac{x}{6}$.

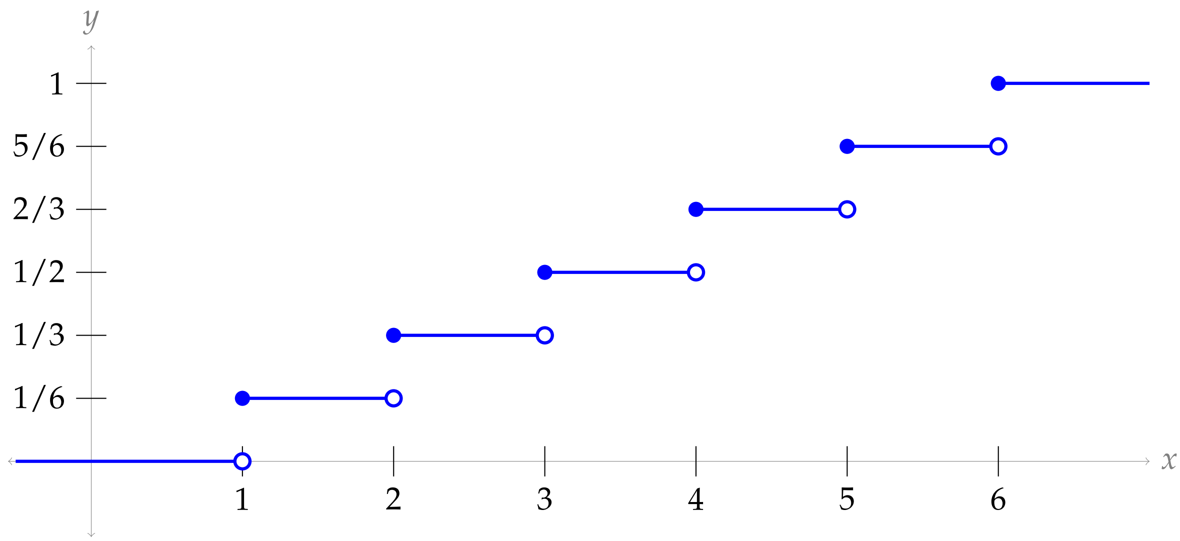
- If $x < 1$, then $F(x) = Pr(D \leqslant x) = 0$, because the dice never ever rolls numbers less than 1.

- If $x > 6$, then $F(x) = Pr(D \leqslant x) = 1$, because the dice always rolls numbers less than x.

- If (say) $x = 2.5$, then $F(2.5) = Pr(D \leqslant 2.5) = Pr(D \leqslant 2)$ because X is never in the interval $(2, 2.5]$. So, $F(2.5) = F(2)$. We can extrapolate this to $F(x) = F(2)$ for every x in the interval $[2, 3)$.

All together:

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{6} & 1 \leqslant x < 2 \\ \frac{1}{3} & 2 \leqslant x < 3 \\ \frac{1}{2} & 3 \leqslant x < 4 \\ \frac{2}{3} & 4 \leqslant x < 5 \\ \frac{5}{6} & 5 \leqslant x < 6 \\ 1 & x \geqslant 6 \end{cases}$$



Since its cumulative distribution function (CDF) is not continuous (note the jump discontinuities), D is not a continuous random variable.

S-9:

- Since Z is never less than -4:
  - if $x < -4$, then $F(x) = Pr(Z \leqslant x) = 0$, and
  - $F(-4) = Pr(Z \leqslant -4) = Pr(Z = -4) = \frac{1}{2}$
- Since Z is never in the interval $(-4, -2)$:
  - if $-4 < x < -2$, then $F(x) = Pr(Z \leqslant x) = Pr(Z = -4) = \frac{1}{6}$, and
  - $F(-2) = Pr(Z \leqslant -2) = Pr(Z = -4 \text{ or } Z = -2) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$
- Since Z is never in the interval $(-2, -1)$:
  - if $-2 < x < -1$, then $F(x) = Pr(Z \leqslant x) = Pr(Z \leqslant -2) = F(-2) = \frac{5}{6}$, and
  - $F(-1) = Pr(Z \leqslant -1) = Pr(Z = -4 \text{ or } Z = -2 \text{ or } Z = -1) = \frac{5}{6} + \frac{1}{6} = 1$
- If $x > -1$, then $F(x) = 1$

All together:

$$F(x) = \begin{cases} 0 & x < -4 \\ \frac{1}{2} & -4 \leqslant x < -2 \\ \frac{5}{6} & -2 \leqslant x < -1 \\ 1 & -1 \leqslant x \end{cases}$$

S-10: From Properties of a CDF (Corollary 4.3.10 in the text) we see that four things need to be true. For convenience, we'll go through them in a different order than that in the theorem.

4. $\lim\limits_{x \to -\infty} F(x) = 0$. This is already true for any values of A and B.

3. $\lim\limits_{x\to\infty} F(x) = 1$. For our particular function, $\lim\limits_{x\to\infty} F(x) = A$, so we must have $A = 1$.

1. $0 \leqslant F(x) \leqslant 1$ for all real x. Since $A = 1$, $F(x) \geqslant 0$ as long as $x^3 + B > 0$ for all $x \geqslant 0$. So, B needs to be a positive number.

2. $F(x)$ is nondecreasing. To check this, we can consider $F'(x)$ when $x > 0$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^3}{x^3+B}\right] &= \frac{(x^3+B)(3x^2) - x^3(3x^2)}{(x^3+B)^2}\\ &= \frac{3Bx^2}{(x^3+B)^2}\end{aligned}$$

When B is a positive number, this derivative is positive, so $F(x)$ is nondecreasing.

All together, we must have $A = 1$, and B must be a positive number.

S-11:

- First, let's see what needs to happen for F to be continuous.

$$\begin{aligned}\lim_{x\to 0^+} F(x) &= \lim_{x\to 0^+}\left[A + \frac{Bx}{x+1}\right] = A\\ \lim_{x\to 0^-} F(x) &= \lim_{x\to 0^-}\left[C + \frac{Dx}{1-x}\right] = C\end{aligned}$$

  So, choosing $A = C$ makes $F(x)$ continuous. From now on, we'll write

$$F(x) = \begin{cases} A + \frac{Bx}{x+1} & x \geqslant 0\\ A + \frac{Dx}{1-x} & x < 0\end{cases}$$

- Properties of a CDF are given in Corollary 4.3.10 in the text. Let's consider the limits.

$$0 = \lim_{x\to -\infty} F(x) = \lim_{x\to -\infty}\left[A + \frac{Dx}{1-x}\right] = A - D$$

  This tells us $D = A$.

$$1 = \lim_{x\to \infty} F(x) = \lim_{x\to \infty}\left[A + \frac{Bx}{x+1}\right] = A + B$$

  That tells us $B = 1 - A$

- CDFs are non-decreasing. If you're familiar with the graphs of the two functions in the piecewise definition, you can easily see that this gives us the requirements $B \geqslant 0$ and $D \geqslant 0$. We can also get there by looking at the derivative of $F(x)$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{Dx}{1-x}\right] &= \frac{(1-x)D - Dx(-1)}{(1-x)^2} = \frac{D}{(1-x)^2}\\ \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{Bx}{x+1}\right] &= \frac{(x+1)B - Bx(1)}{(x+1)^2} = \frac{B}{(x+1)^2}\end{aligned}$$

The derivatives are nonnegative when their parameters are positive. So, we have $D \geqslant 0$ and $B > 0$.

All together, we need $A = C = D$, $B = 1 - A$, $A \geqslant 0$, and $B \geqslant 0$. The last three are fulfilled when $B = 1 - A$ for values of A in $[0, 1]$.

S-12: First, since $F(x) = 0$ whenever $x < 5$, we know W is never less than 5. Similarly, since $F(12) = 1$, we know W is always 12 or less. That is, W only takes values in the interval $[5, 12]$.

Furthermore, within the intervals $(5, 6)$, $(6, 8)$, and $(8, 12)$, $F(x)$ is constant. That means W never falls inside these intervals, so the sample space of W is $\mathcal{S} = \{5, 6, 8, 12\}$.

Using Corollary 4.3.3, we see that subtracting different values of F tells us about the probability mass function (PMF).

- $F(5) - F(x) = \frac{1}{4}$ for any $x < 5$. To see this more concretely: $\frac{1}{4} = F(5) - F(4.99999) = Pr(4.99999 < W \leqslant 5)$. This tells us $Pr(W = 5) = \frac{1}{2}$.
- $F(6) - F(x) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ for any $5 < x < 6$. Again, to see this more concretely: $\frac{1}{12} = F(6) - F(5.99999) = Pr(5.99999 < W \leqslant 6)$. This tells us $Pr(W = 6) = \frac{1}{12}$.
- Similarly, $Pr(X = 8) = F(8) - F(6) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$
- $Pr(X = 12) = F(12) - F(8) = 1 - \frac{1}{2} = \frac{1}{2}$

All together:

x	$Pr(W = x)$
5	$\frac{1}{4}$
6	$\frac{1}{12}$
8	$\frac{1}{6}$
12	$\frac{1}{2}$

Note: it's tempting to memorize "subtract values of $F(x)$ to get $f(x)$." This is, indeed, a good fact to have in your pocket. But you should also *understand why it works that way,* from the definitions and properties of the cumulative distribution function (CDF).

S-13: Comparing to Question 12: where $F(x)$ is constant, there are no values in the sample space. So $\mathcal{S} = \{1, 2, 4\}$. The size of the jump discontinuity of $F(x)$ is the probability that $Y = x$. We see the biggest jump at $x = 2$, followed by $x = 4$, with the smallest jump at $x = 1$. So, $x = 2$ is most likely; $x = 4$ second most likely; and $x = 1$ is least likely.

$$\mathcal{S} = \{2, 4, 1\}$$

Solutions to Exercises 4.4 — Jump to TABLE OF CONTENTS

S-1:

(a) First, let's think about what $f(x) = 4x(x-1)(x-2)$ looks like on the interval $[0,1]$.

- $f(x)$ is cubic function with roots at $x = 0$, $x = 1$, and $x = 2$
- $\lim\limits_{x\to-\infty} f(x) = -\infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$. Knowing the basic shapes cubics take, we know that $f(x)$ will be positive on the interval $(0,1)$.

So $f(0) = 0$, then $f(x)$ increases and decreases, returning to the x-axis at $f(1) = 0$. That means we will have fewer dots close to $x = 0$ and $x = 1$, and more dots in the middle. (If we wanted to be really precise, we could find that $f(x)$ has a local maximum at $x = 1 - \frac{1}{\sqrt{3}}$. Dot diagrams aren't very precise, so there isn't much point to going that far.)

(b) $f(x) = \begin{cases} \frac{1}{2} & 0 < x < 1 \\ \frac{1}{3} & 1 < x < 2 \\ \frac{1}{6} & 2 < x < 3 \end{cases} = \begin{cases} \frac{3}{6} & 0 < x < 1 \\ \frac{2}{6} & 1 < x < 2 \\ \frac{1}{6} & 2 < x < 3 \end{cases}$

Comparing the intervals $(0,1)$, $(1,2)$, and $(1,3)$: we expect $(1,2)$ to be twice as dense as $(2,3)$, and we expect $(0,1)$ to be three times as dense as $(2,3)$.

(c) $f(x) = x^2$ increases on the interval $[1, \sqrt[3]{2}]$, so the dots will be sparse near $x = 1$ and dense near $x = \sqrt[3]{2}$.

S-2:

(See Corollaries 4.4.8 and 4.3.10.)

(a) limit at negative infinity is 1: neither. In the case of a cumulative distribution function (CDF), the limit is 0. In the case of a probability density function (PDF), if that limit were 1, then $\int_{-\infty}^{0} f(x)\mathrm{d}x$ would diverge, but we need $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$.

(b) never negative: both.

(c) nondecreasing: cumulative distribution functions (CDFs) only

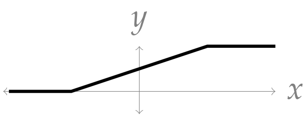
(d) never more than 1: cumulative distribution functions (CDFs) only. It is possible for a probability density function (PDF) to be greater than 1, as long as the area under its curve is only 1.

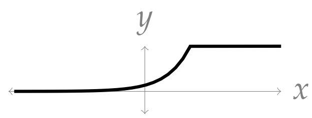
(e) area under the curve gives a probability: probability density functions (PDFs)

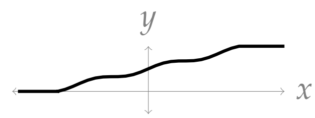
(f) value of function gives a probability: cumulative distribution functions (CDFs) (by definition)

(g) area under the curve from $-\infty$ to ∞ is 1: probability density functions (PDFs)

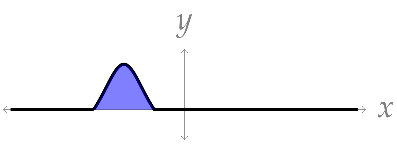
S-3: Cumulative distribution functions (CDFs) are nondecreasing, rising from 0 at the far left to 1 at the far right. This applies to b, c, and e. You can think of the "accumulation" as the function "piling up" as x goes to the right.

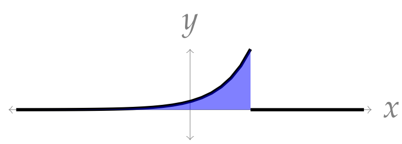


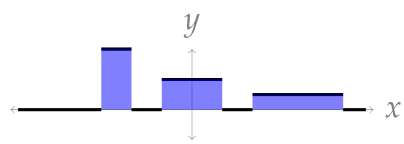




Probability density functions (PDFs) have a finite area (equal to one – but we don't have scales on our axes) underneath their curves. This applies to a, d, and f.



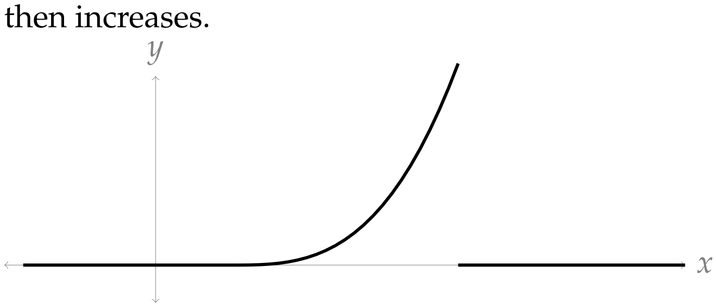




S-4: If the cumulative distribution function (CDF) $F(x)$ is continuous, then the random variable is continuous, and the probability density function (PDF) is $f(x) = F'(x)$.

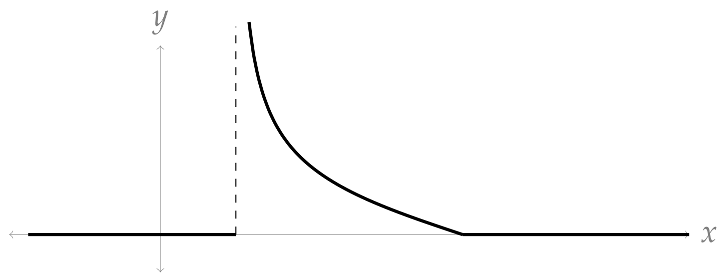
(a) Because the cumulative distribution function (CDF), $F(x)$, is continuous, the variable is continuous. In this case, the probability density function (PDF) is defined as $f(x) = F'(x)$, so we just sketch the derivative of the function shown.

The derivative is 0 outside the middle interval, where the derivative starts close to 0, then increases.



(b) Again the cumulative distribution function (CDF), $F(x)$, is continuous, so the variable is continuous and the probability density function (PDF) is defined as $f(x) = F'(x)$.

The tangent line to the curve seems to be vertical where $F(x)$ first starts increasing, so we'll give our derivative a vertical asymptote there. Then the derivative decreases to 0.



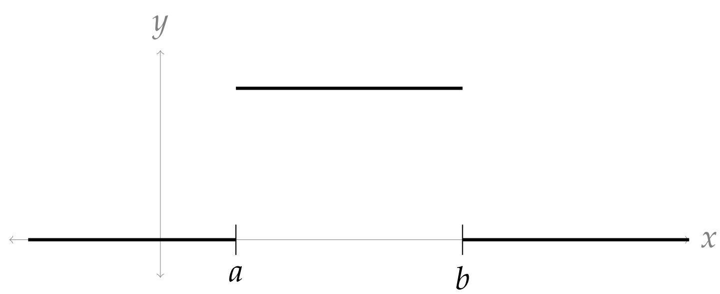
(c) The cumulative distribution function (CDF) of this random variable is not a continuous function, so the variable is not continuous. Note if we did take the derivative of this function, it would be 0 everywhere except the locations of the jumps, where it wouldn't exist. This is a nice illustration of why the probability density function (PDF) is only defined for continuous random variables.

S-5: In each case, we're plotting $F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$, where $f(t)$ is the function shown in the problem. In each case shown, since $f(t)$ is zero when $t < 0$, we see $\int_{-\infty}^{0} f(t)\mathrm{d}t = 0$. So, for positive values of x,

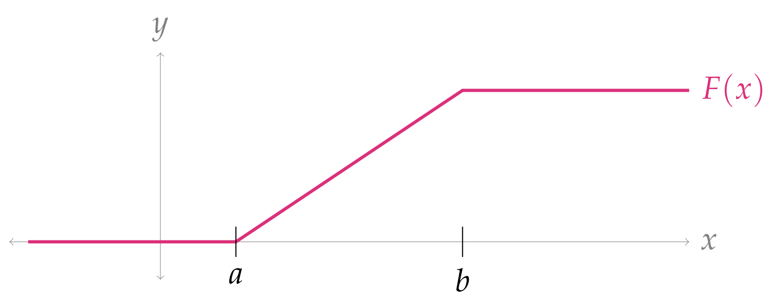
$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_{0}^{x} f(t)\mathrm{d}t$$

In particular, $F(0) = 0$. So rather than trying to estimate area, we can think of $F(x)$ as the antiderivative of $f(x)$ that satisfies $F(0) = 0$.

(a)



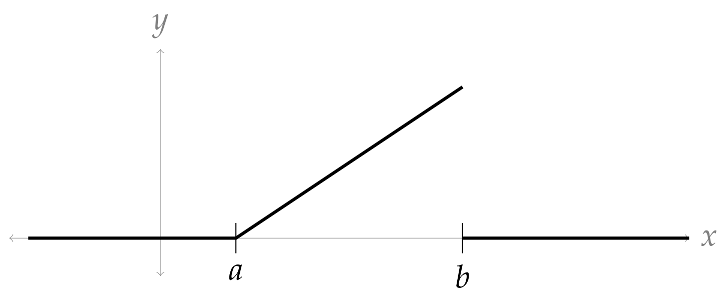
On the interval $[a, b]$, $f(x)$ is constant, so $F(x)$ is a line. Since $f(x)$ is positive, $F(x)$ has positive slope, i.e. it's an increasing line. For $x > b$, $f(x) = 0$, so $F(x)$ turns into a horizontal line.



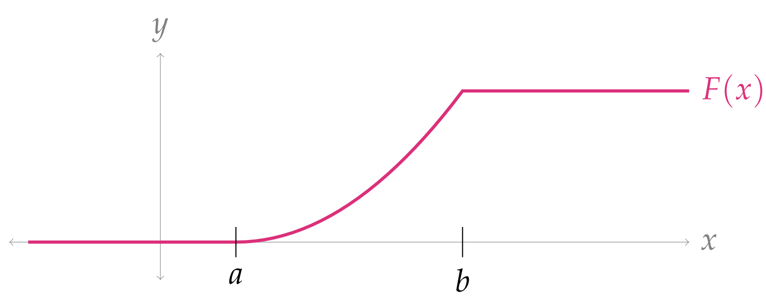
Something to remember is that by definition, since the random variable was given as continuous, $F(x)$ is a continuous function. It's tempting to try to make $F(x) = 0$

when $x > b$. If you're struggling with understanding why that *isn't* the case, go review the proof of Corollary 4.3.10 part 2. This is a case where really remembering and understanding definitions is important. We defined the probability density function (PDF) as the derivative of the cumulative distribution function (CDF); but that definition alone isn't enough to tell us how to find the cumulative distribution function (CDF) from the probability density function (PDF), because there are infinitely many functions that have the same derivative.

(b)

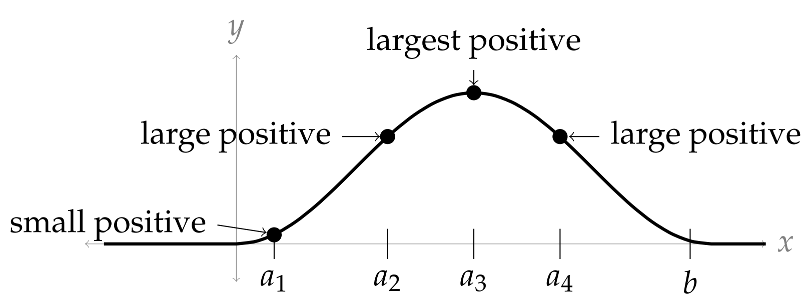


Again, we expect $F(x) = 0$ when $x < a$ and $F(x) = 1$ when $x > b$. On the interval (a, b), the derivative of $F(x)$ starts at $F'(a) = f(a) = 0$, then increases to its maximum at b. To be even more precise, recall an antiderivative of a linear function is a quadratic function.

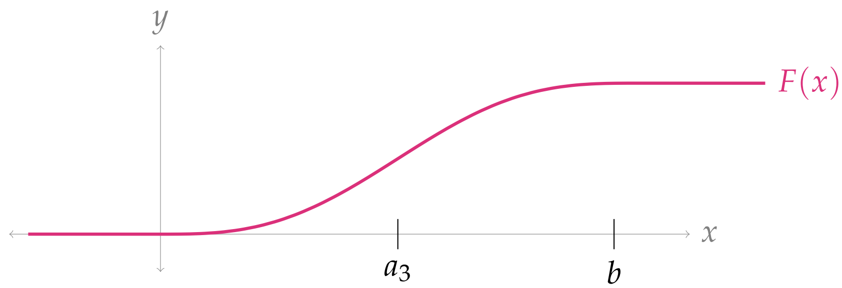


Note the sharp corner at $x = b$. $F(x)$ isn't differentiable there, and indeed $f(x)$ has a discontinuity at $x = b$.

(c)



As with the other parts, $F(x)$ is constant outside the interval $[a, b]$. Inside that interval, $F(x)$ has a derivative that starts out as a small positive number, and then increases until $x = a_3$. After a_3, $f(x)$ is still positive, so $F(x)$ is still increasing; its slope is positive but decreasing until $x = b$.



Note in the sketch above, $F(x)$ is steepest at $x = a_3$, and its derivative is 0 outside of the interval $[0, b]$.

It's a common mistake to see $F'(x)$ decreasing on (a_3, b) and think that $F(x)$ should be decreasing as well. But note that $F'(x)$ is positive on that interval, so $F(x)$ is *increasing* – it's just increasing slower and slower.

S-6: By Corollary 4.4.8,

$$Pr(4 \leqslant W \leqslant 17) = \int_4^{17} f(x)\mathrm{d}x = \int_4^{17} \frac{10/\pi}{1+100x^2}\mathrm{d}x$$

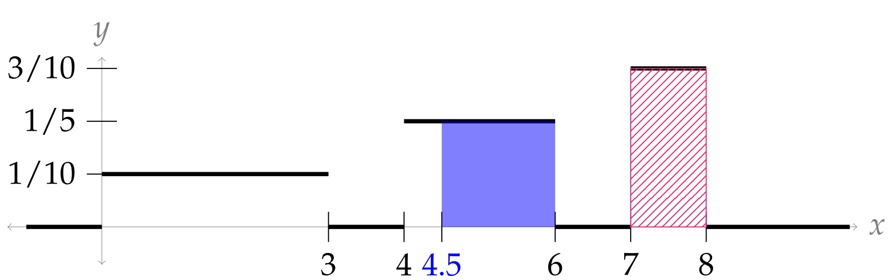
The integrand looks like the derivative of arctangent, so we use the substitution $u = 10x$, $\mathrm{d}u = 10\mathrm{d}x$.

$$\begin{aligned} &= \frac{1}{\pi}\int_4^{17} \frac{1}{1+100x^2} \cdot 10\mathrm{d}x = \frac{1}{\pi}\int_{40}^{170} \frac{1}{1+u^2}\mathrm{d}u \\ &= \frac{1}{\pi}\left[\arctan u\right]_{40}^{170} = \frac{\arctan(170) - \arctan(40)}{\pi} \approx 0.006 \end{aligned}$$

S-7: By Corollary 4.4.8,

$$Pr(Q \geqslant 4.5) = Pr(4.5 \leqslant Q < \infty) = \int_{4.5}^{\infty} f(x)\mathrm{d}x$$

If we wanted to find this area using the Fundamental Theorem of Calculus, we'd need to chop the interval $[4.5, \infty)$ up into subintervals $(4.5, 6)$, $(6, 7)$, $(7, 8)$, and $(8, \infty)$. It's easier to use geometry.



The blue solid rectangle has area $(1.5)(\frac{1}{5}) = 0.3$, and the pink striped rectangle has area $(1)(\frac{3}{10}) = 0.3$, so all together $Pr(Q \geqslant 4.5) = 0.6$.

S-8: Because the two events are disjoint,

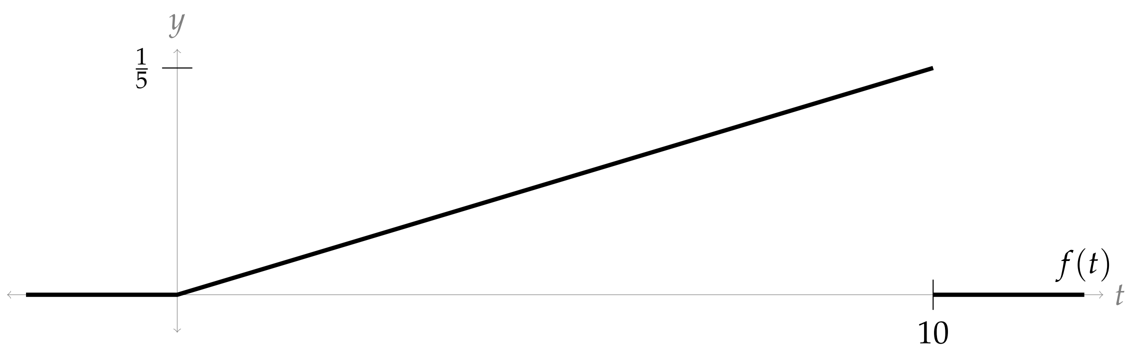
$$Pr\Big(0 < M < 1 \text{ OR } 9 < M < 10\Big) = Pr\Big(0 < M < 1\Big) + Pr\Big(9 < M < 10\Big)$$

Since M is a continuous variable, $Pr\Big(a < M < b\Big) = Pr\Big(a \leqslant M \leqslant b\Big)$.

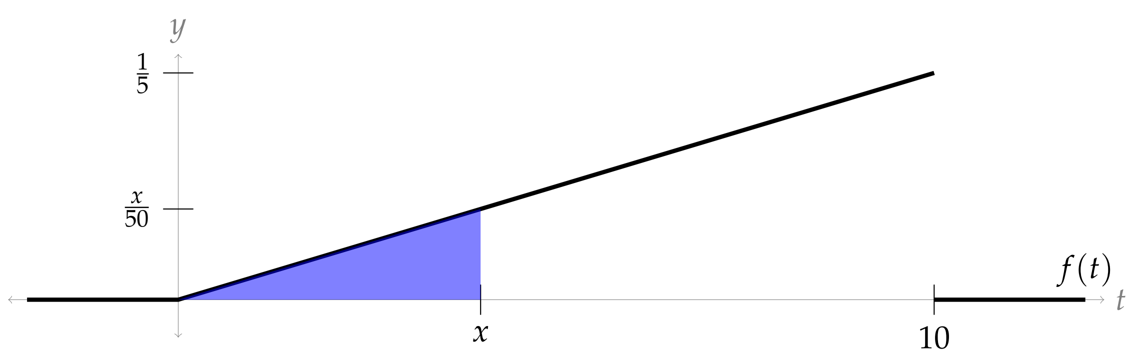
$$= \int_0^1 f(x)\mathrm{d}x + \int_9^{10} f(x)\mathrm{d}x = \int_0^1 \frac{x}{50}\mathrm{d}x + \int_9^{10} \frac{x}{50}\mathrm{d}x$$

$$= \left[\frac{x^2}{100}\right]_0^1 + \left[\frac{x^2}{100}\right]_9^{10} = \left[\frac{1}{100} - 0\right] + \left[\frac{100}{100} - \frac{81}{100}\right] = \frac{2}{10}$$

S-9: By Corollary 4.4.10, $F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$.



- There is no area under $f(t)$ when $t \leqslant 0$. So, if $x \leqslant 0$, then $F(x) = 0$.
- There is no area under $f(t)$ on the interval $[10, \infty)$. So, if $x \geqslant 10$, then $F(x) = F(10)$. By the properties of a cumulative distribution function (CDF), we know this has to be 1. We can also check that by noting the area under the entire curve $f(t)$ is a triangle with area $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(10)\left(\frac{1}{5}\right) = 1$.
- For x in the interval $(0, 10)$, again we can use the area of a triangle to find $F(x)$:



From the definition of our probability density function (PDF, we see the shaded triangle has height $\frac{x}{50}$, so its area is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(x)\left(\frac{x}{50}\right) = \frac{x^2}{100}$. So, for x in the interval $(0, 10)$, $f(x) = \frac{x^2}{100}$.

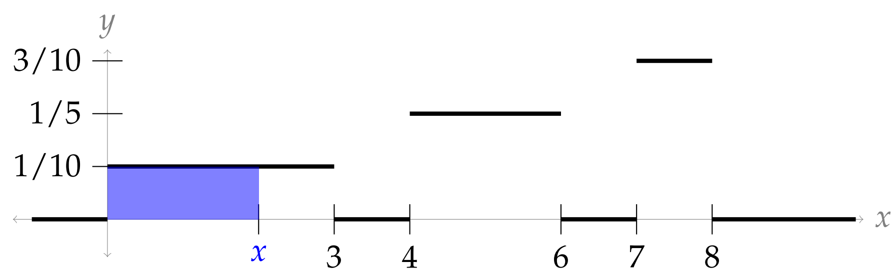
All together, $F(x) = \begin{cases} 0 & x \leqslant 0 \\ \frac{x^2}{100} & 0 < x < 10 \\ 1 & x \geqslant 10 \end{cases}$

S-10: The method to solve this problem is similar to Question 9.

- $F(x) = 0$ when $x \leqslant 0$
- $F(x) = 1$ when $x \geqslant 8$
- There is no area under $f(t)$ on the interval $[3,4]$ or the interval $[6,7]$, so $F(x) = F(3)$ when $3 \leqslant x \leqslant 4$ and $F(x) = F(6)$ when $6 \leqslant x \leqslant 7$.
- If $0 < x < 3$, then

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = 0 + \int_0^x \frac{1}{10}\mathrm{d}t = \frac{x}{10}$$

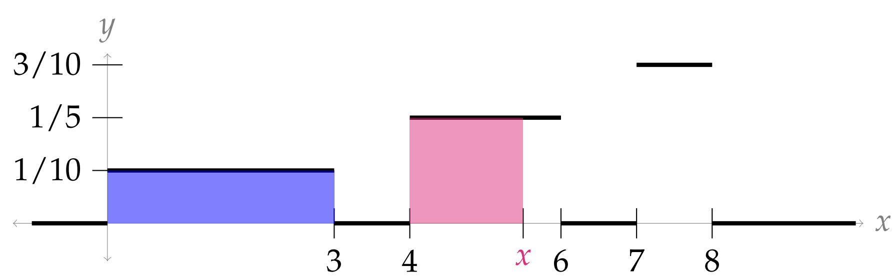
In particular, $F(3) = \frac{3}{10}$.



- If $4 < x < 6$, then

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = 0 + \int_0^3 f(t)\mathrm{d}t + 0 + \int_4^x f(t)\mathrm{d}t = 0.3 + \frac{1}{5}(x-4) = \frac{x}{5} - \frac{1}{2}$$

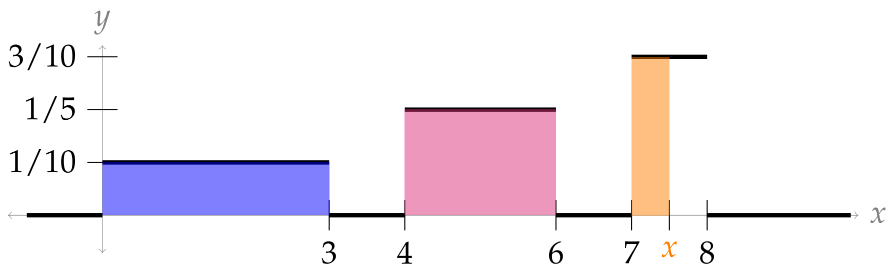
In particular, $F(6) = \frac{7}{10}$.



- If $7 < x < 8$, then

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\mathrm{d}t = 0 + \int_0^3 f(t)\mathrm{d}t + 0 + \int_4^6 f(t)\mathrm{d}t + 0 + \int_7^x f(t)\mathrm{d}t \\ &= \frac{7}{10} + \frac{3}{10}(x-7) = \frac{3x}{10} - \frac{7}{5} \end{aligned}$$

In particular, $F(8) = 1$.



All together, $F(x) = \begin{cases} 0 & x \leqslant 0 \\ \frac{x}{10} & 0 < x \leqslant 3 \\ \frac{3}{10} & 3 < x \leqslant 4 \\ \frac{x}{5} - \frac{1}{2} & 4 \leqslant x < 6 \\ \frac{7}{10} & 6 \leqslant x < 7 \\ \frac{3x}{10} - \frac{7}{5} & 7 \leqslant x < 8 \\ 1 & 8 \leqslant x \end{cases}$

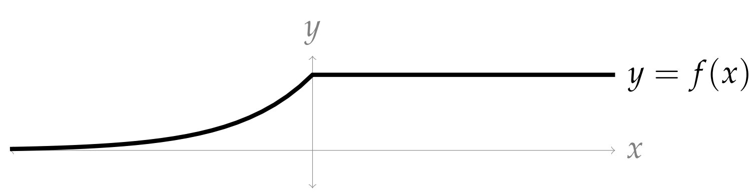
It's a common mistake to focus on only one triangle at a time, and forget the triangles to the left of the current one. For this question, it's important to remember the C in CDF stands for *cumulative*. That's why we're adding up all the area under the curve from $-\infty$ to x.

S-11: X is continuous if $F(X)$ is continuous (Definition 4.3.6 in the text). Note $\lim\limits_{x \to 0^-} F(x) = e^0 = 1 = \lim\limits_{x \to 0^+} F(x)$. So, $F(x)$ is continuous, so X is a continuous random variable.

Continuous variables have PDFs, not PMFs. Definition 4.4.1 in the text says the PDF of a continuous random variable is the derivative of its CDF, where that derivative exists. So:

$$\begin{aligned} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \begin{cases} e^x & x \leqslant 0 \\ 1 & x > 0 \end{cases} \\ &= \begin{cases} e^x & x < 0 \\ 0 & x > 0 \end{cases} \end{aligned}$$

We note here that for some of the subtler parts of the question, it's convenient to sketch a graph of $y = f(x)$.



From the picture, it's easy to see that $f(x)$ is continuous, and that its derivative does not exist at $x = 0$.

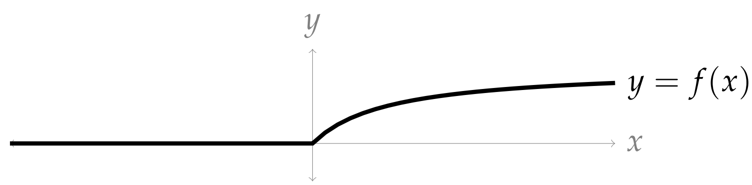
S-12:

X is continuous if $F(X)$ is continuous (Definition 4.3.6 in the text). Note $\lim\limits_{x\to 0^-} F(x) = 0 = \lim\limits_{x\to 0^+} F(x)$. So, $F(x)$ is continuous, so X is a continuous random variable.

Continuous variables have PDFs, not PMFs. Definition 4.4.1 in the text says the PDF of a continuous random variable is the derivative of its CDF, where that derivative exists. So:

$$\begin{aligned} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\begin{cases} \frac{x}{x+1} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \\ &= \begin{cases} \frac{(x+1)-x}{(x+1)^2} & x > 0 \text{ (quotient rule)} \\ 0 & x < 0 \end{cases} \\ &= \begin{cases} \frac{1}{(x+1)^2} & x > 0 \\ 0 & x < 0 \end{cases} \end{aligned}$$

We note here that for some of the subtler parts of the question, it's convenient to sketch a graph of $y = f(x)$.



From the picture, it's easy to see that $f(x)$ is continuous, and that its derivative does not exist at $x = 0$.

S-13: Properties of a PDF are given in Corollary 4.4.8 in the text. In particular, the area under the entire function is 1.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\mathrm{d}x \\ &= 0 + \int_{-1}^{b} e^x \mathrm{d}x \\ &= e^x\Big|_{-1}^{b} \\ &= e^b - e^{-1} \\ 1 + \frac{1}{e} &= e^b \\ \ln\left(1 + \frac{1}{e}\right) &= b \end{aligned}$$

S-14: Properties of a PDF are given in Corollary 4.4.8 in the text. In particular, the area under the entire function is 1. **Solution 1:**

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{A}{x^2+4}\mathrm{d}x
\end{aligned}
$$

Let $x = 2\tan\theta$, $\mathrm{d}x = 2\sec^2\theta\mathrm{d}\theta$. Note $\theta = \arctan\left(\frac{x}{2}\right)$ and $\lim\limits_{x\to\pm\infty} \arctan\left(\frac{x}{2}\right) = \pm\frac{\pi}{2}$

$$
\begin{aligned}
1 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{A}{4\tan^2\theta+4} \cdot 2\sec^2\theta\mathrm{d}\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2A}{4\sec^2\theta} \cdot \sec^2\theta\mathrm{d}\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{A}{2}\mathrm{d}\theta \\
&= \frac{A}{2}\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{A}{2}(\pi) \\
\frac{2}{\pi} &= A
\end{aligned}
$$

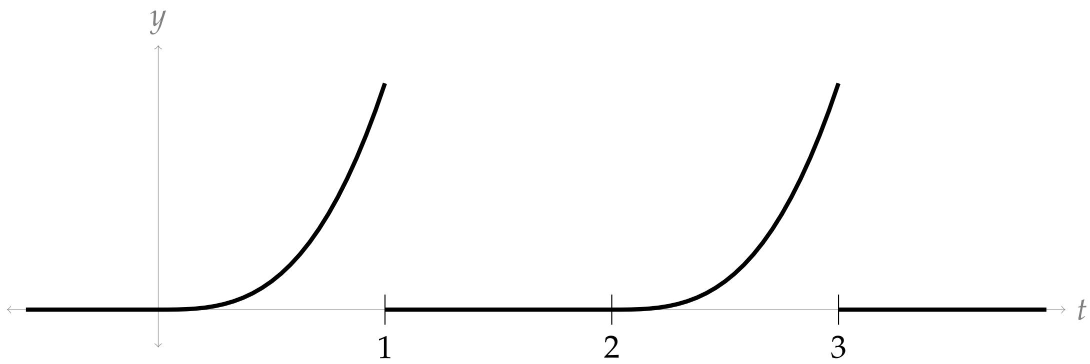
Solution 2:

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{A}{x^2+4}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{A}{4\left(\frac{x}{2}\right)^2+4}\mathrm{d}x \\
&= \frac{A}{4}\int_{-\infty}^{\infty} \frac{1}{\left(\frac{x}{2}\right)^2+1}\mathrm{d}x
\end{aligned}
$$

Let $u = \frac{x}{2}$, $\mathrm{d}u = \frac{1}{2}\mathrm{d}x$.

$$
\begin{aligned}
1 &= \frac{A}{4}\int_{-\infty}^{\infty} \frac{1}{u^2+1} \cdot 2\mathrm{d}u \\
&= \frac{A}{2}\left[\lim_{a\to-\infty} \arctan u\Big|_a^0 + \lim_{b\to\infty} \arctan u\Big|_0^b\right] \\
&= \frac{A}{2}\left[0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0\right] \\
&= \frac{\pi}{2}A \\
\frac{2}{\pi} &= A
\end{aligned}
$$

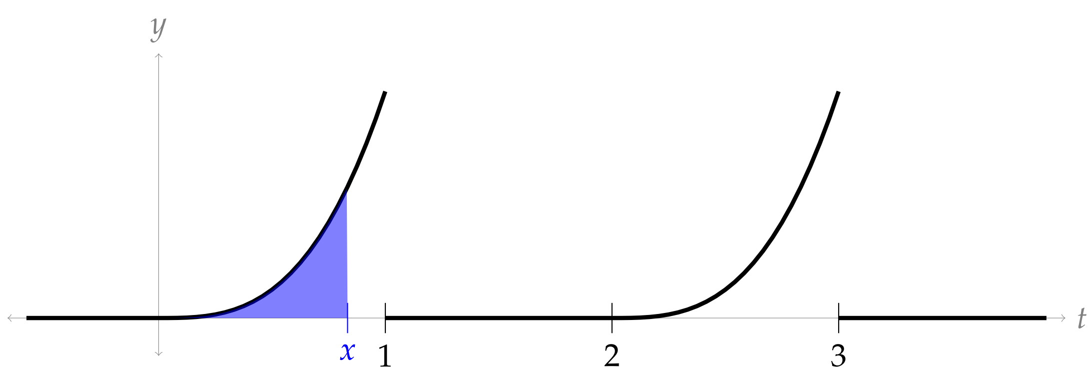
S-15:



The intervals that are most interesting are $[0,1]$ and $[2,3]$; on all other intervals, $F(x)$ is constant.

- If x is on the interval $[0,1]$, then

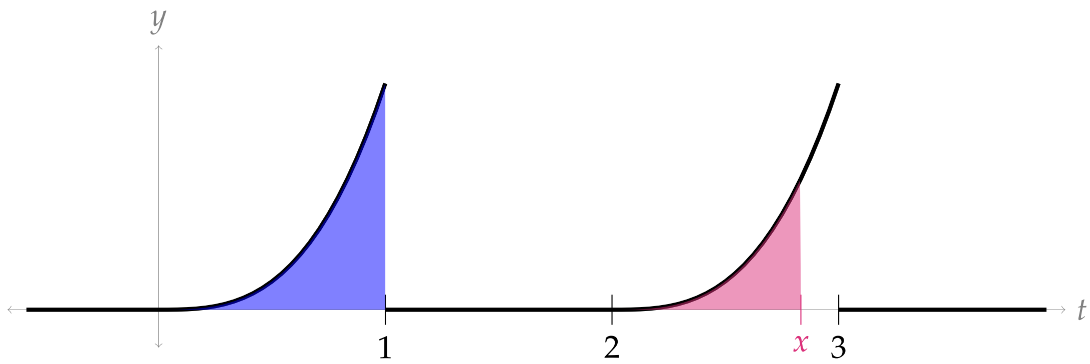
$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = 0 + \int_0^x 2t^3 \mathrm{d}t = \left[\frac{2t^4}{4}\right]_0^x = \frac{x^4}{2}$$
$$F(1) = \frac{1}{2}$$



- Note the function $2(x-2)^2$ is the result of shifting the function $2x^2$ to the right by two.

  If x is on the interval $[2,3]$, then

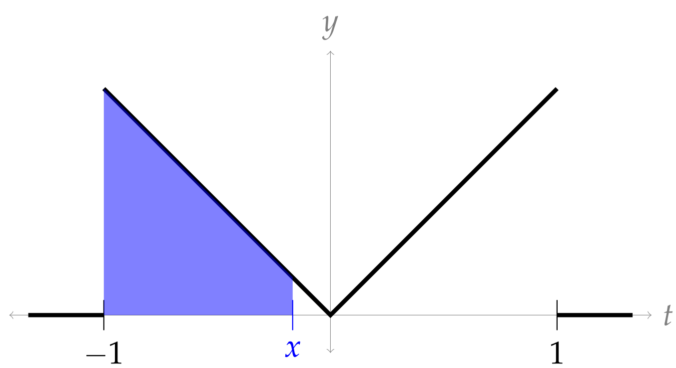
$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = 0 + \frac{1}{2} + 0 + \int_2^x f(t)\mathrm{d}t = \frac{1}{2} + \frac{(x-2)^4}{2}$$
$$F(3) = 1$$



All together, $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^4}{2} & 0 \leqslant x \leqslant 1 \\ \frac{1}{2} & 1 < x < 2 \\ \frac{1}{2} + \frac{(x-2)^4}{2} & 2 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$

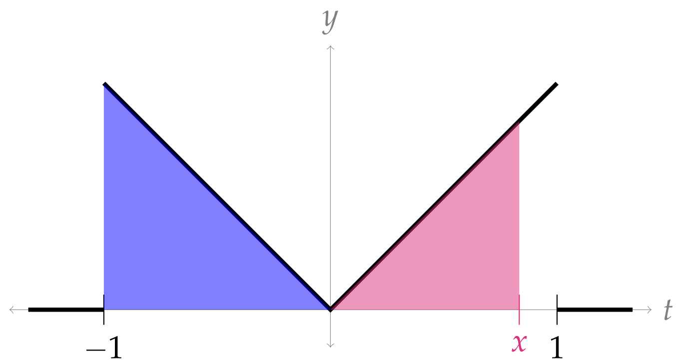
S-16: Remember we define $|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$. So, we'll be considering the intervals $[-1, 0]$ and $(0, 1]$ separately.

- First, suppose $-1 \leqslant x \leqslant 0$.



$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\mathrm{d}t = 0 + \int_{-1}^{x} -t\mathrm{d}t = \left[-\frac{t^2}{2}\right]_{-1}^{x} \\ &= -\frac{x^2}{2} + \frac{1}{2} = \frac{1}{2} - \frac{x^2}{2} \\ F(0) &= \frac{1}{2} \end{aligned}$$

- Now, consider $0 < x \leqslant 1$.



$$\begin{aligned}F(x) &= \int_{-\infty}^{x} f(t)\mathrm{d}t = 0 + \frac{1}{2} + \int_0^x t\mathrm{d}t = \frac{1}{2} + \left[\frac{t^2}{2}\right]_0^x \\ &= \frac{1}{2} + \frac{x^2}{2} \\ F(1) &= 1\end{aligned}$$

All together, $F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} - \frac{x^2}{2} & -1 \leqslant x \leqslant 0 \\ \frac{1}{2} + \frac{x^2}{2} & 0 < x \leqslant 1 \\ 1 & x > 1 \end{cases}$

S-17:

(a) Since $f(x)$ is a probability density function (PDF), the area under the entire curve must equal 1. In particular, since $f(x)$ is 0 outside the interval $[0, 200]$:

$$\begin{aligned}1 &= \int_0^{200} cx^2(200 - x)\mathrm{d}x = c\int_0^{200} (200x^2 - x^3)\mathrm{d}x \\ &= c\left[\frac{200}{3}x^3 - \frac{1}{4}x^4\right]_0^{200} = c\left[\frac{200^4}{3} - \frac{200^4}{4}\right] \\ &= 200^4 c\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{200^4}{12}c \\ c &= \frac{12}{200^4}\end{aligned}$$

(b) If $M < 5$, then $M_1 = 0$; if $5 \leqslant M < 15$, then $M_1 = 10$; if $15 \leqslant M < 25$, then $M_1 = 20$; etc. So, the sample space for M_1 is $\mathcal{S}_1 = \{0, 10, 20, \ldots, 200\}$.

(c) We'll be evaluating $Pr(a \leqslant M < b)$ for different values of a and b. If the interval $[a, b]$

is in the interval $[0, 200]$, then following our work on the previous part:

$$
\begin{aligned}
Pr(a \leqslant M < b) &= \int_a^b cx^2(200 - x)\mathrm{d}x = c\int_a^b \left(200x^2 - x^3\right)\mathrm{d}x \\
&= c\left[\frac{200}{3}x^3 - \frac{1}{4}x^4\right]_a^b \\
&= c\left[\frac{200}{3}(b^3 - a^3) - \frac{1}{4}(b^4 - a^4)\right] \\
&= c\left[\frac{200}{3}(b-a)(b^2 + ab + a^3) - \frac{1}{4}(b^2 - a^2)(b^2 + a^2)\right] \\
&= c\left[\frac{200}{3}(b-a)(b^2 + ab + a^2) - \frac{1}{4}(b+a)(b-a)(b^2 + a^2)\right]
\end{aligned}
$$

Now, let's compute $Pr(M_1 = m)$ for the different m in $\mathcal{S}_1$.

- $m = 0$:

$$
\begin{aligned}
Pr(M_1 = 0) = Pr(0 \leqslant M < 5) &= \int_0^5 cx^2(200 - x)\mathrm{d}x \\
&= c\left[\frac{200}{3}(5^3 - 0) - \frac{1}{4}(5^4 - 0)\right] = \frac{157}{40^4}
\end{aligned}
$$

- $m = 20$:

$$
\begin{aligned}
Pr(M_1 = 20) = Pr(15 \leqslant M \leqslant 20) &= \int_{15}^{20} cx^2(200 - x)\mathrm{d}x \\
&= c\left[\frac{200}{3}(20^3 - 15^3) - \frac{1}{4}(20^4 - 15^4)\right] \\
&= c\left[\frac{200}{3}(5^3)(4^3 - 3^3) - \frac{1}{4}(5^4)(4^4 - 3^4)\right] \\
&= c\left[\frac{40}{3}(5^4)(37) - \frac{1}{4}(5^4)(175)\right] = \frac{3595}{40^4}
\end{aligned}
$$

- All other m in S_1, i.e. $m = 10, 20, 30, \ldots, 180$:

$$
Pr(M_1 = m) = Pr(m - 5 \leqslant M < m + 5) = \int_{m-5}^{m+5} cx^2(200 - x)\mathrm{d}x
$$

  Setting $a = m - 5$ and $b = m + 5$, we have $b + a = 2m$ and $b - a = 10$

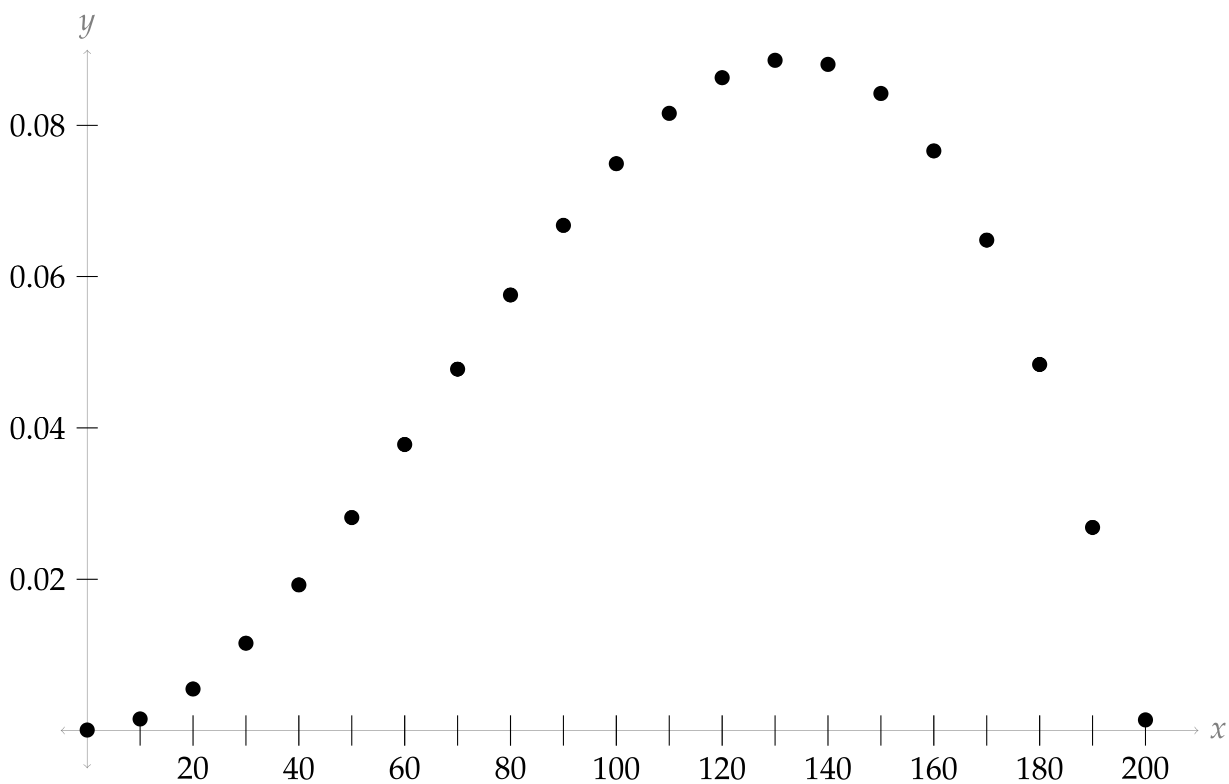
$$
\begin{aligned}
&= c\left[\frac{200}{3}(b-a)(b^2 + ab + a^2) - \frac{1}{4}(b+a)(b-a)(b^2 + a^2)\right] \\
&= c\left[\frac{200}{3}(10)(b^2 + ab + a^2) - \frac{1}{4}(2m)(10)(b^2 + a^2)\right] \\
&= c\left[\frac{2000}{3}(b^2 + ab + a^2) - 5m(b^2 + a^2)\right]
\end{aligned}
$$

Plugging in $a = m - 5$ and $b = m - 5$ and simplifying the quadratics:

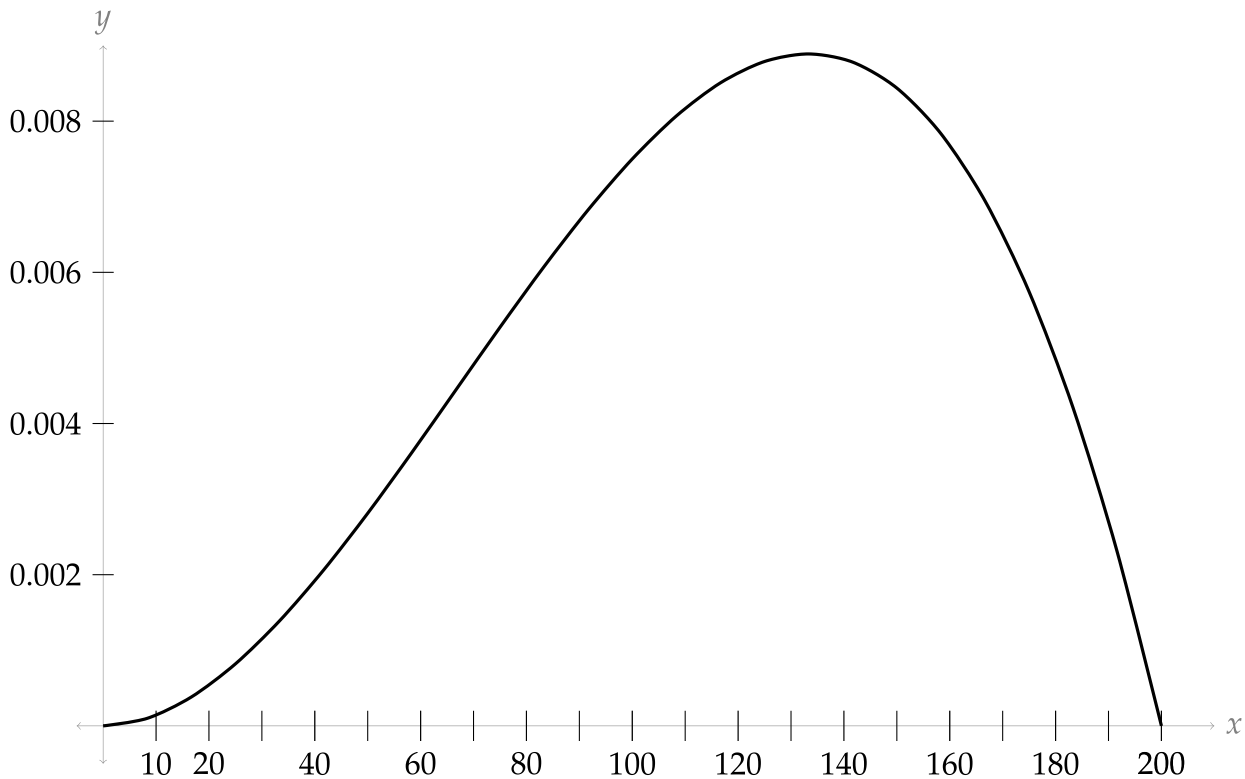
$$\begin{aligned}
&= c\left[\frac{2000}{3}(3m^2+25) - 5m(2m^2+50)\right] \\
&= \frac{12}{200^4}\left[-10m^3 + 2,000m^2 - 250m + \frac{50,000}{3}\right]
\end{aligned}$$

(d) From our work in the last part, we see the probability mass function (PMF) of M_1 is:

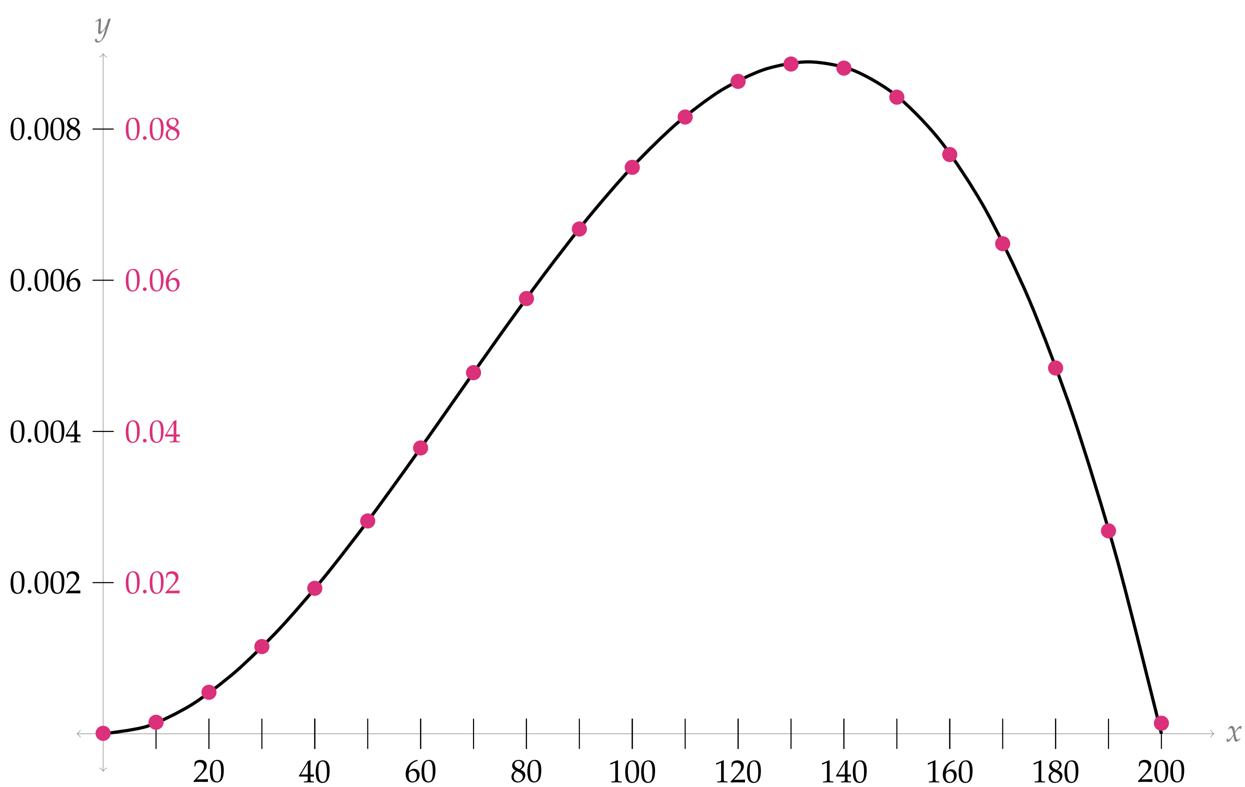
$$Pr(M_1 = m) = \begin{cases} \frac{157}{40^4} & m = 0 \\ \frac{12}{200^4}\left[-10m^3 + 2,000m^2 - 250m + \frac{50,000}{3}\right] & m = 10, 20, \ldots, 180 \\ \frac{3595}{40^4} & m = 200 \end{cases}$$



Now, we sketch $f(x) = \frac{12}{200^4}x^2(200 - x)$.



Notice the scales differ by a factor of 10, but other than that the values of the probability mass function (PMF) seem to approximate the values of the probability density function (PDF). Below we sketch both functions on the same axis, although the y-scales differ.



We took a discrete variable as an approximation of a continuous variable, and then the probability mass function (PMF) for the discrete variable approximates the probability density function (PDF) for the continuous variable, but on a different scale. Since we collapsed intervals of length 10 in $\mathcal{S}$ into each[14] discrete point of $\mathcal{S}_1$, the scales[15] differ by a factor of 10.

S-18: We know $Pr(0 \leqslant X \leqslant 1) = \int_0^1 f(x)\mathrm{d}x$. Equation 3.9.9 gives us Simpson's Rule:

$$\int_a^b f(x)\,\mathrm{d}x \approx \Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots$$
$$\cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]\frac{\Delta x}{3}$$

where $\Delta x = \frac{b-a}{2}$. So,

$$\begin{aligned} Pr(0 \leqslant X \leqslant 1) &= \int_0^1 f(x)\mathrm{d}x \\ &\approx \left[f(0) + 4f\left(\tfrac{1}{4}\right) + 2f\left(\tfrac{1}{2}\right) + 4f\left(\tfrac{3}{4}\right) + f(1)\right]\tfrac{1}{3} \\ &= \left[\sqrt{\frac{2}{\pi}} + 4\left(\sqrt{\frac{2}{\pi}}e^{-1/8}\right) + 2\left(\sqrt{\frac{2}{\pi}}e^{-1/2}\right) + 4\left(\sqrt{\frac{2}{\pi}}e^{-9/8}\right) + \left(\sqrt{\frac{2}{\pi}}e^{-2}\right)\right]\frac{1}{3} \end{aligned}$$

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S-1: False.

Indeed, it is often the case that $\mathbb{E}(X)$ is not even in the sample space of X. For example, imagine X results from a fair coin flip, where $X = 0$ when the flip is tails and $X = 1$ when the flip is heads. Then $\mathbb{E}(X) = \frac{1}{2}$, which is *never* a value of X.

S-2: Let X be the random variable resulting in a dice roll. If the dice is fair, then

$$E(X) = \sum_{x=1}^{6} x \cdot Pr(X = x) = \sum_{x=1}^{6} \frac{x}{6} = 3.5$$

Our interpretation of expected value is a long-term average: if we roll a fair dice a large number of times, we expect the average to be close to 3.5. One million seems like a lot of rolls, and our average doesn't seem that close to 3.5, so the dice does seem rigged.

Notice the vagaries here: how many rolls is "a lot?" How close to 3.5 is "close?" There's a lot of statistics that goes into trying to, say, test that a dice is actually fair. Those statistics, however, are beyond the scope of this class. That's why we're asking such general, vague questions.

14 except at $m = 0$ and $m = 20$, where the intervals only had length 5
15 except at the endpoints

S-3: If we think of $\mathbb{E}(X)$ as a long term average, then since every trial of X results in $X = a$, our average will always be a.

Alternatively, $\mathbb{E}(X) = \sum_{x=a} a \cdot Pr(X = a) = a \cdot 1 = a$

S-4: Z does not have to be uniformly distributed. For any probability density function (PDF) $f(x)$ that has *even symmetry*, the function $xf(x)$ has odd symmetry, so

$$\mathbb{E}(Z) = \int_{-1}^{1} xf(x)\mathrm{d}x = 0$$

If $f(x)$ is a constant function on the interval $[-1, 1]$, then Z is uniformly distributed, but this isn't the only function with even symmetry. For example, we could have $f(x) = \frac{3}{2}x^2$.

S-5:

To do the problem, first let the random variable X be the number of days the men's soccer team plays soccer per week. X has sample space $\{0, 1, 2\}$.

Definition 4.5.1 says:
Given a discrete random variable X, the **expected value of** X, denoted $\mathbb{E}(X)$, is given by

$$\sum x \cdot Pr(X = x)$$

where the sum is taken over every possible value of X.

In this case,

$$\begin{aligned}\mathbb{E}(X) &= 0 \cdot Pr(X = 0) + 1 \cdot Pr(X = 1) + 2 \cdot Pr(X = 2)\\ &= 0 \cdot 0.2 + 1 \cdot 0.5 + 2 \cdot 0.3 = 1.1\end{aligned}$$

The expected value is 1.1. The soccer team would, on the average, expect to play soccer 1.1 days per week. The number 1.1 is the long-term average or expected value if the men's soccer team plays soccer week after week after week.

S-6: We use the definition of expected value:

$$\begin{aligned}\mathbb{E}(X) &= \sum_{x=0}^{5} Pr(X = x) = 0 \cdot \frac{4}{50} + 1 \cdot \frac{8}{50} + 2 \cdot \frac{16}{50} + 3 \cdot \frac{14}{50} + 4 \cdot \frac{6}{50} + 5 \cdot \frac{2}{50}\\ &= \frac{116}{50} = 2.32\end{aligned}$$

S-7: Let X be the amount of money given to you after one round, where $X = -1$ if you loose (because your total money went down by one dollar). The only values X can take are -1 (if you loose) and 4 (if you win). Then $Pr(X = -1) = \frac{3}{4}$ and $Pr(X = 4) = \frac{1}{4}$, so

$$\mathbb{E}(X) = \frac{3}{4}(-1) + \frac{1}{4}(4) = \frac{1}{4}$$

If you were to play N times, where N is a large number, then your average winnings per turn would be close to $\frac{1}{4}$ (twenty-five cents). That is, on average, you earned \$0.25 each time you played. So, your *total* winnings would be $\frac{N}{4}$.

S-8: Let X be the amount of money won after one game, where X is negative if you lose. The sample space of X is $\{-10, 1, 7\}$. Its expected value is

$$\mathbb{E}(X) = (-10)\frac{2}{5} + 1\frac{2}{5} + 7\frac{1}{5} = -\frac{11}{5}$$

If you play the game a lot of times, then your average winnings per game is $-\frac{11}{5}$ dollars. So, your total winnings are actually a *loss* of $-\frac{11}{5}N$ dollars, where N is the number of games.

S-9: Using Definition 4.5.1,

$$\begin{aligned}\mathbb{E}(M) &= \int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x = 0 + \int_0^{100} x \cdot \frac{x}{5000}\mathrm{d}x \\ &= \left[\frac{x^3}{15000}\right]_0^{100} = \frac{10^6}{15000} = \frac{1000}{15} = 66 + \frac{2}{3}\end{aligned}$$

The sample space of M is the interval $\mathcal{S} = [0, 100]$, so by Theorem 4.5.7, $0 \leqslant \mathbb{E}(X) \leqslant 100$. The PDF of M is increasing on its sample space, so by Theorem 4.5.8, $\mathbb{E}(X) > 50$. All together, we must have $50 < \mathbb{E}(M) \leqslant 100$, and indeed this is the case.

S-10: Using Definition 4.5.1,

$$\mathbb{E}(N) = \int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x = 0 + \int_{-1}^{1} x \cdot \frac{2/\pi}{x^2+1}\mathrm{d}x$$

The function $c\frac{x}{x^2+1}$ has odd symmetry, so

$$\int_{-1}^{1} x \cdot \frac{2/\pi}{x^2+1}\mathrm{d}x = 0$$

So, $\mathbb{E}(N) = 0$.

Note $\mathcal{S} = [-1, 1]$, so by Theorem 4.5.7, $-1 \leqslant \mathbb{E}(N) \leqslant 1$, which is true of 0. Since $f(x)$ is not increasing, nor is it decreasing, we cannot use Theorem 4.5.8.

S-11: Using Definition 4.5.1,

$$\begin{aligned}\mathbb{E}(P) &= \int_{-\infty}^{\infty} x \cdot f(x)\mathrm{d}x = 0 + \int_0^1 x \cdot \frac{2}{3}x(1-x)\mathrm{d}x + \int_2^3 x \cdot \frac{2}{3}(x-2)\mathrm{d}x \\ &= \frac{2}{3}\int_0^1 (x^2 - x^3)\mathrm{d}x + \frac{2}{3}\int_2^3 (x^2 - 2x)\mathrm{d}x \\ &= \frac{2}{3}\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 + \frac{2}{3}\left[\frac{1}{3}x^3 - x^2\right]_2^3 = \frac{17}{18}\end{aligned}$$

Since $f(x)$ is neither increasing nor decreasing, we can't apply Theorem 4.5.8. However, by Theorem 4.5.7, we need $0 \leqslant \mathbb{E}(P) \leqslant 3$, so our answer passes that check.

S-12: Using Definition 4.5.1,

$$\mathbb{E}(Q) = \int_1^e x \cdot \left(\frac{e}{e-2}\right) \cdot \frac{\ln x}{x^2} \mathrm{d}x = \frac{e}{e-2} \int_1^e \ln x \cdot \frac{1}{x} \mathrm{d}x$$

Let $u = \ln x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$

$$\begin{aligned} &= \frac{e}{e-2} \int_{\ln 1}^{\ln e} u \mathrm{d}u = \frac{e}{e-2} \int_0^1 u \mathrm{d}u = \frac{e}{e-2} \left[\frac{1}{2}u^2\right]_0^1 \\ &= \frac{e}{e-2} \left(\frac{1}{2}\right) \end{aligned}$$

Q takes on values from the interval $[1, e]$, so we expect $1 \leqslant \mathbb{E}(Q) \leqslant e$.

Using a calculator: we expect $1 \leqslant \mathbb{E}(Q) < 2.72$ and $\mathbb{E}(Q) \approx 1.9$.

By hand:

$$\begin{aligned} e &< 4 \\ \implies -4 &< -e \\ \implies 2e - 4 &< 2e - e = e \\ \implies 1 &< \frac{e}{2e-4}, \text{ as desired} \end{aligned}$$

S-13: Since Y is uniformly distributed over $[a, b]$, its PDF is a constant on that interval: $f(x) = c$, $a \leqslant x \leqslant b$. In order for $f(x)$ to be a PDF, we must have the area underneath it equal to 1, so

$$1 = c(b-a) \qquad \implies \qquad c = \frac{1}{b-a}$$

Now, we can compute $\mathbb{E}(Y)$:

$$\begin{aligned} \mathbb{E}(Y) &= \int_a^b x \cdot \frac{1}{b-a} \mathrm{d}x = \frac{1}{2(b-a)} x^2 \Big|_a^b = \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{(b+a)(b-a)}{2(b-a)} = \frac{b+a}{2} \end{aligned}$$

It makes intuitive sense that, if Y is uniformly distributed on $[a, b]$, then its long-term average will be exactly in the middle of that interval.

S-14:

(a) Corollary 4.4.8 has properties of PDFs.

- $f_p(x) \geqslant 0$ for all real x in the domain of $f_p(x)$: this is true for any positive p
- We need $\int_1^\infty f_p(x)\mathrm{d}x = 1$; in particular, that means $\int_1^\infty \frac{1}{x^p}\mathrm{d}x$ has to converge. From Example 3.10.10 in the text, this happens when $p > 1$.

So, this setup only makes sense when $p > 1$.

(b) For $p > 1$:

$$\begin{aligned}
1 &= \int_1^\infty f_p(x)\mathrm{d}x = a_p \int_1^\infty x^{-p}\mathrm{d}x = a_p \lim_{b\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^b \\
&= \frac{a_p}{1-p}\lim_{b\to\infty}\left[b^{1-p} - 1\right] = \frac{a_p}{1-p}(-1) = \frac{a_p}{p-1} \\
a_p &= p-1
\end{aligned}$$

(c) For $p > 1$:

$$\mathbb{E}(X_p) = \int_1^\infty x \cdot \frac{a_p}{x^p}\mathrm{d}x = (p-1)\int_1^\infty \frac{1}{x^{p-1}}\mathrm{d}x$$

Again from Example 3.10.10 in the text, this integral converges when $p-1 > 1$, i.e. when $p > 2$.

S-15:

$$\mathbb{E}(A) = \int_0^1 x^2 e^x \mathrm{d}x$$

Use integration by parts with $u = x^2$, $\mathrm{d}v = e^x\mathrm{d}x$; $\mathrm{d}u = 2x\mathrm{d}x$, $v = e^x$

$$= \left[x^2 e^x\right]_0^1 - \int_0^1 2xe^x\mathrm{d}x = e - 2\int_0^1 xe^x\mathrm{d}x$$

Use integration by parts with $u = x$, $\mathrm{d}v = e^x\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = e^x$

$$\begin{aligned}
&= e - 2\left([xe^x]_0^1 - \int_0^1 e^x\mathrm{d}x\right) = e - 2\left(e - \left[e^1 - e^0\right]\right) \\
&= e - 2
\end{aligned}$$

Note: $0 \leqslant e - 2 \leqslant 1$, so it passes our first check.

S-16:

$$\mathbb{E}(B) = \int_1^2 x \cdot \frac{2}{\ln^2 2}\left(\frac{\ln x}{x}\right)\mathrm{d}x = \frac{2}{\ln^2 2}\int_1^2 \ln x\mathrm{d}x$$

We use integration by parts to evaluate $\int \ln x\mathrm{d}x$, as in Example 3.5.8 in the text. Let $u = \ln x$, $\mathrm{d}v = \mathrm{d}x$; $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, $v = x$

$$\begin{aligned}
&= \frac{2}{\ln^2 2}\left(x\ln x\Big|_1^2 - \int_1^2 1\mathrm{d}x\right) = \frac{2}{\ln^2 2}\left(2\ln 2 - \left[x\right]_1^2\right) \\
&= \frac{2}{\ln^2 2}(2\ln 2 - 1)
\end{aligned}$$

S-17:

$$\mathbb{E}(C) = \int_2^b \frac{x^2}{x^2-1}\mathrm{d}x = \int_2^b \frac{x^2-1+1}{x^2-1}\mathrm{d}x = \int_2^b \left(1+\frac{1}{x^2-1}\right)\mathrm{d}x$$

To use integration by parial fractions, we factor

$$= \int_2^b \left(1+\frac{1}{(x+1)(x-1)}\right)\mathrm{d}x$$

Intermediate calculation:

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$
$$x=1 \implies B=\frac{1}{2} \qquad x=-1 \implies A=-\frac{1}{2}$$

Returning to the integral:

$$\begin{aligned}
\mathbb{E}(C) &= \int_2^b \left(1+\frac{1/2}{x-1}-\frac{1/2}{x+1}\right)\mathrm{d}x = \left[x+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|\right]_2^b \\
&= \left[x+\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)\right]_2^b \\
&= (b-2)+\frac{1}{2}\ln\left(\frac{b-1}{b+1}\right)-\frac{1}{2}\ln\left(\frac{1}{3}\right) \\
&= (b-2)+\frac{1}{2}\ln\left(3\frac{b-1}{b+1}\right)
\end{aligned}$$

S-18:

$$\begin{aligned}
\mathbb{E}(D) &= \frac{4}{4-\pi}\int_0^{\pi/4} x\cdot\tan^2 x\mathrm{d}x = \frac{4}{4-\pi}\left(\int_0^{\pi/4} x(\sec^2 x-1)\mathrm{d}x\right) \\
&= \frac{4}{4-\pi}\left(\int_0^{\pi/4} x\cdot\sec^2 x\mathrm{d}x - \int_0^{\pi/4} x\mathrm{d}x\right)
\end{aligned}$$

Use integration by parts with $u = x$, $\mathrm{d}v = \sec^2 x\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = \tan x$

$$= \frac{4}{4-\pi}\left(\left[x\tan x\right]_0^{\pi/4} - \int_0^{\pi/4}\tan x\mathrm{d}x - \left[\frac{1}{2}x^2\right]_0^{\pi/4}\right)$$

$$= \frac{4}{4-\pi}\left(\frac{\pi}{4} - \int_0^{\pi/4}\tan x\mathrm{d}x - \left[\frac{\pi^2}{32}\right]\right)$$

$$= \frac{4}{4-\pi}\left(\frac{\pi}{4} - \left[\ln|\sec x|\right]_0^{\pi/4} - \left[\frac{\pi^2}{32}\right]\right)$$

$$= \frac{4}{4-\pi}\left(\frac{\pi}{4} - \left[\ln\sqrt{2}\right] - \left[\frac{\pi^2}{32}\right]\right)$$

$$= \frac{1}{4-\pi}\left(\pi - 2\ln 2 - \frac{\pi^2}{8}\right)$$

S-19: Using the definition of expectation,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\cdot f(x)\mathrm{d}x = \int_{-\infty}^{\infty} cxe^{bx^2}\mathrm{d}x$$

The integrand has odd symmetry, so since the integral does converge, it will be equal to 0. That's the easiest way to find $\mathbb{E}(X)$.

$$= 0$$

However, we could also evaluate the integral with the substitution $u = bx^2$, $\mathrm{d}u = 2bx\mathrm{d}x$. To avoid carrying limits throughout our work, let's first evaluate the indefinite integral.

$$\int cxe^{bx^2}\mathrm{d}x = \int \frac{c}{2b}e^u\mathrm{d}u = \frac{c}{2b}e^u + C = \frac{c}{2b}e^{bx^2} + C$$

$$\int_{-\infty}^{\infty} cxe^{bx^2}\mathrm{d}x = \lim_{A\to-\infty}\left[\int_A^0 cxe^{bx^2}\mathrm{d}x\right] + \lim_{B\to\infty}\left[\int_0^B cxe^{bx^2}\mathrm{d}x\right]$$

$$= \lim_{A\to-\infty}\left[\frac{c}{2b}e^{bx^2}\Big|_A^0\right] + \lim_{B\to\infty}\left[\frac{c}{2b}e^{bx^2}\Big|_0^B\right]$$

$$= \frac{c}{2b}\lim_{A\to-\infty}\left[1 - e^{bA^2}\right] + \frac{c}{2b}\lim_{B\to\infty}\left[e^{bB^2} - 1\right]$$

To evalute the limits, remember b is *negative*, so $\lim\limits_{x\to\pm\infty} bx^2 = -\infty$. Hence $\lim\limits_{x\to\pm\infty} e^{bx^2} = 0$.

$$= \frac{c}{2b}[1-0] + \frac{c}{2b}[0-1] = 0$$

S-20: Sonic will pay \$0 with probability $\frac{999}{1000}$ (should Fred's boat stay afloat) or \$15,000 with probability $\frac{1}{1000}$. So, their expected payment is $\frac{15{,}000}{1000} = 15$ dollars.

Since Fred paid \$1000, the expected profit is $1000 - 15 = 985$.

S-21: It seems a safe assumption that the salary of \$0 means Andrea is unemployed, so that probability is 15%.

$$\mathbb{E} = 0\left(\frac{15}{100}\right) + 2500\left(\frac{20}{100}\right) + 3000\left(\frac{30}{100}\right) + 3500\left(\frac{20}{100}\right) + 4000\left(\frac{15}{100}\right) = 2700$$

S-22:

(a) Suppose Riley invests x dollars in Asset A and y dollars in Asset B. The expected return from Asset A is $1.2x$, while the expected return from Asset B is $4y\,(0.2) + y\,(0.8) = 1.6y$. So, her expected overall return is $1.2x + 1.6y$.

(b) If Riley invests \$300, then we can set $y = 300 - x$. In a bad year, her return is

$$1.2x + 1(300 - x) = 300 + 0.2x$$

This is the worst-case scenario. To satisfy Riley's risk aversion, we need this to be at least \$350.

$$\begin{aligned} 350 &\leqslant 300 + 0.2x \\ 50 &\leqslant 0.2x \\ x &\geqslant 250 \end{aligned}$$

So, Riley needs to invest *at least* \$250 of her \$300 in Asset A.

Now, let's decide how to allocate the remaining \$50. Using our answer from part (a), if she invests y dollars in Asset B, then her expected return is $1.2(300 - y) + 1.6y$, or $360 + 0.4y$. This is a line with positive slope: it is a maximum where y is a maximum, i.e. at $y = 50$. So, her optimal investment is \$50 in Asset B and \$250 in Asset A.

Solutions to Exercises 4.6 — Jump to TABLE OF CONTENTS

S-1:

A - 4: we introduced expected value in Section 4.5.1 precisely as a long-term average.
B -6: Section 4.6.1 motivated the study of "average difference from the average;" in defining variance, we replaced "difference from the average" with "squared difference from the average."
C -5: because variance introduces squares, to get our numbers to have the right kind of size, standard deviation undoes the squaring, trying to get back to the original motivation of "average difference from the average" (but with functions that are differentiable).
D -1 The probability mass function (PMF) is exactly the function (usually written as a table) describing the probability (likelihood) of each event in the sample space.
E -3: The robability density function (PDF) differs from the probability mass function (PMF) because in a continuous function, the probability of any one value happening is 0.

However, it serves a similar purpose, helping us decide which regions are more or less likely.
F -2: this is from the definition of a cumulative distribution function (CDF)

S-2: A high standard deviation means X takes values that are very spread out. So, we let X have sample space $\mathbb{S} = \{-100, 100\}$ let it be equally likely to take either value.

$$\mathbb{E}(X) = \sum_x xPr(X = x) = -100(1/2) + 100(1/2) = 0$$
$$\mathbb{E}(X^2) = \sum_x x^2 Pr(X = x) = 100^2(1/2) + 100^2(1/2) = 100^2$$

Using Corollary 4.6.5:

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 100^2$$
$$\sigma(X) = \sqrt{\mathrm{Var}(X)} = 100$$

We not that Corollary 4.6.9 tells us $\sigma(X) \leqslant \frac{100-(-100)}{2} = 100$, so we've given X the highest possible standard deviation, given its restricted range.

S-3:

(a) Since X is uniformly distributed, its probability density function (PDF) is constant on the interval $[a, b]$:
$$f(x) = c, \quad a \leqslant x \leqslant b$$
If $f(x)$ is a probability density function (PDF), the area underneath it must be one. Conveniently, this area is a rectangle.
$$1 = c(b - a) \quad \implies \quad c = \frac{1}{b - a}$$
So, we have our probability density function (PDF): $f(x) = \frac{1}{b-a}$, $a \leqslant x \leqslant b$. Now we can apply our definitions.

$$\mathbb{E}(X) = \int_a^b \frac{x}{b-a}\mathrm{d}x = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$
$$\mathbb{E}(X^2) = \int_a^b \frac{x^2}{b-a}\mathrm{d}x = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{1}{3}(b^2+ab+a^2)$$
$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}(b+a)^2 = \frac{(b-a)^2}{12}$$
$$\sigma(X) = \sqrt{\mathrm{Var}(X)} = \frac{b-a}{2\sqrt{3}}$$

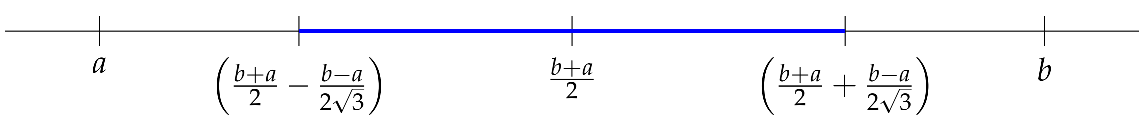
Note: had we wanted to calculate $\mathrm{Var}(X)$ the other way, it would go like this:

$$\begin{aligned}
\mathrm{Var}(X) &= \int_a^b (x - \mathbb{E}(X))\, f(x)\mathrm{d}x = \frac{1}{b-a}\int_a^b \left(x - \frac{b+a}{2}\right)^2 \\
&= \frac{1}{b-a}\int_a^b \left(x^2 - (b+a)x + \frac{(b+a)^2}{4}\right) \\
&= \frac{1}{b-a}\left[\frac{1}{3}x^3 - \frac{b+a}{2}x^2 + \frac{b^2+2ab+a^2}{4}x\right]_a^b \\
&= \frac{1}{b-a}\left[\frac{1}{3}b^3 - \frac{b+a}{2}b^2 + \frac{b^2+2ab+a^2}{4}b - \frac{1}{3}a^3 + \frac{b+a}{2}a^2 - \frac{b^2+2ab+a^2}{4}a\right] \\
&= \frac{1}{b-a}\left[\frac{b^3-3ab^2+3a^2b-a^3}{12}\right] = \frac{(b-a)^3}{12(b-a)} = \frac{(b-a)^2}{12}
\end{aligned}$$

(b) The values that fall within one standard deviation of $\mathbb{E}(X)$ are those on the interval $[\mathbb{E}(X) - \sigma(X) \quad , \quad \mathbb{E}(X) + \sigma(X)]$. In this case,

$$2\sigma(X) = \frac{b-a}{\sqrt{3}}$$

So the interval in question is the middle part of the interval $[a, b]$, with length $\frac{1}{\sqrt{3}}$ times the length of the whole interval. That's a little over half the entire interval $[a, b]$ included in the interval $[\mathbb{E}(X) - \sigma(X) \quad , \quad \mathbb{E}(X) + \sigma(X)]$.



S-4: The probability mass function (PMF) of this variable is that $Pr(X = s) = 1$. So, $\mathbb{E}(X) = s$; $\mathbb{E}(X^2) = s^2$; $\mathrm{Var}(X) = s^2 - s^2 = 0$; $\sigma(X) = 0$.

Since variance is a measure of how much the variable "varies," it makes sense that the variance here is 0.

S-5: If $f(x)$ has even symmetry, then $xf(x)$ has odd symmetry, so $\mathbb{E}(X) = \int_{-a}^{a} xf(x)\mathrm{d}x = 0$.

This works intuitively: the values taken by X are evenly distributed between positive and negative values, so these should cancel each other out in the long-term average.

S-6:

(a) Remember $F(x) = Pr(X \leqslant x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$. Since our PDF is in three pieces (less than 0; from 0 to $\frac{\pi}{2}$; greater than $\frac{\pi}{2}$) we'll use these intervals for finding $F(x)$.

$$\begin{aligned}
x < 0: &\qquad F(x) = \int_{-\infty}^{x} 0\mathrm{d}t = 0 \\
0 \leqslant x \leqslant \frac{\pi}{2}: &\qquad F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_{0}^{x} \sin t\mathrm{d}t = -\cos x + \cos 0 = 1 - \cos x \\
x > \frac{\pi}{2}: &\qquad F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_{0}^{\frac{\pi}{2}} \sin t\mathrm{d}t + \int_{\frac{\pi}{2}}^{x} 0\mathrm{d}t = 1
\end{aligned}$$

All together,

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \cos x & 0 \leqslant x \leqslant \frac{\pi}{2} \\ 1 & x > \frac{\pi}{2} \end{cases}$$

(b) Before we compute $\mathrm{Var}(X)$, we need to know $\mathbb{E}(X)$.

$$\mathbb{E}(X) = \int_{0}^{\frac{\pi}{2}} x \cdot \sin x\mathrm{d}x$$

We use integration by parts with $u = x$, $\mathrm{d}v = \sin x\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = -\cos x$

$$\begin{aligned}
&= \Big[-x\cos x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos x\mathrm{d}x \\
&= \sin\frac{\pi}{2} - \sin 0 = 1
\end{aligned}$$

Now we can find $\mathrm{Var}(X)$

$$\mathbb{E}(X^2) = \int_0^{\frac{\pi}{2}} x^2 \sin x\mathrm{d}x$$

We use integration by parts with $u = x^2$, $\mathrm{d}v = \sin x\mathrm{d}x$; $\mathrm{d}u = 2x\mathrm{d}x$, $v = -\cos x$

$$= \underbrace{\Big[-x^2\cos x\Big]_0^{\frac{\pi}{2}}}_{0} + 2\int_0^{\frac{\pi}{2}} x\cos x\mathrm{d}x$$

We use integration by parts with $u = x$, $\mathrm{d}v = \cos x\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = \sin x$

$$\begin{aligned}
&= 2\left[\underbrace{x\sin x\Big|_0^{\frac{\pi}{2}}}_{\pi/2} + \int_0^{\frac{\pi}{2}} -\sin x\mathrm{d}x\right] \\
&= 2\left[\frac{\pi}{2} + (\cos(\pi/2) - \cos 0)\right] = 2\left[\frac{\pi}{2} - 1\right] = \pi - 2
\end{aligned}$$

So, $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = (\pi - 2) - 1 = \pi - 3$

Note: we can also use the other method to compute $\mathrm{Var}(X)$:

$$\begin{aligned}\mathrm{Var}(X) &= \int_{-\infty}^{\infty} f(x)\cdot(x-\mathbb{E}(X))^2\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x\cdot(x-1)^2\,\mathrm{d}x\\ &= \int_0^{\frac{\pi}{2}} \left(x^2\sin x - 2x\sin x + \sin x\right)\mathrm{d}x\\ &= \int_0^{\frac{\pi}{2}} x^2\sin x\mathrm{d}x - 2\int_0^{\frac{\pi}{2}} x\sin x\mathrm{d}x + \int_0^{\frac{\pi}{2}} \sin x\mathrm{d}x\end{aligned}$$

The second integral is $\mathbb{E}(X)$; the third integral is equal to $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$, which is 1 by Corollary 4.4.8 part 3.

$$= \int_0^{\frac{\pi}{2}} x^2\sin x\mathrm{d}x - 2 + 1$$

Use integration by parts with $u = x^2$, $\mathrm{d}v = \sin x\mathrm{d}x$; $\mathrm{d}u = 2x\mathrm{d}x$, $v = -\cos x$

$$= \underbrace{-x^2\cos x\Big|_0^{\frac{\pi}{2}}}_{0} + 2\int_0^{\frac{\pi}{2}} x\cos x\mathrm{d}x - 1$$

Use integration by parts with $u = x$, $\mathrm{d}v = \cos x\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = \sin x$

$$\begin{aligned}&= 2\left[x\sin x\Big|_0^{\frac{\pi}{2}} - \underbrace{\int_0^{\frac{\pi}{2}} \sin x\mathrm{d}x}_{1}\right] - 1\\ &= 2\left[\frac{\pi}{2} - 1\right] - 1 = \pi - 3\end{aligned}$$

(c) $\sigma(X) = \sqrt{\pi - 3}$

S-7: To find $\mathbb{E}(X)$, it's easiest to note that $f(x)$ is an even function, so $xf(x)$ is an odd function, so $\mathbb{E}(X) = \int_{-1}^{1} xf(x)\mathrm{d}x = 0$. But we can also compute $\mathbb{E}(X)$ using the (piecewise) definition of the absolute value function.

$$\begin{aligned}|x| &= \begin{cases} -x & x < 0\\ x & x \geqslant 0\end{cases}\\ \mathbb{E}(X) &= \int_{-1}^{1} xf(x)\mathrm{d}x = \int_{-1}^{0} x(1+x)\mathrm{d}x + \int_0^1 x(1-x)\mathrm{d}x\\ &= \int_{-1}^{0} x + x^2\mathrm{d}x + \int_0^1 x - x^2\mathrm{d}x = \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1\\ &= \left[0 - \frac{1}{6}\right] + \left[\frac{1}{6} - 0\right] = 0\end{aligned}$$

To compute variance, we'll now find $\mathbb{E}(X^2)$.

$$\begin{aligned}
\mathbb{E}(X^2) &= \int_{-1}^{1} x^2 f(x)\mathrm{d}x = \int_{-1}^{0} x^2(1-x)\mathrm{d}x + \int_{0}^{1} x^2(1+x)\mathrm{d}x \\
&= \int_{-1}^{0} x^2 - x^3\mathrm{d}x + \int_{0}^{1} x^2 + x^3\mathrm{d}x = \left[\frac{x^3}{3} - \frac{1}{4}x^4\right]_{-1}^{0} + \left[\frac{x^3}{3} + \frac{1}{4}x^4\right]_{0}^{1} \\
&= -\left[0 - (\frac{-1}{3} - \frac{1}{4})\right] + \left[(\frac{1}{3} + \frac{1}{4}) - 0\right] = \frac{1}{6}
\end{aligned}$$

So, $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \dfrac{1}{6} - 0 = \dfrac{1}{6}$

Alternately, we can compute variance the other way.

$$\begin{aligned}
\mathrm{Var}(X) &= \int_{-1}^{1} (x - \mathbb{E}(X))^2 f(x)\mathrm{d}x = \int_{-1}^{0} x^2(1-x)\mathrm{d}x + \int_{0}^{1} x^2(1+x)\mathrm{d}x \\
&= \left[\frac{x^3}{3} - \frac{1}{4}x^4\right]_{-1}^{0} + \left[\frac{x^3}{3} + \frac{1}{4}x^4\right]_{0}^{1} = \frac{1}{6}
\end{aligned}$$

Now, $\sigma(X) = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}}$.

S-8:

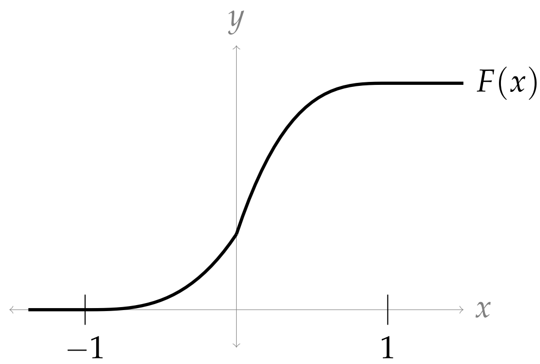
(a) X is continuous if and only if $F(x)$ is continuous. The only places where $F(x)$ looks like it might possibly be discontinuous are at $x = -1, 0, 1$. Recall by definition, $F(x)$ is continuous at the point $x = a$ if $\lim\limits_{x\to a} F(x) = F(a)$.

- $\lim\limits_{x\to -1^-} F(x) = 0$, $\lim\limits_{x\to -1^+} F(x) = \lim\limits_{x\to -1^+} \frac{1}{3}(x+1)^3 = 0 = F(-1)$, so $F(-1) = \lim\limits_{x\to -1} F(x)$. That is, $F(x)$ is continuous at $x = -1$.

- $\lim\limits_{x\to 0^-} F(x) = \lim\limits_{x\to 0^-} \frac{1}{3}(x+1)^3 = \frac{1}{3}$;
  $\lim\limits_{x\to 0^+} F(x) = \lim\limits_{x\to 0^+} \left[\frac{2}{3}(x-1)^3 + 1\right] = -\frac{2}{3} + 1 = \frac{1}{3} = F(1)$, so $\lim\limits_{x\to 0} F(x) = F(0)$, o.e. $F(x)$ is continuous at $x = 0$.

- $\lim\limits_{x\to 1^-} F(x) = \lim\limits_{x\to 1^-} \left[\frac{2}{3}(x-1)^3 + 1\right] = 1 = F(1) = \lim_{x\to 1^+} F(x)$, so $F(x)$ is continuous at $x = 1$.

Since $F(x)$ is a continuous function, X is a continuous variable.



(b) In order to compute the expectation and variance, we need the probability density function (PDF) for X, $f(x)$. Recall we defined $f(x) = F'(x)$, so:

$$f(x) = \begin{cases} 0 & x < -1 \\ (x+1)^2 & -1 \leqslant x < 0 \\ 2(x-1)^2 & 0 < x \leqslant 1 \\ 0 & x > 1 \end{cases}$$

We use this function to compute the expectation and variance, which means we'll need to split up our integral over the intervals $[-1, 0]$ and $[0, 1]$.

$$\begin{aligned} \mathbb{E}(X) &= \int_{-1}^{0} x \underbrace{(x+1)^2}_{f(x)} \mathrm{d}x + \int_{0}^{1} x \cdot \underbrace{2(x-1)^2}_{f(x)} \mathrm{d}x \\ &= \int_{-1}^{0} \left(x^3 + 2x^2 + x\right) \mathrm{d}x + \int_{0}^{1} \left(2x^3 - 4x^2 + 2x\right) \mathrm{d}x \\ &= \left[\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^4 - \frac{4}{3}x^3 + x^2\right]_{0}^{1} \\ &= \left[-\frac{1}{4} + \frac{2}{3} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{4}{3} + 1\right] = \frac{1}{12} \end{aligned}$$

Next:

$$\begin{aligned} \mathbb{E}(X^2) &= \int_{-1}^{0} x^2 \underbrace{(x+1)^2}_{f(x)} \mathrm{d}x + \int_{0}^{1} x^2 \cdot \underbrace{2(x-1)^2}_{f(x)} \mathrm{d}x \\ &= \int_{-1}^{0} \left(x^4 + 2x^3 + x^2\right) \mathrm{d}x + \int_{0}^{1} \left(2x^4 - 4x^3 + 2x^2\right) \mathrm{d}x \\ &= \left[\frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_{-1}^{0} + \left[\frac{2}{5}x^5 - x^4 + \frac{2}{3}x^3\right]_{0}^{1} \\ &= \left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right] + \left[\frac{2}{5} - 1 + \frac{2}{3}\right] = \frac{1}{10} \end{aligned}$$

So,

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{10} - \left(\frac{1}{12}\right)^2 = \frac{67}{720} \\
\sigma(X) &= \sqrt{\frac{67}{720}} = \frac{\sqrt{67}}{12\sqrt{5}}
\end{aligned}$$

(The other method of computing $\mathrm{Var}(X)$ is significantly more obnoxious.)

S-9:

$$\begin{aligned}
\mathbb{E}(T) &= 1 \cdot Pr(T = 1) + 2 \cdot Pr(T = 2) + 3 \cdot Pr(T = 3) \\
&= 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right) \\
&= \frac{7}{4}
\end{aligned}$$

Using the definition of variance,

$$\begin{aligned}
\mathrm{Var}(T) &= \sum_{x=1}^{3} Pr(T = x) \cdot \left(x - \frac{7}{4}\right)^2 \\
&= \frac{1}{2}\left(-\frac{3}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{5}{4}\right)^2 \\
&= \frac{11}{16} \\
\sigma(T) &= \frac{\sqrt{11}}{4}
\end{aligned}$$

Alternately, we could have computed $\mathrm{Var}(T)$ using $\mathbb{E}(T^2)$:

$$\begin{aligned}
\mathbb{E}(T^2) &= \sum_{x=1}^{3} x^2 \cdot Pr(T = x) = 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{4} \\
&= \frac{15}{4} \\
\mathrm{Var}(T) &= \mathbb{E}(T^2) - [\mathbb{E}(T)]^2 = \frac{15}{4} - \left(\frac{7}{4}\right)^2 = \frac{11}{16}
\end{aligned}$$

S-10:

$$
\begin{aligned}
\mathbb{E}(S) &= \sum_x x \cdot Pr(S = x) \\
&= -5 \cdot \frac{1}{9} - 4 \cdot \frac{2}{9} + 2 \cdot \frac{1}{9} + 7 \cdot \frac{5}{9} \\
&= \frac{8}{3} \\
\mathbb{E}(X^2) &= \sum_x x^2 \cdot Pr(S = x) \\
&= 25 \cdot \frac{1}{9} + 16 \cdot \frac{2}{9} + 4 \cdot \frac{1}{9} + 49 \cdot \frac{5}{9} \\
&= 34 \\
\mathrm{Var}(S) &= \mathbb{E}(S^2) - [\mathbb{E}(S)]^2 = 34 - \left(\frac{8}{3}\right)^2 = \frac{242}{9}
\end{aligned}
$$

Alternately, we could have computed

$$
\begin{aligned}
\mathrm{Var}(S) &= \sum_x \left(x - \frac{8}{3}\right)^2 \cdot Pr(S = x) \\
&= \left(-5 - \frac{8}{3}\right)^2 \cdot \frac{1}{9} + \left(-4 - \frac{8}{3}\right)^2 \cdot \frac{2}{9} + \left(2 - \frac{8}{3}\right)^2 \cdot \frac{1}{9} + \left(7 - \frac{8}{3}\right)^2 \cdot \frac{5}{9} \\
&= \frac{242}{9}
\end{aligned}
$$

Finally,

$$
\sigma(S) = \sqrt{\mathrm{Var}(S)} = \frac{11\sqrt{2}}{3}
$$

S-11: The first thing we have to do is find the probability mass function (PMF). (The cumulative distribution function (CDF) is discontinuous, so U is not a continuous random variable. Therefore we don't consider a probability density function (PDF).)

- At $x = 0$, $Pr(U \leqslant x)$ jumps from 0 to $\frac{1}{2}$, so $Pr(U = 0) = \frac{1}{2}$
- At $x = 2$, $Pr(U \leqslant x)$ jumps from $\frac{1}{2}$ to $\frac{2}{3}$, so $Pr(U = 2) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$
- At $x = 3$, $Pr(U \leqslant x)$ jumps from $\frac{2}{3}$ to $\frac{3}{4}$, so $Pr(U = 3) = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$
- At $x = 4$, $Pr(U \leqslant x)$ jumps from $\frac{3}{4}$ to 1, so $Pr(U = 4) = 1 - \frac{4}{3} = \frac{1}{4}$

All together:

x	$Pr(U = x)$
0	$\frac{1}{2}$
2	$\frac{1}{6}$
3	$\frac{1}{12}$
4	$\frac{1}{4}$

Now, we can find the expected value, variance, and standard deviation of U.

$$\begin{aligned}
\mathbb{E}(U) &= 0 \cdot Pr(U = 0) + 2 \cdot Pr(U = 2) + 3 \cdot Pr(U = 3) + 4 \cdot Pr(U = 4) \\
&= 0 + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{4} = \frac{19}{12} \\
\mathbb{E}(U^2) &= 0^2 \cdot Pr(U = 0) + 2^2 \cdot Pr(U = 2) + 3^2 \cdot Pr(U = 3) + 4^2 \cdot Pr(U = 4) \\
&= 0 + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{12} + 16 \cdot \frac{1}{4} = \frac{65}{12} \\
\mathrm{Var}(U) &= \mathbb{E}(U^2) - [\mathbb{E}(U)]^2 = \frac{65}{12} - \left(\frac{19}{12}\right)^2 = \frac{419}{144} \\
\sigma(U) &= \sqrt{\mathrm{Var}(U)} = \frac{\sqrt{419}}{12}
\end{aligned}$$

Had we instead wanted to use the definition of variance, we could have computed it thus:

$$\begin{aligned}
\mathrm{Var}(U) &= \sum_x \left(x - \frac{19}{12}\right)^2 Pr(U = x) \\
&= \left(\frac{19}{12}\right)^2 \cdot \frac{1}{2} + \left(\frac{24-19}{12}\right)^2 \cdot \frac{1}{6} + \left(\frac{36-19}{12}\right)^2 \cdot \frac{1}{12} + \left(\frac{48-19}{12}\right)^2 \cdot \frac{1}{4} \\
&= \frac{419}{12^2}
\end{aligned}$$

S-12: Because the calculation will come up more than once, we start with the antiderivative of $\sin^3 x$.

$$\int \sin^3 x \mathrm{d}x = \int (1 - \cos^2 x) \sin x \mathrm{d}x$$

substitution $u = \cos x$, $\mathrm{d}u = -\sin x \mathrm{d}x$

$$= \int (1 - u^2)(-1)\mathrm{d}u = -u + \frac{1}{3}u^3 + C = \frac{1}{3}\cos^3 x - \cos x + C$$

To find $\mathbb{E}(Z)$:

$$\begin{aligned}
\mathbb{E}(Z) &= a \int_0^{\pi} \sin^3 x \mathrm{d}x = a \left[\frac{1}{3}\cos^3 x - \cos x\right]_0^{\pi} \\
&= a \left[\frac{1}{3}(-1) - (-1) - \frac{1}{3}(1) + 1\right] = \frac{4}{3}a
\end{aligned}$$

To find $\mathbb{E}(Z^2)$:

$$\mathbb{E}(Z^2) = a\int_0^{\pi} x \cdot \sin^3 x \mathrm{d}x$$

Integrate by parts with $u = x$, $\mathrm{d}v = \sin^3 x\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = \frac{1}{3}\cos^3 x - \cos x$

$$\begin{aligned}
&= a\left(\left[x\left(\frac{1}{3}\cos^3 x - \cos x\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{1}{3}\cos^3 x - \cos x\right)\mathrm{d}x\right)\\
&= a\left(\frac{2}{3}\pi - \int_0^{\pi}\frac{1}{3}\cos^3 x\mathrm{d}x + \int_0^{\pi}\cos x\mathrm{d}x\right)\\
&= a\left(\frac{2}{3}\pi - \int_0^{\pi}\frac{1}{3}(1-\sin^2 x)\cos x\mathrm{d}x + 0\right)
\end{aligned}$$

Substituting $u = \sin x$, $\mathrm{d}u = \cos x\mathrm{d}x$

$$\begin{aligned}
&= a\left(\frac{2}{3}\pi - \int_0^{0}\frac{1}{3}(1-u^2)\mathrm{d}u + 0\right)\\
&= \frac{2\pi}{3}a
\end{aligned}$$

So,

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathbb{E}(Z^2) - [\mathbb{E}(Z)]^2 = \frac{2\pi}{3}a - \left(\frac{4}{3}a\right)^2\\
\sigma(X) &= \sqrt{\frac{2\pi}{3}a - \left(\frac{4}{3}a\right)^2}
\end{aligned}$$

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S-1: (a) The values of the sequence seem to be getting closer and closer to -2, so we guess the limit of this sequence is -2.
(b) Overall, the values of the sequence seem to be getting extremely close to 0, so we approximate the limit of this sequence as 0. It doesn't matter that the sequence changes signs, or that the numbers are sometimes farther from 0, sometimes closer.
(c) This limit does not exist. The sequence is sometimes 0, sometimes -2, and not consistently staying extremely near to either one.

S-2: True. We consider the end behaviour of the sequences, which does not depend on any finite number of terms at their beginning.

S-3: (a) We follow the arithmetic of limits, Theorem 5.1.8 in the text: $\dfrac{A-B}{C}$
(b) Since $\lim\limits_{n\to\infty} c_n$ is some real number, and n grows without bound, $\lim\limits_{n\to\infty}\dfrac{c_n}{n} = 0$.

(c) We note $\lim_{n\to\infty} a_{2n+5} = \lim_{n\to\infty} a_n$, so $\dfrac{a_{2n+5}}{b_n} = \dfrac{A}{B}$.

S-4: There are many possible answers. One is:

$$a_n = \begin{cases} 3000 - n & \text{if } n \leqslant 1000 \\ -2 + \frac{1}{n} & \text{if } n > 1000 \end{cases}$$

where we have a series that looks different before and after its thousandth term. Note every term is smaller than the term preceding it.

Another sequence with the desired properties is:

$$a_n = \frac{1,002,001}{n} - 2$$

When $n \leqslant 1000$, $a_n \geqslant \frac{1{,}002{,}001}{1000} - 2 > \frac{1{,}002{,}000}{1{,}000} - 2 = 1000$. That is, $a_n > 1000$ when $n \leqslant 1000$. As n gets larger, a_n gets smaller, so $a_{n+1} < a_n$ for all n. Finally, $\lim_{n\to\infty} a_n = 0 - 2 = -2$.

S-5: One possible answer is $a_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, -1, \ldots\}$.
Another is $a_n = n(-1)^n = \{-1, 2, -3, 4, -5, 6, -7, \ldots\}$.

S-6: If the terms of a sequence are alternating sign, but the limit of the sequence exists, the limit must be zero. (If it were a positive number, the negative terms would not get very close to it; if it were a negative number, the positive terms would not get very close to it.)

This gives us the idea to modify an answer from Question 5. One possible sequence:

$$a_n = \frac{(-1)^n}{n} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots\right\}$$

S-7:

(a) Since $-1 \leqslant \sin n \leqslant 1$ for all n, one potential set of upper and lower bound is

$$\frac{-1}{n} \leqslant \frac{\sin n}{n} \leqslant \frac{1}{n}$$

Note $\lim_{n\to\infty} \dfrac{-1}{n} = \lim_{n\to\infty} \dfrac{1}{n}$, so these are valid comparison sequences for the Squeeze Theorem.

(b) Since $-1 \leqslant \sin n \leqslant 1$ and $-5 \leqslant -5\cos n \leqslant 5$ for all n, we see

$$\begin{aligned} 7 - 1 - 5 &\leqslant 7 + \sin n - 5\cos n \leqslant 7 + 1 + 5 \\ 1 &\leqslant 7 + \sin n - 5\cos n \leqslant 13 \end{aligned}$$

This gives us the idea to try the bounds

$$\frac{n^2}{13e^n} \leqslant \frac{n^2}{e^n(7+\sin n - 5\cos n)} \leqslant \frac{n^2}{e^n}$$

We check that $\lim_{n\to\infty} \frac{n^2}{13e^n} = \lim_{n\to\infty} \frac{n^2}{e^n}$ (they're both 0–you can verify using l'Hôpital's rule), so these are indeed reasonable bounds to choose to use with the Squeeze Theorem.

(c) Since $(-n)^{-n} = \frac{1}{(-n)^n} = \frac{(-1)^n}{n^n}$, we see

$$\frac{-1}{n^n} \leqslant (-n)^{-n} \leqslant \frac{1}{n^n}$$

Since both $\lim_{n\to\infty} \frac{-1}{n^n}$ and $\lim_{n\to\infty} \frac{1}{n^n}$ are 0, these are reasonable bounds to use with the Squeeze Theorem.

S-8:

(a)
- Note $a_n = b_n$, since (in the absence of evidence to the contrary) we assume n begins at one, hence $n = |n|$. Then $a_n = b_n = 1 + \frac{1}{n} = \frac{n+1}{n}$. So, whenever n is a whole number, a_n and b_n are the same as $h(n)$ and $i(n)$. (Be careful here: $h(x) \neq i(x)$ when x is not a whole number.)
- $c_n = e^{-n} = \frac{1}{e^n} = j(n)$
- For any integer n, $\cos(\pi n) = (-1)^n$. So, $d_n = f(n)$.
- Similarly, $e_n = g(n)$.

(b) According to Theorem 5.1.6 in the text, if any of the functions on the right have limits that exist as $x \to \infty$, then these limits match the limits of their corresponding sequences. So, we only have to be suspicious of $f(x)$ and $i(x)$, since these do not converge.

The limit $\lim_{x\to\infty} f(x)$ does not exist, and $f(n) = d_n$; the limit $\lim_{n\to\infty} d_n$ also does not exist. (We generally don't write equality for two things that don't exist: equality refers to numerical value, and these have none.[16])

The limit $\lim_{x\to\infty} i(x)$ does not exist, because $i(x) = 0$ when x is not a whole number, while $i(x)$ approaches 1 when x is a whole number. However, lim $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$.

So, using our answers from part (a), we match the following:

16 The idea "two things that both don't exist are equal" is also rejected because it can lead to contradictions. For example, in the real numbers $\sqrt{-1}$ and $\sqrt{-2}$ don't exist; if we write $\sqrt{-1} = \sqrt{-2}$, then squaring both sides yields the inanity $-1 = -2$.

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{x\to\infty} h(x) = 1$

- $\lim_{n\to\infty} c_n = \lim_{n\to\infty} e_n = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} j(x) = 0$

- $\lim_{n\to\infty} d_n$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} i(x)$ do not exist.

S-9: (a) We want to find odd multiples of π that are close to integers.

Solution 1: One way to do that is to remember that π is somewhat close to $\frac{22}{7}$. Then when we multiply π by a multiple of 7, we should get something close to an integer. In particular, 7π, 21π, and 35π should be reasonably close to $7\left(\frac{22}{7}\right) = 22$, $21\left(\frac{22}{7}\right) = 66$, and $35\left(\frac{22}{7}\right) = 110$, respectively. We check whether they are close enough:

$$7\pi \approx 21.99 \qquad 21\pi \approx 65.97 \qquad 35\pi \approx 109.96$$

So indeed, 22, 66, and 110 are all within 0.1 of some odd multiple of π.

Since the cosine of an odd multiple of π is -1, we expect all of the sequence values to be close to -1. Using a calculator:

$$\begin{aligned} a_{22} &= \cos(22) \approx -0.99996, \\ a_{66} &= \cos(66) \approx -0.99965, \\ a_{110} &= \cos(110) \approx -0.99902 \end{aligned}$$

Solution 2: Alternately, we could have just listed odd multiple of π until we found three that are close to integers.

$\mathbf{2k+1}$	$\mathbf{(2k+1)}$ß
1	3.14
3	9.42
5	15.71
7	21.99
9	28.27
11	34.56
13	40.84
15	47.12
17	53.41
19	59.69
21	65.97
23	72.26
25	78.54
27	84.82
29	91.11
31	97.39
33	103.67
35	109.96

Some earlier odd multiples of π (like 15π and 29π) get fairly close to integers, but not within 0.1.

(b) If $x = \dfrac{2k+1}{2}\pi$ for some integer k (that is, x is an odd multiple of $\pi/2$), then $\cos x = 0$. So, we can either list out the first few terms of a_n until we find three that are very close to 0, or we can use our approximation $\pi \approx \dfrac{22}{7}$ to choose values of n that are close to $\dfrac{2k+1}{2}\pi$.

Solution 1:

$$\frac{2k+1}{2}\pi \approx \frac{(2k+1)\times 22}{2\times 7} = 11\frac{2k+1}{7}$$

So, we expect our values to be close to integers when $2k+1$ is a multiple of 7. For example, $2k+1 = 7$, $2k+1 = 21$, and $2k+1 = 35$.

We check:

$\mathbf{x}$	$\mathbf{n}$	$\mathbf{a_n}$
$7 \times \dfrac{\pi}{2} \approx 10.99557$	11	$a_{11} \approx 0.0044$
$21 \times \dfrac{\pi}{2} \approx 32.98672$	33	$a_{33} \approx -0.0133$
$35 \times \dfrac{\pi}{2} \approx 54.97787$	55	$a_{55} \approx 0.0221$

These seem like values of a_n that are all pretty close to 0.

Solution 2: We could have listed the first several values of a_n, and looked for some that are close to 0.

$\mathbf{n}$	$\mathbf{a_n}$
1	0.54
2	-0.42
3	-0.99
4	-0.65
5	0.28
6	0.96
7	0.75
8	-0.15
9	-0.91
10	-0.84

Oof. Nothing very close yet. Maybe a better way is to list values of $\frac{2k+1}{2}\pi$, and see which ones are close to integers.

$\mathbf{2k+1}$	$\mathbf{\frac{2k+1}{2}}$ß
1	1.57
3	4.71
5	7.85
7	10.996
9	14.14
11	17.28
13	20.42
15	23.56
17	26.70
19	29.85
21	32.99
23	36.13
25	39.27
27	42.41
29	45.55
31	48.69
33	51.84
35	54.98

We find roughly the same candidates we did in Solution 1, depending on what we're ready to accept as "close".

Remark: it is possible to turn the ideas of this question into a rigorous proof that $\lim\limits_{n\to\infty} \cos n$ is undefined.

- Let, for each integer $k \geqslant 1$, n_k be the integer that is closest to $2k\pi$. Then $2k\pi - \frac{1}{2} \leqslant n_k \leqslant 2k\pi + \frac{1}{2}$ so that $\cos(n_k) \geqslant \cos\frac{1}{2} \geqslant 0.8$. Consequently, if $\lim\limits_{n\to\infty} \cos n = c$ exists, we must have $c \geqslant 0.8$.
- Let, for each integer $k \geqslant 1$, n'_k be the integer that is closest to $(2k+1)\pi$. Then $(2k+1)\pi - \frac{1}{2} \leqslant n'_k \leqslant (2k+1)\pi + \frac{1}{2}$ so that $\cos(n'_k) \leqslant -\cos\frac{1}{2} \leqslant -0.8$. Consequently, if $\lim\limits_{n\to\infty} \cos n = c$ exists, we must have $c \leqslant -0.8$.
- It is impossible to have both $c \geqslant 0.8$ and $c \leqslant -0.8$, so $\lim\limits_{n\to\infty} \cos n$ does not exist.

S-10:

(a)

$$a_0 = 4$$
$$a_1 = 10a_0 - 6 = 10 \cdot 4 - 6 = 34$$
$$a_2 = 10a_1 - 6 = 10 \cdot 34 - 6 = 334$$
$$a_3 = 10a_2 - 6 = 10 \cdot 334 - 6 = 3334$$
$$a_4 = 10a_3 - 6 = 10 \cdot 3334 - 6 = 33334$$

(b)

$$b_0 = 1$$
$$b_1 = \frac{b_0}{2} = \frac{1}{2}$$
$$b_2 = \frac{b_1}{2} = \frac{1}{4}$$
$$b_3 = \frac{b_2}{2} = \frac{1}{8}$$
$$b_4 = \frac{b_3}{2} = \frac{1}{16}$$

(c)

$$c_0 = 0$$
$$c_1 = \frac{c_0}{2} = 0$$
$$c_2 = \frac{c_1}{2} = 0$$
$$c_3 = \frac{c_2}{2} = 0$$
$$c_4 = \frac{c_3}{2} = 0$$

(d)

$$d_0 = 1$$
$$d_1 = -1$$
$$d_2 = d_0 - d_1 = 2$$
$$d_3 = d_1 - d_2 = -3$$
$$d_4 = d_2 - d_3 = 5$$

S-11:

(a) $\{1, 1, 1, 1, 1, \ldots\}$

(b) $b_n = n + 1$

$$\begin{aligned} b_0 &= 0 + 1 = 1 \\ b_1 &= 1 + 1 = 2 \\ b_2 &= 2 + 1 = 3 \\ b_3 &= 3 + 1 = 4 \\ b_4 &= 4 + 1 = 5 \end{aligned}$$

(c) $c_n = \tan(\pi n)$

$$\begin{aligned} c_0 &= \tan(0) = 0 \\ c_1 &= \tan(\pi) = 0 \\ c_2 &= \tan(2\pi) = 0 \\ c_3 &= \tan(3\pi) = 0 \\ c_4 &= \tan(4\pi) = 0 \end{aligned}$$

(d) $d_n = (-1)^n$

$$\begin{aligned} d_0 &= (-1)^0 = 1 \\ d_1 &= (-1)^1 = -1 \\ d_2 &= (-1)^2 = 1 \\ d_3 &= (-1)^3 = -1 \\ d_4 &= (-1)^4 = 1 \end{aligned}$$

S-12: Writing out the first few terms and looking for a pattern is the usual way to start these.

(a) $\{a_n\} = \{2, 2^2, 2^4, 2^8, 2^{16}, \ldots\}$. Note the exponents are powers of two. That is, $a_n = 2^{2^n}$.

(b) $\{b_n\} = \{5, -5, 5, -5, 5, -5, \ldots\}$. The sign-switching behaviour is fairly common; we can use the powers of a negative number to achieve it. $b_n = (-1)^n \cdot 5$.

(c) $\{c_n\} = \{8, 8, 8, 8, \ldots\}$; so, $c_n = 8$

S-13:

(a) $\{0, 1, 4, 9, 16, \ldots\}$ looks like the first few squares of whole numbers: $a_n = n^2$

(b) $\{1, -2, 4, -16, 32, \ldots\}$ looks like powers of 2, with alternating signs. Remember we can generate alternating signs using powers of negative numbers: $a_n = (-2)^n$.

(c) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ has a numerator that's one less than the denominator. The numerators and denominators go up by 1 as n goes up by one, so $a_n = \frac{n+1}{n+2}$.

(d) $\{1.5, 2, 2.5, 3, 3.5, 4, \ldots\}$ goes up by one-half as n goes up by one: $a_n = 1.5 + \frac{n}{2}$

S-14: When determining the end behaviour of rational functions, recall from last semester that we can either cancel out the highest power of n from the numerator and denominator, or skip this step and compare the highest powers of the numerator and denominator.

(a) Since the numerator has a higher degree than the denominator, this sequence will diverge to positive or negative infinity; since its terms are positive for large n, its limit is (positive) infinity. (You can imagine that the numerator is growing much, much faster than the denominator, leading the terms to have a very, very large absolute value.)

Calculating the longer way:

$$a_n = \frac{3n^2 - 2n + 5}{4n + 3}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right) = \frac{3n - 2 + \frac{5}{n}}{4 + \frac{3}{n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n - 2 + \frac{5}{n}}{4 + \frac{3}{n}} = \lim_{n\to\infty} \frac{3n - 2 + 0}{4 + 0} = \infty$$

(b) Since the numerator has the same degree as the denominator, as n goes to infinity, this sequence will converge to the ratio of their leading coefficients: $\dfrac{3}{4}$. (You can imagine that the numerator is growing at roughly the same rate as the denominator, so the terms settle into an almost-constant ratio.)

Calculating the longer way:

$$b_n = \frac{3n^2 - 2n + 5}{4n^2 + 3}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \frac{3 - \frac{2}{n} + \frac{5}{n^2}}{4 + \frac{3}{n^2}}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3 - \frac{2}{n} + \frac{5}{n^2}}{4 + \frac{3}{n^2}} = \frac{3 - 0 + 0}{4 + 0} = \frac{3}{4}$$

(c) Since the numerator has a lower degree than the denominator, this sequence will converge to 0 as n goes to infinity. (You can imagine that the denominator is growing much, much faster than the numerator, leading the terms to be very, very small.)

Calculating the longer way:

$$c_n = \frac{3n^2 - 2n + 5}{4n^3 + 3}\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right) = \frac{\frac{3}{n} - \frac{2}{n^2} + \frac{5}{n^3}}{4 + \frac{3}{n^3}}$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\frac{3}{n} - \frac{2}{n^2} + \frac{5}{n^3}}{4 + \frac{3}{n^3}} = \frac{0 - 0 + 0}{4 + 0} = 0$$

S-15: At first glance, we see both the numerator and denominator grow huge as n increases, so we'll need to think a little further to find the limit.

We don't have a rational function, but we can still divide the top and bottom by n^e to get a clearer picture.

$$a_n = \frac{4n^3 - 21}{n^e + \frac{1}{n}} \left(\frac{\frac{1}{n^e}}{\frac{1}{n^e}} \right) = \frac{4n^{3-e} - \frac{21}{n^e}}{1 + \frac{1}{n^{e+1}}}$$

Since $e < 3$, we see $3 - e$ is positive, so $\lim\limits_{n\to\infty} n^{3-e} = \infty$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^{3-e} - \frac{21}{n^e}}{1 + \frac{1}{n^{e+1}}} = \lim_{n\to\infty} \frac{4n^{3-e} - 0}{1 + 0} = \infty$$

S-16: This isn't a rational sequence, but factoring out $\sqrt{n}$ from the top and bottom will still clear things up.

$$b_n = \frac{\sqrt[4]{n} + 1}{\sqrt{9n + 3}} \left(\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \right) = \frac{\frac{1}{\sqrt[4]{n}} + \frac{1}{\sqrt{n}}}{\sqrt{9 + \frac{3}{n}}}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[4]{n}} + \frac{1}{\sqrt{n}}}{\sqrt{9 + \frac{3}{n}}} = \frac{0 + 0}{\sqrt{9 + 0}} = 0$$

S-17: First, let's start with a tempting fallacy.

> The denominator grows without bound, so $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$.

It's certainly true that *if the limit of the numerator is a real number*, and the denominator grows without bound, then the limit of the sequence is zero. However, in our case, the limit of the numerator does not exist. To apply the limit arithmetic rules from the text (Theorem 5.1.8), our limits must actually exist.

A better reasoning looks something like this:

> The denominator grows without bound, and the numerator never gets very large, so $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$.

To quantify this reasoning more precisely, we use the Squeeze Theorem, Theorem 5.1.10 in the text. There are two parts to the Squeeze Theorem: finding two bounding functions, and making sure these functions have the same limit.

- Since $-1 \leqslant \sin n \leqslant 1$ for all n, we choose functions $a_n = \frac{-1}{n}$ and $b_n = \frac{1}{n}$. Then $a_n \leqslant c_n \leqslant b_n$ for all n.
- Both $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = 0$.

So, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

S-18: The denominator of this sequence grows without bound. The numerator is unpredictable: imagine that n is large. When $\sin n$ is close to -1, $n^{\sin n}$ puts a power of n "in the denominator," so we can have $n^{\sin n}$ very close to 0. When $\sin n$ is close to 1, $n^{\sin n}$ is close to n, which is large.

To control for these variations, we'll use the Squeeze Theorem.

- Since $-1 \leqslant \sin n \leqslant 1$ for all n, let $b_n = \frac{n^{-1}}{n^2} = \frac{1}{n^3}$ and $c_n = \frac{n}{n^2} = \frac{1}{n}$. Then $b_n \leqslant a_n \leqslant c_n$.
- Both $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$.

So, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{n^{\sin n}}{n^2} = 0$ as well.

Remark: we also could have used $b_n = 0$ for our lower bound, since $a_n \geqslant 0$ for all n.

S-19:

$$d_n = e^{-1/n} = \frac{1}{e^{1/n}}$$
$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{1}{e^{1/n}} = \frac{1}{e^0} = \frac{1}{1} = 1$$

S-20:

Solution 1: Let's use the Squeeze Theorem. Since $\sin(n^2)$ and $\sin n$ are both between -1 and 1 for all n, we note:

$$1 + 3(-1) - 2(1) \leqslant 1 + 3\sin(n^2) - 2\sin n \leqslant 1 + 3(1) - 2(-1)$$
$$-4 \leqslant 1 + 3\sin(n^2) - 2\sin n \leqslant 6$$

This allows us to choose suitable bounding functions for the Squeeze Theorem.

- Let $b_n = -\frac{4}{n}$ and $c_n = \frac{6}{n}$. From the work above, we see $b_n \leqslant a_n \leqslant c_n$ for all n.
- Both $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$.

So, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{1 + 3\sin(n^2) - 2\sin n}{n} = 0$.

Solution 2: We simplify slightly to begin.

$$a_n = \frac{1 + 3\sin(n^2) - 2\sin n}{n} = \frac{1}{n} + 3 \cdot \frac{\sin(n^2)}{n} - 2 \cdot \frac{\sin n}{n}$$

We apply the Squeeze Theorem to the pieces $\frac{\sin(n^2)}{n}$ and $\frac{\sin n}{n}$.

- Let $b_n = \frac{-1}{n}$ and $c_n = \frac{1}{n}$. Then $b_n \leqslant \frac{\sin(n^2)}{n} \leqslant c_n$, and $b_n \leqslant \frac{\sin n}{n} \leqslant c_n$.
- Both $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$.

So, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin(n^2)}{n} = 0$ and $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Now, using the arithmetic of limits from Theorem 5.1.8 in the text,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left[\frac{1}{n} + 3\cdot\frac{\sin(n^2)}{n} - 2\cdot\frac{\sin n}{n}\right]\\ &= 0 + 3\cdot 0 - 2\cdot 0 = 0\end{aligned}$$

S-21: First, we note that both numerator and denominator grow without bound. So, we have to decide whether one outstrips the other, or whether they reach a stable ratio.

Solution 1: Let's try dividing the numerator and denominator by 2^n (the dominant term in the denominator; this is the same idea behind factoring out the leading term in rational expressions).

$$b_n = \frac{e^n}{2^n + n^2}\left(\frac{\frac{1}{2^n}}{\frac{1}{2^n}}\right) = \frac{\left(\frac{e}{2}\right)^n}{1 + \frac{n^2}{2^n}}$$

Since $e > 2$, we see $\frac{e}{2} > 1$, and so $\lim_{n\to\infty}\left(\frac{e}{2}\right)^n = \infty$. Since exponential functions grow much, much faster than polynomial functions, we also see $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$. So,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\left(\frac{e}{2}\right)^n}{1 + \frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{\left(\frac{e}{2}\right)^n}{1+0} = \infty$$

Solution 2: Since the numerator and denominator both increase without bound, we apply l'Hôpital's rule. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{2^x\} = 2^x \ln 2$.

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \underbrace{\frac{e^n}{2^n + n^2}}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}\\ &= \lim_{n\to\infty} \underbrace{\frac{e^n}{2^n \ln 2 + 2n}}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}\\ &= \lim_{n\to\infty} \underbrace{\frac{e^n}{2^n (\ln 2)^2 + 2}}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}\\ &= \lim_{n\to\infty} \frac{e^n}{2^n (\ln 2)^3}\\ &= \frac{1}{(\ln 2)^3}\lim_{n\to\infty}\left(\frac{e}{2}\right)^n\\ &= \infty\end{aligned}$$

Since $e > 2$, we see $\dfrac{e}{2} > 1$, and so $\lim\limits_{n\to\infty} \left(\dfrac{e}{2}\right)^n = \infty$.

S-22: First, we simplify. Remember $n! = n(n-1)(n-2)\cdots(2)(1)$ for any whole number n, so $(k+1)! = (k+1)k!$.

$$a_k = \frac{k!\sin^3 k}{(k+1)!} = \frac{k!\sin^3 k}{(k+1)k!} = \frac{\sin^3 k}{k+1}$$

Now, we can use the Squeeze Theorem.

- $-1 \leqslant \sin k \leqslant 1$ for all k, so $-1 \leqslant \sin^3 k \leqslant 1$. Let $b_k = \frac{-1}{k+1}$ and $c_k = \frac{1}{k+1}$. Then $b_k \leqslant a_k \leqslant c_k$.
- Both $\lim\limits_{k\to\infty} b_k = 0$ and $\lim\limits_{k\to\infty} c_k = 0$.

So, by the Squeeze Theorem, also $\lim\limits_{k\to\infty} a_k = 0$.

S-23: Note $\lim\limits_{n\to\infty} (-1)^n$ doesn't exist, but $-1 \leqslant (-1)^n \leqslant 1$ for all n. Let's use the Squeeze Theorem.

- Let $a_n = -\sin\left(\frac{1}{n}\right)$ and $b_n = \sin\left(\frac{1}{n}\right)$. Then $a_n \leqslant (-1)^n \sin\left(\frac{1}{n}\right) \leqslant b_n$.
- Both $\lim\limits_{n\to\infty} -\sin\left(\frac{1}{n}\right) = 0$ and $\lim\limits_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$, since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\sin 0 = 0$.

By the Squeeze Theorem, the sequence $\left\{(-1)^n \sin\frac{1}{n}\right\}$ converges to 0.

S-24: First, we note that $\lim\limits_{n\to\infty} \dfrac{6n^2+5n}{n^2+1} = 6$. We see this either by comparing the leading terms in the numerator and denominator, or by factoring out n^2 from the top and the bottom.

Second, since $\lim\limits_{n\to\infty} \dfrac{1}{n^2} = 0$, we see $\lim\limits_{n\to\infty} \cos\left(\dfrac{1}{n^2}\right) = \cos 0 = 1$.

Using arithmetic of limits, Theorem 5.1.8 in the text, we conclude

$$\lim_{n\to\infty}\left[\frac{6n^2+5n}{n^2+1} + 3\cos(1/n^2)\right] = 6 + 3(1) = 9.$$

S-25: (There are infinitely many potential answers to these questions.)

(a) $\{a_n\} = \{1, 4, 16, 37, \cdots\}$. The first three terms are powers of four, so we could use the sequence $b_n = 4^n$. Then $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$, but $a_3 \neq b_3$ so the sequences are indeed different.

(b) $\{c_n\} = \{3, 6, 9, 18, \cdots\}$. The first three terms are consecutive multiples of three, so we could use the sequence $d_n = 3(n+1)$. Then $c_0 = d_0$, $c_1 = d_1$, and $c_2 = d_2$, but $c_3 \neq d_3$ so the sequences are indeed different.

(c) $\{e_n\} = \{0, 0, 0, 6, \cdots\}$. We could use the sequence $f_n = 0$. Then $e_0 = f_0$, $e_1 = f_1$, and $e_2 = f_2$, but $e_3 \neq f_3$ so the sequences are indeed different.

S-26: If there are only negative values in the sequence, then a_0 itself is negative.

Note $5x - x^2 = x(5-x)$. That's a parabola pointing down with roots at $x = 0$ and $x = 5$. So if x is negative, then $x(5-x)$ is negative as well.

So: if a_0 is negative, then also a_1 is negative; so then also a_2 is negative by the same logic, and so on.

S-27: Let's take stock: $\sin(1/n) \to \sin(0) = 0$ as $n \to \infty$, so $\ln(\sin(1/n)) \to -\infty$. However, $\ln(2n) \to \infty$. So, we have some tension here: the two pieces behave in ways that pull the terms of the sequence in different directions. (Recall we *cannot* conclude anything like "$-\infty + \infty = 0$.")

We try using logarithm rules to get a clearer picture.

$$\ln\left(\sin\frac{1}{n}\right) + \ln(2n) = \ln\left(2n\sin\left(\frac{1}{n}\right)\right)$$

Still, we have indeterminate behaviour: $2n\sin(1/n)$ is the product of $2n$, which grows without bound, and $\sin(1/n)$, which approaches zero. In the past, we learned that we can handle the indeterminate form $0 \cdot \infty$ with l'Hôpital's rule (after a little algebra), but there's a slicker way. Note $1/n \to 0$ as $n \to \infty$. If we write $\frac{1}{n} = x$, then this piece of our limit resembles something familiar.

$$2n\sin\left(\frac{1}{n}\right) = 2\left(\frac{\sin x}{x}\right)$$

If $n \to \infty$, then $x = \frac{1}{n} \to 0$.

$$\lim_{n\to\infty} 2n\sin\left(\frac{1}{n}\right) = 2\lim_{x\to 0}\frac{\sin x}{x}$$

That limit is familiar:

$$= 2(1) = 2$$

Then:

$$\lim_{n\to\infty} \ln\left(2n\sin\left(\frac{1}{n}\right)\right) = \ln 2$$

Note: if you have forgotten that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$, you can also evaluate this limit using l'Hôpital's rule:

$$\underbrace{\lim_{x\to 0}\frac{\sin x}{x}}_{\substack{\text{num}\to 0\\ \text{den}\to 0}} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$$

S-28: First, although this sequence is not defined for some small values of n, it is defined as long as $n \geqslant 5$, so it's not a problem to take the limit as $n \to \infty$. Second, we notice that our limit has the indeterminate form $\infty - \infty$. Since this form is indeterminate, more work is needed to find our limit, if it exists.

A standard trick we saw last semester with functions of this form was to multiply and divide by the conjugate of the expression, $\sqrt{n^2+5n} + \sqrt{n^2-5n}$. Then the denominator will be the *sum* of two similar things, rather than their difference. See the work below to find out why that is helpful.

$$\begin{aligned}\sqrt{n^2+5n} - \sqrt{n^2-5n} &= \left(\sqrt{n^2+5n} - \sqrt{n^2-5n}\right)\left(\frac{\sqrt{n^2+5n} + \sqrt{n^2-5n}}{\sqrt{n^2+5n} + \sqrt{n^2-5n}}\right) \\ &= \frac{(n^2+5n) - (n^2-5n)}{\sqrt{n^2+5n} + \sqrt{n^2-5n}} \\ &= \frac{10n}{\sqrt{n^2+5n} + \sqrt{n^2-5n}}\end{aligned}$$

Now, we'll cancel out n from the top and the bottom. Note $n = \sqrt{n^2}$.

$$\begin{aligned}&= \frac{10n}{\sqrt{n^2+5n} + \sqrt{n^2-5n}}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right) \\ &= \frac{10n}{\sqrt{n^2+5n} + \sqrt{n^2-5n}}\left(\frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2}}}\right) \\ &= \frac{10}{\sqrt{1+\frac{5}{n}} + \sqrt{1-\frac{5}{n}}}\end{aligned}$$

Now, the limit is clear.

$$\lim_{n\to\infty} \frac{10}{\sqrt{1+\frac{5}{n}} + \sqrt{1-\frac{5}{n}}} = \frac{10}{\sqrt{1+0} + \sqrt{1+0}} = \frac{10}{1+1} = 5$$

S-29: First, although this sequence is not defined for some small values of n, it is defined as long as $n \geqslant \sqrt{2.5}$, so it's not a problem to take the limit as $n \to \infty$. Second, we notice that our limit has the indeterminate form $\infty - \infty$. Since this form is indeterminate, more work is needed to find our limit, if it exists.

In Question 28, we saw a similar limit, and made use of the conjugate. However, in this case, there's an easier path: let's factor out n from each term.

$$\begin{aligned}\sqrt{n^2+5n} - \sqrt{2n^2-5} &= \sqrt{n^2\left(1+\frac{5}{n}\right)} - \sqrt{n^2\left(2-\frac{5}{n^2}\right)} \\ &= n\sqrt{1+\frac{5}{n}} - n\sqrt{2-\frac{5}{n^2}} \\ &= n\left(\sqrt{1+\frac{5}{n}} - \sqrt{2-\frac{5}{n^2}}\right)\end{aligned}$$

Now, the limit is clear.

$$\lim_{n\to\infty}\left[\sqrt{n^2+5n}-\sqrt{2n^2-5}\right] = \lim_{n\to\infty}\left[n\left(\sqrt{1+\frac{5}{n}}-\sqrt{2-\frac{5}{n^2}}\right)\right]$$
$$= \lim_{n\to\infty}\left[n\left(\sqrt{1+0}-\sqrt{2-0}\right)\right]$$
$$= \lim_{n\to\infty}\left[n\,(-1)\right] = -\infty$$

Remark: check Question 28 to see whether a similar trick would work there. Why or why not?

S-30: First, we note that we have in indeterminate form: as n grows, $2+\frac{1}{n}\to 2$, so $n\left[\left(2+\frac{1}{n}\right)^{100}-2^{100}\right]$ has the form $\infty\cdot 0$. To overcome this difficulty, we could use some algebra and l'Hôpital's rule, but there's a slicker way. If we let $h=\frac{1}{n}$, then $h\to 0$ as $n\to\infty$, and our limit looks like:

$$\lim_{n\to\infty} n\left[\left(2+\frac{1}{n}\right)^{100}-2^{100}\right] = \lim_{h\to 0}\frac{(2+h)^{100}-2^{100}}{h}$$

This reminds us of the definition of a derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{100}\right\} = \lim_{h\to 0}\frac{(x+h)^{100}-x^{100}}{h}$$

So, if we set $f(x)=x^{100}$, our limit is simply $f'(2)$. That is, $\left[100x^{99}\right]_{x=2}=100\cdot 2^{99}$.

S-31: Using the definition of a derivative,

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

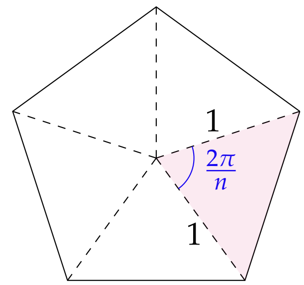
We want $n\to\infty$, so we set $h=\frac{1}{n}$.

$$= \lim_{\frac{1}{n}\to 0}\frac{f\left(a+\frac{1}{n}\right)-f(a)}{\frac{1}{n}}$$
$$= \lim_{n\to\infty} n\left[f\left(a+\frac{1}{n}\right)-f(a)\right]$$

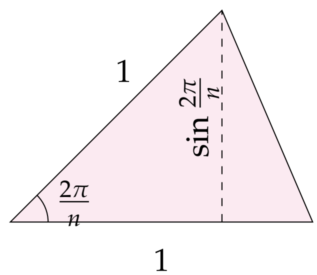
We also could have chosen $h=-\frac{1}{n}$, which leads to the following:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{-\frac{1}{n}\to 0}\frac{f\left(a-\frac{1}{n}\right)-f(a)}{-1/n}$$
$$= \lim_{n\to\infty} -n\left(f\left(a-\frac{1}{n}\right)-f(a)\right)$$
$$= \lim_{n\to\infty} n\left(f(a)-f\left(a-\frac{1}{n}\right)\right)$$

S-32: (a) To find the area A_n, note that the figure with n sides can be divided up into n isosceles triangles, each with two sides of length 1 and angle between them of $\frac{2\pi}{n}$:



Each of these triangles has area $\frac{1}{2}\sin\left(\frac{2\pi}{n}\right)$:



All together, the area of the n-sided figure is $A_n = \dfrac{n}{2}\sin\left(\dfrac{2\pi}{n}\right)$.

(b) We will discuss two ways to find $\lim\limits_{n\to\infty} A_n$, which has the indeterminate form $\infty \times 0$.

First, note that as $n \to \infty$, our figures look more and more like a circle of radius 1. So, we see A_n is approaching the area of a circle of radius 1. That is, $\lim\limits_{n\to\infty} A_n = \pi$.

Alternately, we can make use of the limit $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. Let $x = \frac{2\pi}{n}$. Note if $n \to \infty$, then $x \to 0$.

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n}{2}\sin\left(\frac{2\pi}{n}\right) = \lim_{n\to\infty} \frac{\pi}{\frac{2\pi}{n}}\sin\left(\frac{2\pi}{n}\right)$$
$$= \lim_{x\to 0} \pi\frac{\sin x}{x} = \pi \times 1 = \pi$$

S-33: We'll define a sequence $\{e_n\}_{n=0}^{\infty}$ where e_n is the number of readers left after each equation. We set $e_0 = 7{,}000{,}000{,}000$ (seven billion).

Under the first assumption, $e_{n+1} = \frac{e_n}{2}$. In particular:

- $e_0 = 7{,}000{,}000{,}000$: there have been no equations, and no lost readers
- $e_1 = \frac{e_0}{2}$: there has been one equation, so half of the readers checked out
- $e_2 = \frac{e_1}{2} = \frac{e_0}{2^2}$: there have been two equations, so half of the readers from e_1 (which was already half of the original number) are left
- In general, $e_n = \frac{e_0}{2^n} = \frac{7{,}000{,}000{,}000}{2^n}$

If there is one reader left, then

$$\begin{aligned}1 = e_n &= \frac{7,000,000,000}{2^n} \\ 2^n &= 7,000,000,000 \\ n &= \log_2(7,000,000,000) \approx 32.7\end{aligned}$$

n only makes sense as an integer, so we see unsurprisingly the author took some poetic license. More precisely,

$$\begin{aligned}e_{32} &= \frac{7,000,000,000}{2^32} \approx 1.6 \\ e_{33} &= \frac{7,000,000,000}{2^32} \approx 0.8\end{aligned}$$

So the number of equations in the book was probably 32 or 33.

Now, let $\{f_n\}_{n=0}^{\infty}$ be the sequence of readers after n equations under the more generous assumptions of the second paragraph. Let's look for a pattern.

- After the first equation, half the readers are lost, so half remain:

$$f_1 = \frac{f_0}{2}$$

- After the second equation, one-third of those readers are lost, so two-thirds remain:

$$f_2 = \frac{2}{3} \cdot f_1 = \frac{2}{3} \cdot \frac{f_0}{2} = \frac{f_0}{3}$$

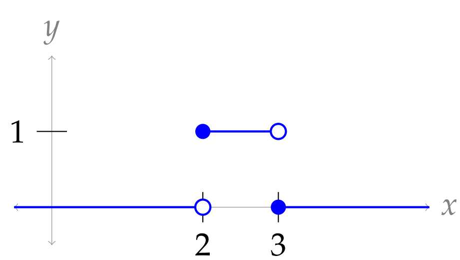
- After the third equation, one-fourth of the readers are lost, so three-fourths remain:

$$f_3 = \frac{3}{4} \cdot f_2 = \frac{3}{4} \cdot \frac{f_0}{3} = \frac{f_0}{4}$$

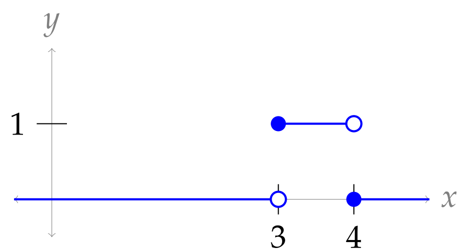
We see a nice pattern: for $n > 0$, $f_n = \frac{7,000,000,000}{n}$. So if there were 32 or 33 equations, the readers remaining number $\frac{7,000,000,000}{32} = 218,750,000$ or $\frac{7,000,000,000}{32} \approx 212,121,212$.

S-34:

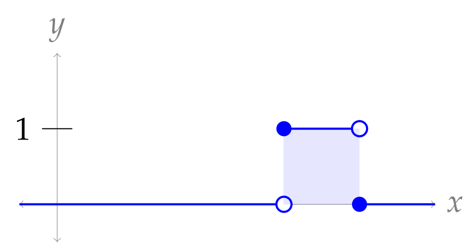
(a) $f_2(x) = \begin{cases} 1 & 2 \leqslant x < 3 \\ 0 & \text{else} \end{cases}$



(b) $f_3(x) = \begin{cases} 1 & 3 \leqslant x < 4 \\ 0 & \text{else} \end{cases}$



(c) For any n, $f_n(x) = 1$ for an interval of length 1, and $f_n(x) = 0$ for all other x. So, the area under the curve is a square of side length one.



Then $A_n = \int_0^\infty f_n(x)\,\mathrm{d}x = 1$ for all n. That is, the sequence $\{A_n\}$ is simply $\{1, 1, \ldots, 1\}$, a sequence of all 1s.

(d) Given the description above, $\lim_{n\to\infty} A_n = 1$.

(e) For any fixed x, recall $\{f_n(x)\} = \{0, \ldots, 0, 1, 0, \ldots 0, 0, 0, 0, 0, \ldots\}$. In particular, there are infinitely many zeroes at its end. So, $\lim_{n\to\infty} f_n(x) = 0$. Then $g(x) = 0$ for every x.

(f) Given the description above, $\displaystyle\int_0^\infty g(x)\,\mathrm{d}x = \int_0^\infty 0\,\mathrm{d}x = 0$.

Remark: what we've shown here is that, for this particular $f_n(x)$,

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,\mathrm{d}x \neq \int_0^\infty \lim_{n\to\infty} f_n(x)\,\mathrm{d}x$$

That is, we can't necessarily swap a limit with an integral (which is, in this case, another limit, since the integral is improper). The interested reader can look up "uniform convergence" to learn about the conditions under which these *can* be swapped.

S-35: If we naively try to find the limit, we run up against the indeterminate form 1^∞. We'd like to use l'Hôpital's rule, but we don't have the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$–we'll need to use a logarithm. Additionally, l'Hôpital's rule applies to differentiable functions defined for real numbers–so we'll consider a function, rather than the sequence.

Note the terms of the sequence are all positive.

Solution 1: Define $x = \frac{1}{n}$, and $f(x) = \left(1 + 3x + 5x^2\right)^{1/x}$. Then $b_n = f\left(\frac{1}{n}\right) = f(x)$, and

$$\lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{x\to 0^+} f(x).$$

If this limit exists, it is equal to $\lim_{n\to\infty} b_n$.

$$\begin{aligned}
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \left(1+3x+5x^2\right)^{1/x} \\
\lim_{x\to 0^+} \ln[f(x)] &= \lim_{x\to 0^+} \ln\left[\left(1+3x+5x^2\right)^{1/x}\right] = \underbrace{\lim_{x\to 0^+} \frac{\ln\left[1+3x+5x^2\right]}{x}}_{\substack{\text{num}\to 0\\ \text{den}\to 0}} \\
&= \lim_{x\to 0^+} \frac{\frac{3+10x}{1+3x+5x^2}}{1} = 3 \\
\lim_{x\to 0^+} f(x) &= e^3
\end{aligned}$$

Since the limit exists, $\lim_{n\to\infty} b_n = e^3$.

Solution 2: If we didn't see the nice simplifying trick of letting $x=\frac{1}{n}$, we can still solve the problem using $g(x) = \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$:

$$\begin{aligned}
g(x) &= \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x \\
\ln[g(x)] &= x\ln\left[1+\frac{3}{x}+\frac{5}{x^2}\right] = \underbrace{\frac{\ln\left[1+\frac{3}{x}+\frac{5}{x^2}\right]}{1/x}}_{\substack{\text{num}\to 0\\ \text{den}\to 0}}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \ln[g(x)] &= \lim_{x\to\infty} \frac{\frac{-\frac{3}{x^2}-\frac{10}{x^3}}{1+\frac{3}{x}+\frac{5}{x^2}}}{\frac{-1}{x^2}} = \lim_{x\to\infty} x^2 \frac{\frac{3}{x^2}+\frac{10}{x^3}}{1+\frac{3}{x}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{10}{x}}{1+\frac{3}{x}+\frac{5}{x^2}} = \frac{3+0}{1+0+0} = 3 \\
\lim_{x\to\infty} f(x) &= e^3
\end{aligned}$$

Since the limit exists, $\lim_{n\to\infty} b_n = e^3$

S-36:

(a) When $a_1 = 4$, we see $a_2 = \dfrac{4+8}{3} = 4$, and so on. That is, $a_n = 4$ for every n. So, $\lim_{n\to\infty} a_n = 4$.

(b) Cross-multiplying, we see $3x = x+8$, hence $x = 4$.

(c) In order for our sequence to converge to 4, the terms should be getting infinitely close to 4. So, we find the relationship between $a_{n+1}-4$ and $a_n - 4$.

$$\begin{aligned}
a_{n+1} &= \frac{a_n+8}{3} \\
a_{n+1}-4 &= \frac{a_n+8}{3} - 4 = \frac{a_n-4}{8}
\end{aligned}$$

So, the distance between our sequence terms and the number 4 is decreasing by a factor of 8 each term. This implies that the terms get infinitely close to 4 as n grows. That is, $\lim_{n\to\infty} a_n = 4$.

S-37:

(a) Since w_1 has the highest frequency, w_2 has the next-highest frequency, and so on, we know f_1 is larger than the other members of its sequence, f_2 is the next largest, etc. So, $\{f_n\}$ is a decreasing sequence.

(b) The most-used word in a language is w_1, while the n-th most used word in a language is w_n. So, we re-state the law as:

$$f_1 = nf_n$$

Then we can rewrite this fomula a little more naturally as $f_n = \frac{1}{n}f_1$.

(c) Then $f_3 = \frac{1}{3}f_1$. In this case, we expect the third-most used word to account for $\frac{1}{3}(6\%) = 2\%$ of all words.

(d) From (b), we know $f_{10} = \frac{1}{10}f_1$. Note $f_1 = 6f_6 = 6(0.3\%)$. Then:

$$f_{10} = \frac{1}{10}f_1 = \frac{1}{10}6f_6 = \frac{1}{10}(6)(0.3\%) = \frac{1.8}{10}\% = 0.18\%$$

So, f_{10} should be 0.18% of all words.

(e) The use of the word "frequency" in the statement of Zipf's law implies $f_n = \frac{\text{uses of } w_n}{\text{total number of words}}$. The question asks for the total uses of w_n. If we call this quantity t_n, and the total number of all words is T, then Zipf's law tells us $\frac{t_n}{T} = \frac{1}{n}\frac{t_1}{T}$, hence $t_n = \frac{1}{n}t_1$.

With this notation, the problem states $t_1 = 22,038,615$, $w_1 = \texttt{the}$, $w_2 = \texttt{be}$, and $w_3 = \texttt{and}$.

Following Zipf's law, $t_n = \frac{1}{n}t_1$. So, we expect $t_2 = \frac{t_1}{2} = 11,019,307.5$; since this isn't an integer, let's say we expect $t_2 \approx 11,019,308$. Similarly, we expect $t_3 = \frac{t_1}{3} = 7,346,205$.

Remark: The 450-million-word source material that used "the" 22,038,615 times also contained 12,545,825 instances of "be," and 10,741,073 instances of "and." While Zipf's Law might be a nice model for our data overall, in these few instances it does not appear to be extremely accurate.

S-38:

(a) I comes from simplifying $P(1+r) + rP(1+r) = P[(1+r) + r(1+r)] = P(1+r)^2$. (This is not the only way to simplify the expression.)

For II, we take $\mathrm{I} + r\mathrm{I}$, i.e. $P(1+r)^2 + r \cdot P(1+r)^2$. This simplifies to $(1+r) \cdot P(1+r)^2 = P(1+r)^3$, which goes in III.

(b) This corresponds to $P = 100$ and $r = 0.1$, so it would be $P(1+r)^2 = 100(1.1^2) = 121$ dollars.

(c) Recognizing the pattern in the third column of the table, after n years, the amount in the account will be $P(1+r)^n$ dollars.

(d) Using our answer from above, we solve for P:

$$\begin{aligned} 300 &= P(1+0.1)^3 = P(1.1^3) \\ P &= \frac{300}{1.1^3} \approx 225.394 \end{aligned}$$

Rounding, to the closest cent, in this case, rounds down; so an investment of \$225.39 would yield slightly *less* than \$300 after three years, whereas an investment of \$225.40 would yield slightly more. Both \$225.39 and \$225.40 are acceptable answers, but the latter fits the spirit of the question better: usually people want to have *at least* some amount of money more then they want to be *as close as possible* to that amount.

Solutions to Exercises 5.1.1 — Jump to TABLE OF CONTENTS

S-1:

Let's write our original four notes as multiples of the first note. (We'll use the same notation as a series, although we have only finitely many terms.)

$$\begin{aligned} a_0 &= 100 \\ a_1 &= 110 = 1.1 \cdot a_0 \\ a_2 &= 150 = 1.5 \cdot a_0 \\ a_3 &= 200 = 2 \cdot a_0 \end{aligned}$$

These same multiples are what will be preserved in the physicist's song.

$$\begin{aligned} b_0 &= 150 \\ b_1 &= 1.1 \cdot b_0 = 165 \\ b_2 &= 1.5 \cdot b_0 = 225 \\ b_3 &= 2 \cdot b_0 = 300 \end{aligned}$$

So, the physicist's song uses frequencies 150, 165, 225, and 300.

It's worth nothing that the ratios between all pairs of notes are preserved – not just the ratio between one note and the original note. For example, $\frac{a_3}{a_1} = \frac{20}{11} = \frac{b_3}{b_1}$. So there was nothing special about using the ratio between each note and a_0. We equally well could have used the ratios of consecutive notes:

$$
\begin{array}{lcl}
a_0 = 100 & & b_0 = 150 \\
a_1 = 110 = 1.1 \cdot a_0 & \Longrightarrow & b_1 = 1.1 \cdot b_0 = 165 \\
a_2 = 150 = \dfrac{15}{11} \cdot a_1 & \Longrightarrow & b_2 = \dfrac{15}{11} \cdot b_1 = 225 \\
a_3 = 200 = \dfrac{4}{3} \cdot a_2 & \Longrightarrow & b_3 = \dfrac{4}{3} \cdot b_2 = 300
\end{array}
$$

S-2: The scale isn't even-tempered, because the intervals (ratios) between consecutive notes are not constant. For starters, $\frac{120}{110} \neq \frac{140}{120}$.

The octave is divided into five intervals: (1) from 100 to 120, (2) from 120 to 140, (3) from 140 to 160, (4) from 160 to 180, and (5) from 180 to 200.

It can sometimes be confusing that six notes give us five intervals! You saw a similar count with Riemann sums in Section 3.1. The $(n+1)$ points $x_0, x_1, \ldots, x_n$ delineate n intervals.

S-3: Let's name the 13 notes from 444 to 888 $\{a_n\}_{n=0}^{12}$. In an even-tempered scale, $\frac{a_{n+1}}{a_n} = r$ for some constant r and every n from 0 to 11. That means our scale will be the initial part of a geometric sequence,

$$a_n = ar^n$$

Since $a_{12} = 888 = 2 \cdot a_0 = r^{12} a_0$, we see $r = 2^{1/12}$. Now we can write down each note's frequency:

0. 444
1. $444 \cdot 2^{1/12} \approx 470.40$
2. $444 \cdot 2^{2/12} \approx 498.37$
3. $444 \cdot 2^{3/12} \approx 528.01$
4. $444 \cdot 2^{4/12} \approx 559.40$
5. $444 \cdot 2^{5/12} \approx 592.67$
6. $444 \cdot 2^{6/12} \approx 627.91$
7. $444 \cdot 2^{7/12} \approx 665.25$
8. $444 \cdot 2^{8/12} \approx 704.81$
9. $444 \cdot 2^{9/12} \approx 746.72$
10. $444 \cdot 2^{10/12} \approx 791.12$
11. $444 \cdot 2^{11/12} \approx 838.16$

12. $444 \cdot 2^{12/12} = 888$

S-4:

Let's name the 11 notes from 100 to 200 $\{a_n\}_{n=0}^{10}$. In an even-tempered scale, $\frac{a_{n+1}}{a_n} = r$ for some constant r and every n from 0 to 9. That means our scale will be the initial part of a geometric sequence,

$$a_n = ar^n$$

Since $a_{10} = 200 = 2 \cdot a_0 = r^{10} a_0$, we see $r = 2^{1/10}$. Now we can write down each note's frequency:

0. 100
1. $100 \cdot 2^{1/10} \approx 107.12$
2. $100 \cdot 2^{2/10} \approx 114.87$
3. $100 \cdot 2^{3/10} \approx 1213.11$
4. $100 \cdot 2^{4/10} \approx 131.95$
5. $100 \cdot 2^{5/10} \approx 141.42$
6. $100 \cdot 2^{6/10} \approx 151.57$
7. $100 \cdot 2^{7/10} \approx 162.45$
8. $100 \cdot 2^{8/10} \approx 174.11$
9. $100 \cdot 2^{9/10} \approx 186.61$
10. $100 \cdot 2^{10/10} = 200$

S-5:

1. As in Example 5.1.14, we'll consider the ratios of pitches.

$$\begin{aligned} a_0 &= 440 & & & b_0 &= 586.\overline{66} \\ a_1 &= 495 = \frac{9}{8} a_0 & &\Longrightarrow & b_1 &= \frac{9}{8} b_0 = 660 \\ a_2 &= 556.875 = \frac{9}{8} a_1 & &\Longrightarrow & b_2 &= \frac{9}{8} b_1 = 742.5 \end{aligned}$$

The pitches in the transposed song all correspond to notes our instrument can play, so this is indeed possible.

2. Proceeding similarly:

$$\begin{aligned} a_0 &= 440 & & & b_0 &= 495 \\ a_1 &= 495 = \frac{9}{8} a_0 & &\Longrightarrow & b_1 &= \frac{9}{8} b_0 = 556.875 \\ a_2 &= 556.875 = \frac{9}{8} a_1 & &\Longrightarrow & b_2 &= \frac{9}{8} b_1 = 626.484375 \end{aligned}$$

Our instrument cannot play the note 626.484375. If we tried to play this transposition, we'd have to change it to use a different note (probably 660), and the song would sound different.

S-6: Our sequence notation makes it really easy to find the interval between two notes: $\frac{a_m}{a_n} = \frac{100 \cdot 2^{n/12}}{100 \cdot 2^{m/12}} = 2^{n-m}$. Notice the interval depends on the difference between the two indices. So if we want our intervals to be the same, we just need the differences of indices to be the same. So, if a_k is our lowest note, the four notes of the song are simply a_k, a_{k+2}, a_{k+5}, and a_{k+7}.

We can check that the ratios between notes are preserved:

$$\frac{a_2}{a_0} = \frac{100 \cdot 2^{2/12}}{100 \cdot 2^{0/12}} = 2^{2/12} \qquad \frac{a_{k+2}}{a_k} = \frac{100 \cdot 2^{(k+2)/12}}{100 \cdot 2^{k/12}} = 2^{2/12}$$

$$\frac{a_5}{a_2} = \frac{100 \cdot 2^{5/12}}{100 \cdot 2^{2/12}} = 2^{3/12} \qquad \frac{a_{k+5}}{a_{k+2}} = \frac{100 \cdot 2^{(k+5)/12}}{100 \cdot 2^{(k+2)/12}} = 2^{3/12}$$

$$\frac{a_7}{a_5} = \frac{100 \cdot 2^{7/12}}{100 \cdot 2^{5/12}} = 2^{2/12} \qquad \frac{a_{k+7}}{a_{k+5}} = \frac{100 \cdot 2^{(k+7)/12}}{100 \cdot 2^{(k+5)/12}} = 2^{2/12}$$

So, the transposition can be played for *any* lowest note a_k. If a_k is a note on the scale, then the rest of the notes of the song are notes on the scale as well.

Note: moving a song up or down while keeping the intervals the same is called transposition. The ease of transposition is one reason why even-tempered scales are popular.

S-7: Remember we need to keep the ratios of frequecies the same. Let's start with the first two notes.

$$\frac{b_{11}}{b_{10}} = \frac{100 \cdot 11}{100 \cdot 10} = \frac{11}{10}$$

If we were to change the notes b_{10} and b_{11} to other notes on the scale, b_n and b_m, then the ratio would have to be preserved:

$$\frac{11}{10} = \frac{b_m}{b_n} = \frac{100 \cdot m}{100 \cdot n} = \frac{m}{n}$$

If we can only use notes from the scale, then for this first ratio to be preserved, we must have $n = \frac{11}{10}m$. In particular, for n to be a whole number, m must be divisible by 10. The smallest whole number divisible by 10 is, of course, 10 itself. So while t is possible to shift the song *higher* using notes from our harmonic scale, it is not possible to shift it lower.

S-8: In each question, we specified subdivisions of an octave. An octave has ratio 2. So, in both cases, we had a geometric series

$$a_n = ar^n$$

where $a_k = 2a_0$ was specified. That's where the 2 comes from.

S-9:

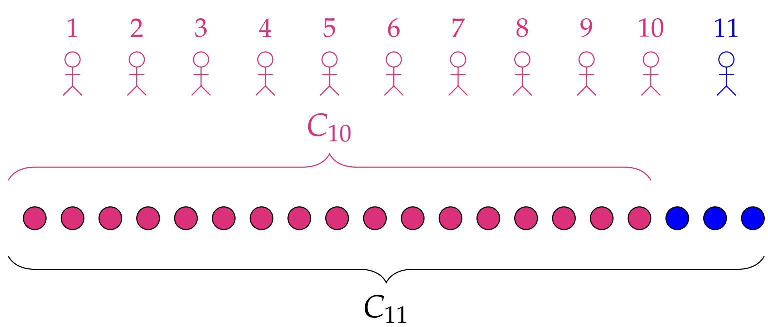
- Pinching at the first position makes the string $\frac{11}{12}$ its original length, so the frequency produced is $\frac{12}{11}$ the original frequency, or $\frac{12}{11}(330) = 360$ Hz
- Pinching at the second position makes the string half its original length, so the frequency produced is double the original frequency, or $2(330) = 660$ Hz
- Pinching at the third position makes the string $\frac{1}{3}$ its original length, so the frequency produced is three times the original frequency, or $3(330) = 990$ Hz

Solutions to Exercises 5.2 — Jump to TABLE OF CONTENTS

S-1: The Nth term of the sequence of partial sums, S_N, is the sum of the first N terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

N	$\mathbf{S_N}$
1	1
2	$1 + \frac{1}{2}$
3	$1 + \frac{1}{2} + \frac{1}{3}$
4	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
5	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

S-2: If there were a total of 17 cookies before Student 11 came, and 20 cookies after, then Student 11 brought 3 cookies.



S-3:

(a) We find $\{a_n\}$ from $\{S_N\}$ using the same logic as Question 2. S_N is the sum of the first N terms of $\{a_n\}$, and S_{N-1} is the sum of all the same terms *except* a_N. So,

$a_N = S_N - S_{N-1}$ when $N \geqslant 2$. Written another way:

$$S_N = a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1} + a_N$$
$$S_{N-1} = a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1}$$

So,

$$\begin{aligned} S_N - S_{N-1} &= \left[a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1} + a_N\right] \\ &\quad - \left[a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1}\right] \\ &= a_N \end{aligned}$$

So, we calculate

$$\begin{aligned} a_N = S_N - S_{N-1} &= \left(\frac{N}{N+1}\right) - \left(\frac{N-1}{N-1+1}\right) \\ &= \frac{N^2}{N(N+1)} - \frac{N^2-1}{N(N+1)} \\ &= \frac{1}{N(N+1)} \end{aligned}$$

Therefore,

$$a_n = \frac{1}{n(n+1)}$$

Remark: the formula given for S_N has $S_0 = 0$, which makes sense: the sum of no terms at all should be 0. However, it is common for a sequence of partial sums to start at $N = 1$. (This fits our definition of a partial sum–we don't really define the "sum of no terms.") In this case, a_1 must be calculated separately from the other terms of $\{a_n\}$. To find a_1, we simply set $a_1 = S_1$, which (to reiterate) might not be the same as $S_1 - S_0$.

(b)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0.$$

That is, the terms we're adding up are getting very, very small as we go along.

(c) By Definition 5.2.3 in the text,

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N}{N+1} = 1$$

That is, as we add more and more terms of our series, our cumulative sum gets very, very close to 1.

S-4: As in Question 3,

$$\begin{aligned}a_N = S_N - S_{N-1} &= \left[(-1)^N + \frac{1}{N}\right] - \left[(-1)^{N-1} + \frac{1}{N-1}\right] \\ &= (-1)^N - (-1)^{N-1} + \frac{1}{N} - \frac{1}{N-1} \\ &= (-1)^N + (-1)^N + \frac{N-1}{N(N-1)} - \frac{N}{N(N-1)} \\ &= 2(-1)^N - \frac{1}{N(N-1)}\end{aligned}$$

Note, however, that a_N is only the same as $S_N - S_{N-1}$ when $N \geqslant 2$: otherwise, we're trying to calculate $S_1 - S_0$, but S_0 is not defined. So, we find a_1 separately:

$$a_1 = S_1 = (-1)^1 + \frac{1}{1} = 0$$

All together:

$$a_n = \begin{cases} 0 & \text{if } n = 1 \\ 2(-1)^n - \frac{1}{n(n-1)} & \text{else} \end{cases}$$

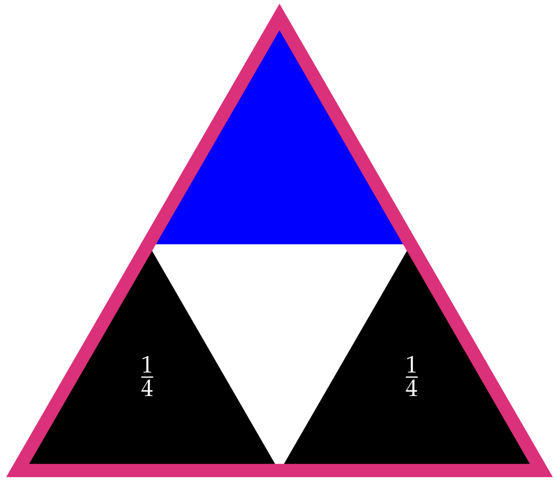
S-5: If $f'(N) < 0$, that means $f(N)$ is decreasing. So, adding more terms makes for a smaller sum. That means the terms we're adding are negative. That is, $a_n < 0$ for all $n \geqslant 2$.

S-6: (a) To generate the pattern, we repeat the following steps:

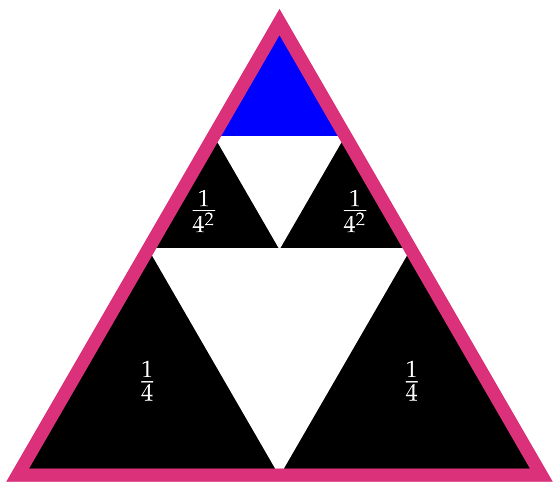
- divide the top triangle into four triangles of equal area,
- colour the bottom two of them black, and
- leave the middle one white.

Every time we repeat this sequence, we divide up a triangle with an area one-quarter the size of our previous triangle, and take two of the four resulting pieces. So, our area should end up as a geometric sum with common ratio $r = \frac{1}{4}$, and coefficient $a = 2$. This is shown more explicitly below.

Since the entire triangle (outlined in red) has area 1, the four smaller triangles below each have area $\frac{1}{4}$. The two black triangles will be added to our total black area; the blue triangle will be subdivided.



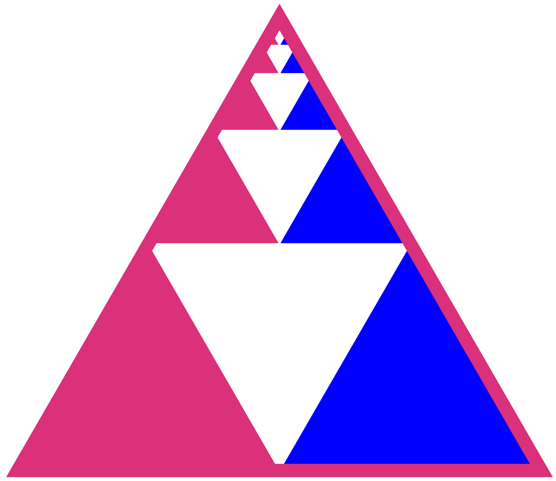
The blue triangle had area $\frac{1}{4}$, so each of the small black triangles below has area $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{4^2}$.



Each time we make another subdivision, we add two black triangles, each with $\frac{1}{4}$ the area of the previous black triangles. So, our total black area is:

$$2\left(\frac{1}{4}\right) + 2\left(\frac{1}{4^2}\right) + 2\left(\frac{1}{4^3}\right) + 2\left(\frac{1}{4^4}\right) + \cdots = \sum_{n=1}^{\infty} \frac{2}{4^n}$$

(b) To evalutate the series, we imagine gathering up all our little triangles and sorting them into three identical piles: the bottom three triangles go in three different piles, the three triangles directly above them go in three different piles, etc. (In the picture below, different colours correspond to different piles.)

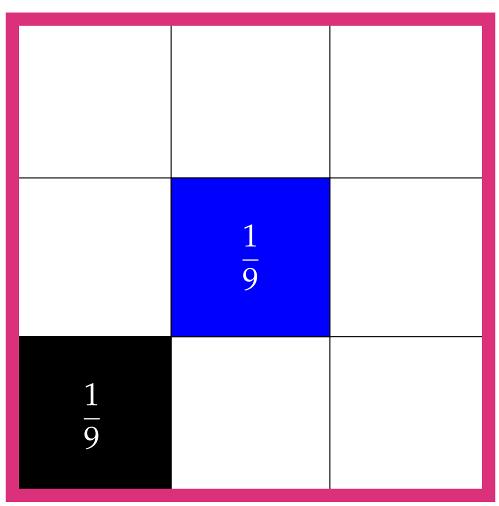


Since the piles all have equal area, each pile has a total area of $\frac{1}{3}$. The black area shaded in the problem corresponds to *two* piles (red and blue above), so

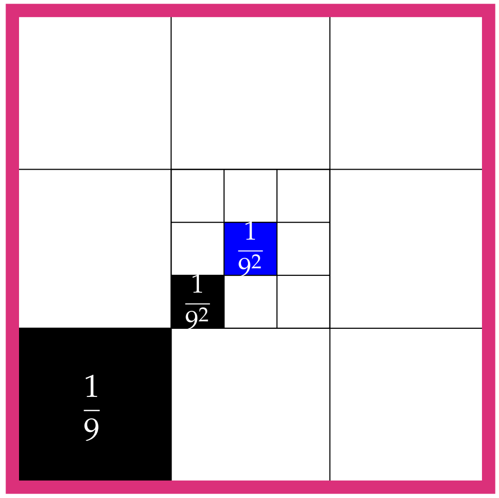
$$\sum_{n=1}^{\infty} \frac{2}{4^n} = \frac{2}{3}$$

S-7: (a) The pattern can be described as follows: divide the innermost square into 9 equal parts (a 3×3 grid), choose one square to be black, and another square to subdivide.

The area of the red (outermost) square is 1, so the area of the largest black square is $\frac{1}{9}$. The area of the central, blue square below is also $\frac{1}{9}$.



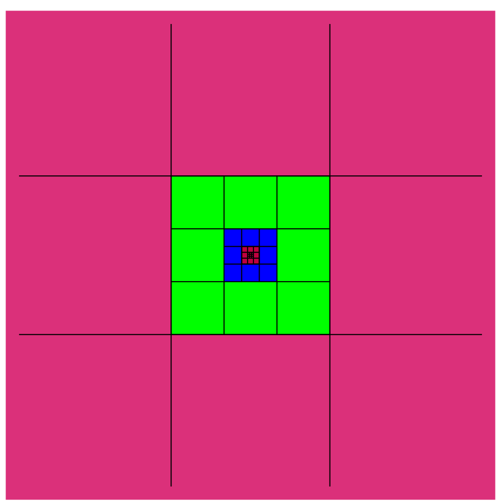
When we subdivide the blue square, the subdivisions each have one-ninth its area, or $\frac{1}{9^2}$.



We continue taking squares that are one-ninth the area of the previous square. So, our total black area is

$$\frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9^n}$$

(b) If we cut up this square along the marks, we can easily share it equally among 8 friends: there are eight squares of area $\frac{1}{9}$ along the outer ring, eight squares of area $\frac{1}{9^2}$ along the next ring in, and so on.



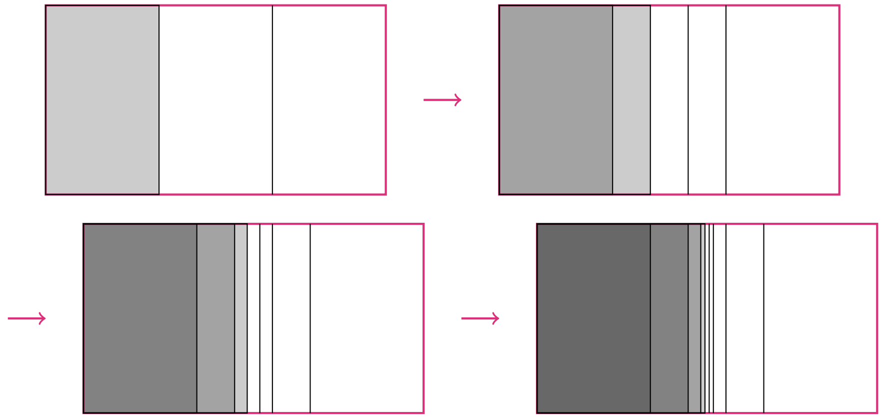
Since the eight friends all get the same total area, the area each friend gets is $\frac{1}{8}$. The area shaded in black in the question corresponds to the pile given to one friend. So,

$$\sum_{n=1}^{\infty} \frac{1}{9^n} = \frac{1}{8}$$

S-8:

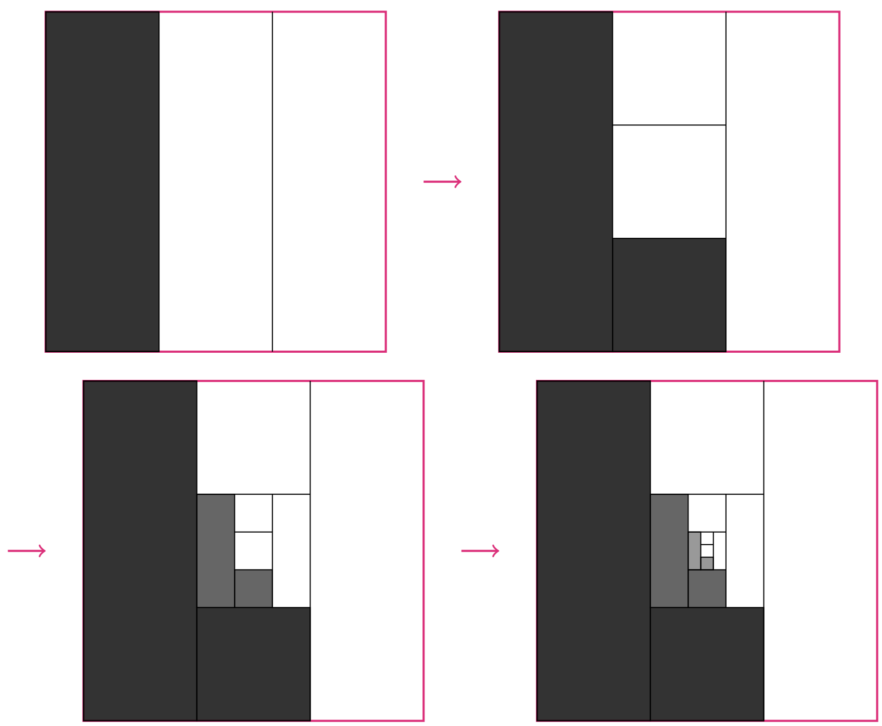
If we start with a shape of area 1, and iteratively divide it into thirds, taking one of the three newly created pieces each time, then the area we take will be equal to the desired series, $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

One way to do this is to start with a rectangle, make three vertical strips, then keep the left strip and subdivide the middle strip.



We see that the total area we take approaches one-half the total area of the figure, so $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$.

Alternately, instead of always taking vertical strips, we could alternate vertical and horizontal slices.



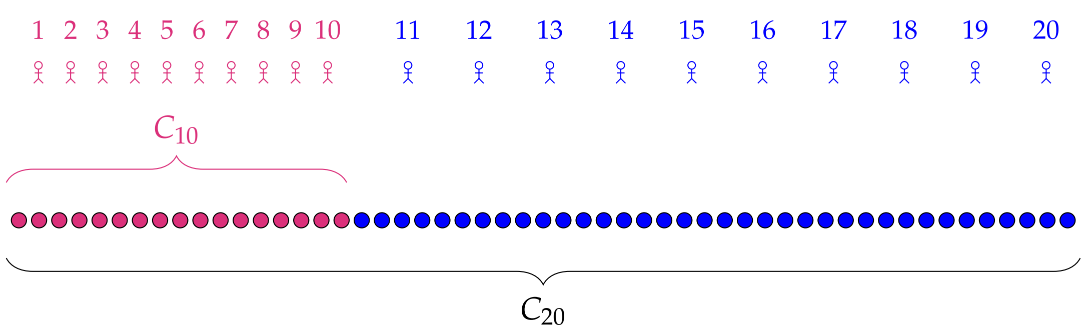
In this setup, we notice that our strips come in pairs: two large vertical strips, two smaller horizontal strips, two smaller vertical strips, etc. We shaed exactly one of each, so the shaded area is one-half the total area: $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$.

Other solutions are possible, as well.

S-9: Theorem 5.2.5 in the text tells us $\sum_{n=0}^{N} ar^n = a\frac{1-r^{N+1}}{1-r}$, for $r \neq 1$. Our geometric sum has $a = 1$, $r = \frac{1}{5}$, and $N = 100$. So:

$$\sum_{n=0}^{100} \frac{1}{5^n} = \frac{1 - \frac{1}{5^{101}}}{1 - \frac{1}{5}} = \frac{5^{101} - 1}{4 \cdot 5^{100}}$$

S-10: After twenty students have brought their cookies, the pile numbers 53 cookies. 17 of these cookies were brought by students one through ten. So, the remainder $(53 - 17 = 36)$ is the number of cookies brought by students 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20, together.



S-11:

Solution 1: Using the ideas of Question 10, we see:

$$\sum_{n=50}^{100} \frac{1}{5^n} = \sum_{n=0}^{100} \frac{1}{5^n} - \sum_{n=0}^{49} \frac{1}{5^n}$$

That is, we want start with the sum of all the terms up to $\frac{1}{5^{100}}$, and then subtract off the ones we actually don't want, which is everything up to $\frac{1}{5^{49}}$. Now, both series are in a form appropriate for Theorem 5.2.5 in the text.

$$\begin{aligned}
\sum_{n=0}^{100} \frac{1}{5^n} - \sum_{n=0}^{49} \frac{1}{5^n} &= \frac{1-\frac{1}{5^{101}}}{1-\frac{1}{5}} - \frac{1-\frac{1}{5^{50}}}{1-\frac{1}{5}} \\
&= \frac{5^{101}-1}{4\cdot 5^{100}} - \frac{5^{50}-1}{4\cdot 5^{49}}\left(\frac{5^{51}}{5^{51}}\right) \\
&= \frac{5^{101}-1}{4\cdot 5^{100}} - \frac{5^{101}-5^{51}}{4\cdot 5^{100}} \\
&= \frac{5^{51}-1}{4\cdot 5^{100}}
\end{aligned}$$

Solution 2: If we write out the first few terms of our series, we see we can factor out a constant to change the starting index.

$$\begin{aligned}
\sum_{n=50}^{100} \frac{1}{5^n} &= \frac{1}{5^{50}} + \frac{1}{5^{51}} + \frac{1}{5^{52}} + \frac{1}{5^{53}} + \cdots + \frac{1}{5^{100}} \\
&= \frac{1}{5^{50}}\left(\frac{1}{5^0} + \frac{1}{5^1} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots + \frac{1}{5^{50}}\right) \\
&= \sum_{n=0}^{50} \frac{1}{5^{50}} \cdot \frac{1}{5^n}
\end{aligned}$$

Now, our sum is in the form of Theorem 5.2.5 in the text with $a = \frac{1}{5^{50}}$, $r = \frac{1}{5}$, and $N = 50$.

$$\begin{aligned}
\sum_{n=0}^{50} \frac{1}{5^{50}} \cdot \frac{1}{5^n} = \frac{1}{5^{50}} \cdot \frac{1-\frac{1}{5^{51}}}{1-\frac{1}{5}} &= \frac{1-\frac{1}{5^{51}}}{4\cdot 5^{49}}\left(\frac{5^{51}}{5^{51}}\right) \\
&= \frac{5^{51}-1}{4\cdot 5^{100}}
\end{aligned}$$

S-12: (a) The table below is a record of our account, with black entries representing the money your friend gives you, and red entries representing the money you give them (which is why the red entries are negative).

$\mathbf{d}$	$-\frac{1}{d+1}$	$\frac{1}{d}$	**total after day** d
1	$-\frac{1}{2}$	1	$\frac{1}{2}$
2	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
3	$-\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{4}$
4	$-\frac{1}{5}$	$\frac{1}{4}$	$\frac{4}{5}$
5	$-\frac{1}{6}$	$\frac{1}{5}$	$\frac{5}{6}$
6	$-\frac{1}{7}$	$\frac{1}{6}$	$\frac{6}{7}$

After the exchange of day n, the amount you're left with is $\$\left(1-\frac{1}{n+1}\right)$. We see this by the cancellation in the table: the $\$\frac{1}{2}$ you gave your friend on day 1 was returned on day 2; the $\$\frac{1}{3}$ you gave your friend on day 2 was returned on day 3, etc.

So, after a long time, you'll have gained close to (but always slightly less than) one dollar.

(b) The series $\displaystyle\sum_{d=1}^{\infty}\left(\frac{1}{d}-\frac{1}{(d+1)}\right)$ describes the scenario in (a), so by our reasoning there,

$$\sum_{d=1}^{\infty}\left(\frac{1}{d}-\frac{1}{(d+1)}\right)=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

(c) Again, let's set up an account book.

$\mathbf{d}$	$\mathbf{d+1}$	$-(\mathbf{d+2})$	**total**
1	2	-3	-1
2	3	-4	-2
3	4	-5	-3
4	5	-6	-4
5	6	-7	-5
6	7	-8	-6

By day d, you've *lost* $\$\,d$ to your so-called friend. As time goes on, you lose more and more.

(d) The series $\displaystyle\sum_{d=1}^{\infty}((d+1)-(d+2))$ exactly describes the scenario in part (c), so it

diverges to $-\infty$. You can also see this by writing
$\sum_{d=1}^{\infty}((d+1)-(d+2)) = \sum_{d=1}^{\infty}(-1) = -1-1-1-1-1-\cdots$.
Be careful to avoid a common mistake with telescoping series: if we look back at our account book, we see that every negative term will cancel with a positive term, with the initial $+2$ as the only term that never cancels. Your friend takes \$3, which they return the next day; then they take \$4, which they return the next day; then they take \$5, which they return the next day, and so on. It's extremely tempting to say that the series adds up to \$2, since every other term cancels out eventually. This is where we lean on Definition 5.2.3 in the text: we evaluate the *partial sums*, which always leave your friend's last withdrawal unreturned. This definition makes sense: saying "I gained two bucks from this exchange" doesn't really capture the reality of your increasing debt.

S-13: Using arithmetic of series, Theorem 5.2.10 in the text, we see

$$\sum_{n=1}^{\infty}(a_n+b_n+c_{n+1}) = A+B+\sum_{n=1}^{\infty}c_{n+1}$$

The question remaining is what do to with the last series. If we write out the terms, we see the difference between $\sum_{n=1}^{\infty}c_n$ and $\sum_{n=1}^{\infty}c_{n+1}$ is simply that the latter is missing c_1:

$$\begin{aligned}\sum_{n=1}^{\infty}c_{n+1} &= c_2+c_3+c_4+c_5+\cdots\\ &= -c_1+c_1+c_2+c_3+c_4+c_5+\cdots\\ &= -c_1+\sum_{n=1}^{\infty}c_n\end{aligned}$$

So,

$$\sum_{n=1}^{\infty}(a_n+b_n+c_{n+1}) = A+B+C-c_1$$

S-14: Theorem 5.2.10 in the text, arithmetic of series, doesn't mention division, because in general it doesn't work the way the question suggests. For example, let $\{a_n\}=\{b_n\}=\frac{1}{2^n}$. Then:

- $\sum_{n=0}^{\infty}a_n = \sum_{n=0}^{\infty}b_n = \frac{1}{1-\frac{1}{2}} = 2$, while
- $\sum_{n=0}^{\infty}\dfrac{a_n}{b_n} = \sum_{n=0}^{\infty}1 = \infty$.

For the statement in the question, we can take $\{a_n\}=\{b_n\}=\frac{1}{2^n}$, $A=B=2$, $\{c_n\}=\{0,0,0,\ldots\}$, and $C=0$. We see the statement is false in this case.

So, in general, the statement given is false.

S-15: We recognize that this is a geometric series:

$$\begin{aligned}1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\cdots &= \frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}+\cdot\\ &= \sum_{n=0}^{\infty}\frac{1}{3^n}\end{aligned}$$

Using Theorem 5.2.5 in the text with $r=\dfrac{1}{3}$ and $a=1$,

$$=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}.$$

S-16: This is a geometric series, with ratio $r=\dfrac{1}{8}$. However, it doesn't start at $k=0$, which is what we're used to.

Solution 1: We write out the first few terms of the series to figure out a convenient constant to factor out.

$$\begin{aligned}\sum_{k=7}^{\infty}\frac{1}{8^k} &= \frac{1}{8^7}+\frac{1}{8^8}+\frac{1}{8^9}+\cdots\\ &= \frac{1}{8^7}\left(\frac{1}{8^0}+\frac{1}{8^1}+\frac{1}{8^2}+\cdots\right)\\ &= \sum_{k=0}^{\infty}\frac{1}{8^7}\cdot\frac{1}{8^n}\end{aligned}$$

We now evaluate the series using Theorem 5.2.5 in the text with $r=\dfrac{1}{8}$, $a=\dfrac{1}{8^7}$.

$$=\frac{1}{8^7}\cdot\frac{1}{1-\frac{1}{8}}=\frac{1}{7\times 8^6}$$

Solution 2: Using the idea of Question 10, we express the series we're interested in as the difference of two series that we can easily evaluate.

$$\sum_{k=7}^{\infty}\frac{1}{8^k}=\sum_{k=0}^{\infty}\frac{1}{8^k}-\sum_{k=0}^{6}\frac{1}{8^k}$$

Using Theorem 5.2.5 in the text,

$$\begin{aligned}&=\frac{1}{1-\frac{1}{8}}-\frac{1-\frac{1}{8^7}}{1-\frac{1}{8}}\\ &=\frac{1}{7\times 8^6}\end{aligned}$$

S-17: We recognize this as a telescoping series.

$\mathbf{k}$	$\frac{6}{k^2}$	$-\frac{6}{(k+1)^2}$	s_k
1	6	$-\frac{6}{4}$	$6-\frac{6}{4}$
2	$\frac{6}{4}$	$-\frac{6}{9}$	$6-\frac{6}{9}$
3	$\frac{6}{9}$	$-\frac{6}{16}$	$6-\frac{6}{16}$
4	$\frac{6}{16}$	$-\frac{6}{25}$	$6-\frac{6}{25}$
5	$\frac{6}{25}$	$-\frac{6}{36}$	$6-\frac{6}{36}$
6	$\frac{6}{36}$	$-\frac{6}{47}$	$6-\frac{6}{47}$
$\vdots$			

When we compute the n^{th} partial sum, i.e. the sum of of the first n terms, successive terms cancel and only the first half of the first term, $\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right)\Big|_{k=1}$, and the second half of the n^{th} term, $\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right)\Big|_{k=n}$, survive. That is:

$$s_n = \sum_{k=1}^{n}\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right) = \frac{6}{1^2}-\frac{6}{(n+1)^2}$$

Therefore, we can see directly that the sequence of partial sums $\{s_n\}$ is convergent:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(6-\frac{6}{(n+1)^2}\right) = 6$$

By Definition 5.2.3 in the text the series is also convergent, with limit 6.

S-18: We recognize that this is a telescoping series, and set up a table to find the sequence of partial sums.

n	$\cos\left(\frac{ß}{n}\right)$	$-\cos\left(\frac{ß}{n+1}\right)$	s_n
3	$\cos\left(\frac{\pi}{3}\right)$	$-\cos\left(\frac{\pi}{4}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{4}\right)$
4	$\cos\left(\frac{\pi}{4}\right)$	$-\cos\left(\frac{\pi}{5}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{5}\right)$
5	$\cos\left(\frac{\pi}{5}\right)$	$-\cos\left(\frac{\pi}{6}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{6}\right)$
6	$\cos\left(\frac{\pi}{6}\right)$	$-\cos\left(\frac{\pi}{7}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{7}\right)$
7	$\cos\left(\frac{\pi}{7}\right)$	$-\cos\left(\frac{\pi}{8}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{8}\right)$
8	$\cos\left(\frac{\pi}{8}\right)$	$-\cos\left(\frac{\pi}{9}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{9}\right)$
$\vdots$			

The Nth partial sum sees every term cancel except the first part of the first term $(\frac{1}{2})$ and the second part of the last term $(-\cos(\frac{\pi}{n+1}))$.

$$\begin{aligned} s_N &= \sum_{n=3}^{N}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right) \\ &= \cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{N+1}\right) \\ &= \frac{1}{2}-\cos\left(\frac{\pi}{N+1}\right). \end{aligned}$$

As $N \to \infty$, the argument $\frac{\pi}{N+1}$ converges to 0, and $\cos x$ is continuous at $x = 0$. By Definition 5.2.3 in the text, the value of the series is

$$\begin{aligned} \lim_{N\to\infty} s_N &= \lim_{N\to\infty}\left[\frac{1}{2}-\cos\left(\frac{\pi}{N+1}\right)\right] \\ &= \frac{1}{2}-\cos(0) = -\frac{1}{2} \end{aligned}$$

S-19: (a) As in Question 2, since

$$\begin{aligned} s_{n-1} &= a_1 + a_2 + \cdots + a_{n-1} \\ s_n &= a_1 + a_2 + \cdots + a_{n-1} + a_n \end{aligned}$$

we can find a_n by subtracting:

$$\begin{aligned}
a_n &= s_n - s_{n-1} \\
&= \frac{1+3n}{5+4n} - \frac{1+3(n-1)}{5+4(n-1)} = \frac{3n+1}{4n+5} - \frac{3n-2}{4n+1} \\
&= \frac{(3n+1)(4n+1)-(3n-2)(4n+5)}{(4n+1)(4n+5)} \\
&= \frac{11}{16n^2+24n+5}
\end{aligned}$$

(b) Using Definition 5.2.3 in the text,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1+3n}{5+4n} = \lim_{n\to\infty} \frac{1/n+3}{5/n+4} = \frac{0+3}{0+4} = \frac{3}{4}$$

The series converges to $\frac{3}{4}$.

S-20: What we have is a geometric series, but we need to get it into the proper form before we can evaluate it.

$$\sum_{n=2}^{\infty} \frac{3\cdot 4^{n+1}}{8\cdot 5^n} = \sum_{n=2}^{\infty} \frac{3\cdot 4\cdot 4^n}{8\cdot 5^n} = \frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n$$

Solution 1: If we factor our $\left(\frac{4}{5}\right)^2$, we can change our index to something more convenient.

$$\begin{aligned}
\frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n &= \frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^2\left(\frac{4}{5}\right)^{n-2} \\
&= \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^2\left(\frac{4}{5}\right)^{n}
\end{aligned}$$

We use Theorem 5.2.5 in the text with $r = \frac{4}{5}$.

$$= \frac{3}{2}\left(\frac{4}{5}\right)^2 \cdot \frac{1}{1-\frac{4}{5}} = \frac{24}{5}$$

Solution 2: Using the idea of Question 10, we view our series as a more convenient series, minus a few initial terms.

$$\begin{aligned}
\frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n &= \frac{3}{2}\left(\left[\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n\right] - \left(\frac{4}{5}\right)^1 - \left(\frac{4}{5}\right)^0\right) \\
&= \frac{3}{2}\left(\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n - \frac{9}{5}\right)
\end{aligned}$$

We use Theorem 5.2.5 in the text with $r = \frac{4}{5}$.

$$= \frac{3}{2}\left(\frac{1}{1-\frac{4}{5}} - \frac{9}{5}\right) = \frac{24}{5}$$

S-21: The number is:

$$\begin{aligned} 0.2 + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots &= \frac{1}{5} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots \\ &= \frac{1}{5} + \frac{3}{10^2}\left(\frac{1}{10^0} + \frac{1}{10^1} + \frac{1}{10^2} + \cdots\right) \\ &= \frac{1}{5} + \frac{3}{10^2}\sum_{n=0}^{\infty}\frac{1}{10^n} \end{aligned}$$

We use Theorem 5.2.5 in the text with $r = \frac{1}{10}$.

$$\begin{aligned} &= \frac{1}{5} + \frac{3}{10^2} \cdot \frac{1}{1-\frac{1}{10}} \\ &= \frac{1}{5} + \frac{1}{30} = \frac{7}{30} \end{aligned}$$

S-22: The number is:

$$\begin{aligned} 2 + \frac{65}{100} + \frac{65}{10000} + \frac{65}{1000000} + \cdots &= 2 + \frac{65}{100} + \frac{65}{100^2} + \frac{65}{100^3} + \cdots \\ &= 2 + \frac{65}{100}\sum_{n=0}^{\infty}\frac{1}{100^n} \end{aligned}$$

We use Theorem 5.2.5 in the text with $r = \frac{1}{100}$.

$$\begin{aligned} &= 2 + \frac{65}{100} \cdot \frac{1}{1-\frac{1}{100}} \\ &= 2 + \frac{65}{99} = \frac{263}{99} \end{aligned}$$

S-23: The number is:

$$\begin{aligned} 0.\overline{321} &= 0.321321321\ldots \\ &= \frac{321}{1000} + \frac{321}{10^6} + \frac{321}{10^9} + \cdots \\ &= \frac{321}{1000}\left(\left(\frac{1}{10^3}\right)^0 + \left(\frac{1}{10^3}\right)^1 + \left(\frac{1}{10^3}\right)^2 + \cdots\right) \\ &= \frac{321}{1000}\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n \end{aligned}$$

We use Theorem 5.2.5 in the text with $r = \dfrac{1}{10^3}$.

$$= \frac{321}{1000} \cdot \frac{1}{1 - \frac{1}{10^3}} = \frac{321}{999} = \frac{107}{333}$$

S-24: We split the sum into two parts.

$$\sum_{n=2}^{\infty} \left(\frac{2^{n+1}}{3^n} + \frac{1}{2n-1} - \frac{1}{2n+1} \right) = \sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n} \quad + \quad \sum_{n=2}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

The first part is a geometric series.

$$\sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=0}^{\infty} \frac{2^{n+3}}{3^{n+2}} = \sum_{n=0}^{\infty} \frac{2^3}{3^2} \cdot \left(\frac{2}{3} \right)^n$$

We use Theorem 5.2.5 in the text with $r = \dfrac{2}{3}$ and $a = \dfrac{8}{9}$.

$$= \frac{8}{9} \cdot \frac{1}{1 - \frac{2}{3}} = \frac{8}{3}$$

The second part is a telescoping series. Let's make a table to see how it cancels.

$\mathbf{n}$	$\frac{1}{2n-1}$	$-\frac{1}{2n+1}$	s_n
2	$\frac{1}{3}$	$-\frac{1}{5}$	$\frac{1}{3} - \frac{1}{5}$
3	$\frac{1}{5}$	$-\frac{1}{7}$	$\frac{1}{3} - \frac{1}{7}$
4	$\frac{1}{7}$	$-\frac{1}{9}$	$\frac{1}{3} - \frac{1}{9}$
5	$\frac{1}{9}$	$-\frac{1}{11}$	$\frac{1}{3} - \frac{1}{11}$
6	$\frac{1}{11}$	$-\frac{1}{13}$	$\frac{1}{3} - \frac{1}{13}$
7	$\frac{1}{13}$	$-\frac{1}{15}$	$\frac{1}{3} - \frac{1}{15}$
$\vdots$			

After adding terms $n = 2$ through $n = N$, the partial sum is

$$s_N = \frac{1}{3} - \frac{1}{2N+1}$$

because all the terms except the first part of the $n = 2$ term, and the last part of the $n = N$ term, cancel. Then:

$$\sum_{n=2}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right) = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{1}{3} - \frac{1}{2N+1}$$
$$= \frac{1}{3}$$

All together,

$$\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n}+\frac{1}{2n-1}-\frac{1}{2n+1}\right) = \sum_{n=2}^{\infty}\frac{2^{n+1}}{3^n} \quad + \quad \sum_{n=2}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$
$$= \frac{8}{3}+\frac{1}{3} = 3$$

S-25: We split the sum into two parts.

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n + \left(-\frac{2}{5}\right)^{n-1}\right] = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty}\left(-\frac{2}{5}\right)^{n-1}$$

Both are geometric series.

$$= \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+1} + \sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n$$
$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n$$

We use Theorem 5.2.5 in the text with $a_1 = \dfrac{1}{3}$ and $r_1 = \dfrac{1}{3}$, then with $a_2 = 1$ and $r_2 = -\dfrac{2}{5}$.

$$= \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}} + \frac{1}{1+\frac{2}{5}}$$
$$= \frac{1}{2}+\frac{5}{7} = \frac{17}{14}$$

S-26: We split the sum into two parts.

$$\sum_{n=0}^{\infty}\frac{1+3^{n+1}}{4^n} = \sum_{n=0}^{\infty}\frac{1}{4^n} + \sum_{n=0}^{\infty}\frac{3^{n+1}}{4^n}$$
$$= \sum_{n=0}^{\infty}\frac{1}{4^n} + 3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$$

Using Theorem 5.2.5 in the text,

$$\begin{aligned} &= \frac{1}{1-\frac{1}{4}} + \frac{3}{1-\frac{3}{4}} \\ &= \frac{4}{3} + 12 = \frac{40}{3} \end{aligned}$$

S-27: Using logarithm rules, we see

$$\sum_{n=5}^{\infty} \ln\left(\frac{n-3}{n}\right) = \sum_{n=5}^{\infty} \left[\ln(n-3) - \ln n\right]$$

which looks like a telescoping series. Let's make a table to figure out the partial sums.

$\mathbf{n}$	$\ln(\mathbf{n-3})$	$-\ln\mathbf{n}$	$\mathbf{s_n}$
5	$\ln 2$	$-\ln 5$	$\ln 2 - \ln 5$
6	$\ln 3$	$-\ln 6$	$\ln 2 + \ln 3 - \ln 5 - \ln 6$
7	$\ln 4$	$-\ln 7$	$\ln 2 + \ln 3 + \ln 4 - \ln 5 - \ln 6 - \ln 7$
8	$\ln 5$	$-\ln 8$	$\ln 2 + \ln 3 + \ln 4 - \ln 6 - \ln 7 - \ln 8$
9	$\ln 6$	$-\ln 9$	$\ln 2 + \ln 3 + \ln 4 - \ln 7 - \ln 8 - \ln 9$
10	$\ln 7$	$-\ln 10$	$\ln 2 + \ln 3 + \ln 4 - \ln 8 - \ln 9 - \ln 10$
11	$\ln 8$	$-\ln 11$	$\ln 2 + \ln 3 + \ln 4 - \ln 9 - \ln 10 - \ln 11$
$\vdots$			

There is a "lag" before the terms cancel, which is why they "build up" more than we saw in past examples. Still, we can clearly see the Nth partial sum:

$$\begin{aligned} \sum_{n=5}^{N} \Big(\ln(n-3) - \ln(n)\Big) &= \ln 2 + \ln 3 + \ln 4 - \ln(N-2) - \ln(N-1) - \ln(N) \\ &= \ln\left(\frac{24}{N(N-1)(N-2)}\right) \end{aligned}$$

when $N \geqslant 7$. So,

$$\begin{aligned} \sum_{n=5}^{\infty} \Big(\ln(n-3) - \ln(n)\Big) &= \lim_{N\to\infty} s_N \\ &= \lim_{N\to\infty} \ln\left(\frac{24}{N(N-1)(N-2)}\right) \\ &= -\infty \end{aligned}$$

S-28: This is a telescoping series. Let's investigate it in the usual way. We leave the fractions in the middle of the table unsimplified, to make the pattern of cancellation clearer, since terms with the same denominator cancel.

$\mathbf{n}$	$\dfrac{\mathbf{2}}{\mathbf{n}}$	$-\dfrac{\mathbf{1}}{\mathbf{n+1}}$	$-\dfrac{\mathbf{1}}{\mathbf{n-1}}$	$\mathbf{s_n}$
2	$\frac{2}{2}$	$-\frac{1}{3}$	$-\frac{1}{1}$	$-\frac{1}{3}$
3	$\frac{2}{3}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{3}-\frac{1}{4}-\frac{1}{2}$
4	$\frac{2}{4}$	$-\frac{1}{5}$	$-\frac{1}{3}$	$\frac{1}{4}-\frac{1}{5}-\frac{1}{2}$
5	$\frac{2}{5}$	$-\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{5}-\frac{1}{6}-\frac{1}{2}$
6	$\frac{2}{6}$	$-\frac{1}{7}$	$-\frac{1}{5}$	$\frac{1}{6}-\frac{1}{7}-\frac{1}{2}$
7	$\frac{2}{7}$	$-\frac{1}{8}$	$-\frac{1}{6}$	$\frac{1}{7}-\frac{1}{8}-\frac{1}{2}$
8	$\frac{2}{8}$	$-\frac{1}{9}$	$-\frac{1}{7}$	$\frac{1}{8}-\frac{1}{9}-\frac{1}{2}$
$\vdots$				

In general, there are three terms with the same denominator, and these cancel out to zero, but it takes a while to gather all three. So, some are left over in the partial sum.

$$s_N = \sum_{n=2}^{N}\left(\frac{2}{n}-\frac{1}{n+1}-\frac{1}{n-1}\right) = \frac{1}{N}-\frac{1}{N+1}-\frac{1}{2}$$

Therefore,

$$\begin{aligned}\sum_{n=2}^{\infty}\left(\frac{2}{n}-\frac{1}{n+1}-\frac{1}{n-1}\right) &= \lim_{N\to\infty} s_N\\ &= \lim_{N\to\infty}\left[\frac{1}{N}-\frac{1}{N+1}-\frac{1}{2}\right]\\ &= -\frac{1}{2}\end{aligned}$$

S-29: Before you start, make sure your remember the relevant definitions. For the PDF, $f(x) = Pr(X = x)$; for the CDF, $F(x) = Pr(X \leqslant x)$.

- If $x < 1$, then $Pr(X \leqslant x) = 0$, so $F(x) = 0$ for all x in the interval $(-\infty, 1)$.

- If x is a natural number (1, 2, 3, etc.), then:

$$F(x) = Pr(X \leqslant x) = \sum_{n=1}^{x} Pr(X = n) = \sum_{n=1}^{x} \left(\frac{1}{2}\right)^n$$

  This is a geometric partial sum. Note it starts at $n = 1$, not at $n = 0$.

$$\begin{aligned}
&= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^x \\
&= \frac{1}{2}\left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \cdots + \left(\frac{1}{2}\right)^{x-1}\right] \\
&= \frac{1}{2}\sum_{n=0}^{x-1}\left(\frac{1}{2}\right)^n = \frac{1}{2}\left(\frac{1-(1/2)^x}{1-1/2}\right) \\
&= 1 - \frac{1}{2^x}
\end{aligned}$$

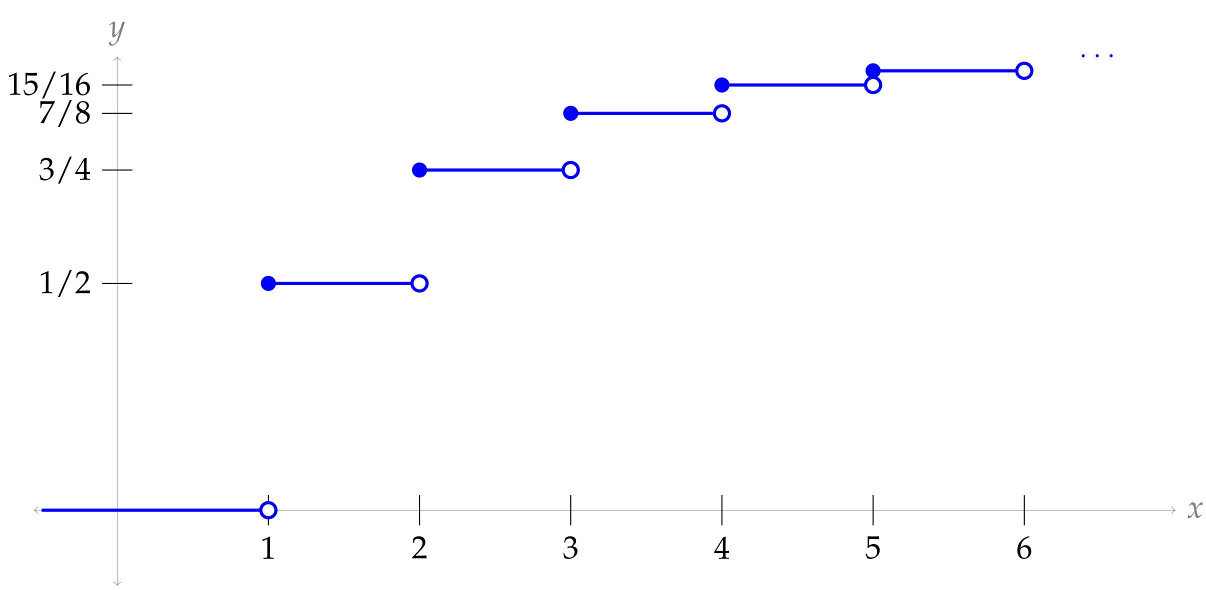
- To get a feel for what happens for positive non-integers, let's take a few samples.
  - If $X \leqslant 1.5$, then really $X = 1$, so $Pr(X \leqslant 1.5) = Pr(X \leqslant 1)$.
  - If $X \leqslant 2.5$, then really $X = 1$ or $X = 2$, so $Pr(X \leqslant 2.5) = Pr(X \leqslant 2)$.

  Helpful notation: rounding x down to the nearest integer is $\lfloor x \rfloor$. With this notation, for $x > 1$, $F(x) = Pr(X \leqslant x) = Pr(X \leqslant \lfloor x \rfloor) = F(\lfloor x \rfloor)$.

All together:

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - \frac{1}{2^{\lfloor x \rfloor}} & x \geqslant 1 \end{cases}$$

Sketched:



S-30: The volume of a sphere of radius $\dfrac{1}{\pi^n}$ is

$$v_n = \frac{4}{3}\pi\left(\frac{1}{\pi^n}\right)^3 = \frac{4\pi}{3}\left(\frac{1}{\pi^3}\right)^n$$

So, the volume of all the spheres together is:

$$\begin{aligned}\sum_{n=1}^{\infty} v_n &= \sum_{n=1}^{\infty}\frac{4\pi}{3}\left(\frac{1}{\pi^3}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{4\pi}{3}\left(\frac{1}{\pi^3}\right)^{n+1} \\ &= \sum_{n=0}^{\infty}\frac{4}{3\pi^2}\left(\frac{1}{\pi^3}\right)^n\end{aligned}$$

We use Theorem 5.2.5 in the text with $a = \dfrac{4}{3\pi^2}$ and $r = \dfrac{1}{\pi^3}$.

$$= \frac{4}{3\pi^2}\cdot\frac{1}{1-\frac{1}{\pi^3}} = \frac{4\pi}{3\left(\pi^3-1\right)}$$

S-31: Let's make a table. Keep in mind $\cos^2\theta + \sin^2\theta = 1$.

$\mathbf{n}$	$\dfrac{\sin^2 \mathbf{n}}{\mathbf{2^n}}$	$\dfrac{\cos^2(\mathbf{n}+\mathbf{1})}{\mathbf{2^{n+1}}}$	$\mathbf{s_n}$
3	$\dfrac{\sin^2 3}{2^3}$	$\dfrac{\cos^2 4}{2^4}$	$\dfrac{\sin^2 3}{2^3} + \dfrac{\cos^2 4}{2^4}$
4	$\dfrac{\sin^2 4}{2^4}$	$\dfrac{\cos^2 5}{2^5}$	$\dfrac{\sin^2 3}{2^3} + \dfrac{1}{2^4} + \dfrac{\cos^2 5}{2^5}$
5	$\dfrac{\sin^2 5}{2^5}$	$\dfrac{\cos^2 6}{2^6}$	$\dfrac{\sin^2 3}{2^3} + \dfrac{1}{2^4} + \dfrac{1}{2^5} + \dfrac{\cos^2 6}{2^6}$
6	$\dfrac{\sin^2 6}{2^6}$	$\dfrac{\cos^2 7}{2^7}$	$\dfrac{\sin^2 3}{2^3} + \dfrac{1}{2^4} + \dfrac{1}{2^5} + \dfrac{1}{2^6} + \dfrac{\cos^2 7}{2^7}$
7	$\dfrac{\sin^2 7}{2^7}$	$\dfrac{\cos^2 8}{2^8}$	$\dfrac{\sin^2 3}{2^3} + \dfrac{1}{2^4} + \dfrac{1}{2^5} + \dfrac{1}{2^6} + \dfrac{1}{2^7} + \dfrac{\cos^2 8}{2^8}$
$\vdots$			

This gives us an equation for the partial sum s_N, when $N \geqslant 4$:

$$\begin{aligned}s_N &= \sum_{n=3}^{N}\left(\frac{\sin^2 n}{2^n} + \frac{\cos^2(n+1)}{2^{n+1}}\right) \\ &= \frac{\sin^2 3}{2^3} + \left(\sum_{n=4}^{N}\frac{1}{2^n}\right) + \frac{\cos^2(N+1)}{2^{N+1}}\end{aligned}$$

Using Definition 5.2.3 in the text, our series evaluates to:

$$\begin{aligned}\lim_{N\to\infty} s_N &= \lim_{N\to\infty}\left[\frac{\sin^2 3}{2^3} + \left(\sum_{n=4}^{N}\frac{1}{2^n}\right) + \frac{\cos^2(N+1)}{2^{N+1}}\right]\\ &= \frac{\sin^2 3}{8} + \left[\lim_{N\to\infty}\frac{\cos^2(N+1)}{2^{N+1}}\right] + \sum_{n=4}^{\infty}\frac{1}{2^n}\end{aligned}$$

We evaluate the limit using the Squeeze Theorem; the series is geometric.

$$\begin{aligned}&= \frac{\sin^2 3}{8} + 0 + \sum_{n=0}^{\infty}\frac{1}{2^{n+4}}\\ &= \frac{\sin^2 3}{8} + \frac{1}{2^4}\sum_{n=0}^{\infty}\frac{1}{2^n}\end{aligned}$$

Using Theorem 5.2.5 in the text,

$$\begin{aligned}&= \frac{\sin^2 3}{8} + \frac{1}{2^4}\frac{1}{1-\frac{1}{2}}\\ &= \frac{\sin^2 3}{8} + \frac{1}{8} \approx 0.1275\end{aligned}$$

S-32:

(a)

$$\begin{aligned}i(i+1)(i+2)-(i-1)i(i+1) &= i(i^2+3i+2)-i(i^2-1)\\ &= i(i^2-i^2+3i+2+1)\\ &= i(3i+3) = 3i^2+3i\end{aligned}$$

(b) We'll start by evaluating the telescoping sum

$$\sum\big(i(i+1)(i+2)-(i-1)i(i+1)\big)$$

We can use tables, but this is actually a simpler relationship than some of the other examples we've seen.

$\mathbf{n}$	$\mathbf{i(i+1)(i+2)}$	$\mathbf{-(i-1)i(i+1)}$	$\mathbf{s_n}$
1	$1 \cdot 2 \cdot 3$	$-0 \cdot 1 \cdot 2$	$1 \cdot 2 \cdot 3$
2	$2 \cdot 3 \cdot 4$	$-1 \cdot 2 \cdot 3$	$2 \cdot 3 \cdot 4$
3	$3 \cdot 4 \cdot 5$	$-2 \cdot 3 \cdot 4$	$3 \cdot 4 \cdot 5$
4	$4 \cdot 5 \cdot 6$	$-3 \cdot 4 \cdot 5$	$4 \cdot 5 \cdot 6$
$\vdots$			
n			$n \cdot (n+1) \cdot (n+2)$

So:

$$\begin{aligned}
\sum_{i=1}^{n}(3i^2+3i) &= \sum \Big(i(i+1)(i+2)-(i-1)i(i+1)\Big) = n(n+1)(n+2) \\
3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i &= n(n+1)(n+2) \\
\sum_{i=1}^{n} i^2 &= \frac{1}{3}n(n+1)(n+2) - \sum_{i=1}^{n} \\
&= \frac{1}{3}n(n+1)(n+2) - \frac{n(n+1)}{2} \\
&= n(n+1)\left(\frac{n+2}{3}-\frac{1}{2}\right) \\
&= n(n+1)\left(\frac{2(n+2)-3}{6}\right) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

(c) First, note:

$$\begin{aligned}
i^2(i+1)^2-(i-1)^2i^2 &= i^2\Big(i^2+2i+1-(i^2-2i+1)\Big) \\
&= i^2\Big(4i\Big) = 4i^3
\end{aligned}$$

So,

$$\sum_{i=1}^{n} 4i^3 = \sum_{i=1}^{n} \left(i^2(i+1)^2 - (i-1)^2 i^2\right)$$
$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\sum_{i=1}^{n} \left(i^2(i+1)^2 - (i-1)^2 i^2\right)$$

This is a telescoping series.

$\mathbf{n}$	$\mathbf{i^2(i+1)^2}$	$\mathbf{-(i-1)^2 i^2}$	$\mathbf{s_n}$
1	$1^2 \cdot 2^2$	$-0^2 \cdot 1^2$	$1^2 \cdot 2^2$
2	$2^2 \cdot 3^2$	$-1^2 \cdot 2^2$	$2^2 \cdot 3^2$
3	$3^2 \cdot 4^2$	$-2^2 \cdot 3^2$	$3^2 \cdot 4^2$
4	$4^2 \cdot 5^2$	$-3^2 \cdot 4^2$	$4^2 \cdot 5^2$
$\vdots$			
n			$n^2 \cdot (n+1)^2$

All together,

$$\sum_{i=1}^{n} i^3 = \frac{1}{4} \cdot n^2 \cdot (n+1)^2$$

S-33: Since $\{\mathscr{S}_M\}$ is the sequence of partial sums of $\sum_{N=1}^{\infty} S_N$, we can find $\{S_N\}$ from $\{\mathscr{S}_M\}$ as in Question 3:

$$S_N = \mathscr{S}_N - \mathscr{S}_{N-1} = \frac{N+1}{N} - \frac{N}{N-1} = -\frac{1}{N(N-1)} \qquad \text{if } N \geqslant 2,$$
$$S_1 = \mathscr{S}_1 = 2$$

Similarly, we find $\{a_n\}$ from $\{S_N\}$. Do be careful: S_N only follows the formula we found above when $N \geqslant 2$. In the next line, we use an expression containing S_{n-1}; in order for the subscript to be at least two (so the formula fits), we need $n \geqslant 3$.

$$\begin{aligned}
a_n &= S_n - S_{n-1} = -\frac{1}{n(n-1)} - \frac{-1}{(n-1)(n-2)} \\
&= \frac{2}{n(n-1)(n-2)} && \text{if } n \geqslant 3, \\
a_2 &= S_2 - S_1 = -\frac{1}{2(2-1)} - 2 = -\frac{5}{2}, \\
a_1 &= S_1 = 2
\end{aligned}$$

All together,

$$a_n = \begin{cases} \frac{2}{n(n-1)(n-2)} & \text{if } n \geqslant 3, \\ -\frac{5}{2} & \text{if } n = 2, \\ 2 & \text{if } n = 1 \end{cases}$$

S-34: $\int_{-\infty}^{\infty} f(x)$ is the area under the entire curve, which (conveniently) is made up of rectangles. Since the left and right half-planes are symmetric, we'll double the area under the curve when $x \geqslant 0$ to get the entire area.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\int_{0}^{\infty} f(x)\mathrm{d}x &= 2\sum_{n=1}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n
\end{aligned}$$

This is a geometric series with $r = \frac{1}{2}$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1$$

So $f(x)$ could indeed be a PDF.

Indeed, $f(x)$ illustrates the surprising property that a PDF need *not* have $\lim\limits_{x \to -\infty} f(x) = 0$.

S-35: We don't know how to evaluate very many sums. This one isn't geometric, so let's hope it's telescoping. The telescoping series we've seen have terms that are differences; let's pull our term apart using partial fractions.

$$\begin{aligned}
\frac{2}{n(n+1)(n+2)} &= \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} = \frac{A(n+1)(n+2) + Bn(n+2) + Cn(n+1)}{n(n+1)(n+2)} \\
2 &= A(n+1)(n+2) + Bn(n+2) + Cn(n+1) \\
n = 0 \implies 2 &= A(1)(2) \implies A = 1 \\
n = -1 \implies 2 &= B(-1)(1) \implies B = -2 \\
n = -2 \implies 2 &= C(-2)(-1) \implies C = 1
\end{aligned}$$

So, we can rewrite our sum as

$$\sum_{n=1}^{1000}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)$$

which looks much more like a telescoping sum.

$$\begin{array}{cccc}
 & \mathbf{\frac{1}{n}} & \mathbf{\frac{-2}{n+1}} & \mathbf{\frac{1}{n+2}} \\
n=1 & \frac{1}{1} & -\frac{2}{2} & \frac{1}{3} \\
n=2 & \frac{1}{2} & -\frac{2}{3} & \frac{1}{4} \\
n=3 & \frac{1}{3} & -\frac{2}{4} & \frac{1}{5} \\
n=4 & \frac{1}{4} & -\frac{2}{5} & \frac{1}{6} \\
\vdots & & & \\
n=997 & \frac{1}{997} & -\frac{2}{998} & \frac{1}{999} \\
n=998 & \frac{1}{998} & -\frac{2}{999} & \frac{1}{1000} \\
n=999 & \frac{1}{999} & -\frac{2}{1000} & \frac{1}{1001} \\
n=1000 & \frac{1}{1000} & -\frac{2}{1001} & \frac{1}{1002}
\end{array}$$

Reading off the remaining terms,

$$\begin{aligned}
\sum_{n=1}^{1000}\frac{2}{n(n+1)(n+2)} &= \left(\frac{1}{1}-\frac{2}{2}+\frac{1}{2}\right)+\left(\frac{1}{1001}-\frac{2}{1001}+\frac{1}{1002}\right) \\
&= \frac{1}{2}-\frac{1}{1001}+\frac{1}{1002}
\end{aligned}$$

S-36: We consider a circle of radius R, with an "inner ring" from $\frac{R}{3}$ to $\frac{2R}{3}$ and an "outer ring" from $\frac{2R}{3}$ to R.

The area of the outer ring is:

$$\pi R^2-\pi\left(\frac{2R}{3}\right)^2=\frac{5}{9}\pi R^2$$

The area of the inner ring is:

$$\pi\left(\frac{2R}{3}\right)^2 - \pi\left(\frac{R}{3}\right)^2 = \frac{3}{9}\pi R^2$$

So, the ratio of the inner ring's area to the outer ring's area is $\frac{3}{5}$.

In our bullseye diagram, if we pair up any red ring with the blue ring just inside it, the blue ring has $\frac{3}{5}$ the area of the red ring. So, the blue portion of the bullseye has $\frac{3}{5}$ the area of the red portion.

Since the circle has area 1, if we let the red portion have area A, then

$$1 = A + \frac{3}{5}A = \frac{8}{5}A$$

So, the red portion has area $\frac{5}{8}$.

Solutions to Exercises 5.3 — Jump to TABLE OF CONTENTS

S-1:

(A) $\lim_{n\to\infty} \frac{1}{n} = 0$, so the divergence test is inconclusive. It's true that this series diverges, but we can't show it using the divergence test.

(B) $\lim_{n\to\infty} \frac{n^2}{n+1} = \infty$, which is not zero, so the divergence test tells us this series diverges.

(C) $\lim_{n\to\infty} \sin n$ does not exist, so in particular it is not zero. Therefore, the divergence test tells us this series diverges.

(D) For all whole numbers n, $\sin(\pi n) = 0$, so $\lim_{n\to\infty} \sin(\pi n) = 0$ and the divergence test is inconclusive.

S-2: Let $f(x)$ be a function with $f(n) = a_n$ for all whole numbers n. In order to apply the integral test (Theorem 5.3.5 in the text) we need $f(x)$ to be positive and decreasing for all sufficiently large values of n.

(A) $f(x) = \frac{1}{x}$, which is positive and decreasing for all $x \geqslant 1$, so the integral test does apply here.

(B) $f(x) = \frac{x^2}{x+1}$, which is not decreasing–in fact, it goes to infinity. So, the integral test does not apply here. (The divergence test tells us the series diverges, though.)

(C) $f(x) = \sin x$, which is neither consistently positive nor consistently decreasing, so the integral test does not apply. (The divergence test tells us the series diverges, though.)

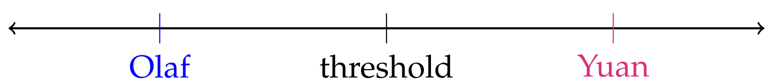
(D) $f(x) = \frac{\sin x + 1}{x^2}$ is positive for all whole numbers n. To determine whether it is decreasing, we consider its derivative.

$$f'(x) = \frac{x^2(\cos x) - (\sin x + 1)(2x)}{x^4} = \frac{x\cos x - 2\sin x - 2}{x^3}$$

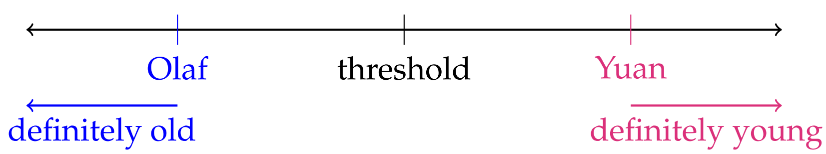
This is sometimes positive, and sometimes negative. (For example, if $x = 100\pi$, $f'(x) = \frac{100\pi - 0 - 2}{(100\pi)^3} > 0$, but if $x = 101\pi$ then $f'(x) = \frac{101\pi(-1) - 0 - 2}{(101\pi)^3} < 0$.) Then $f(x)$ is not a decreasing function, so the integral test does not apply.

S-3: (a) If Olaf is old, and I am *even older*, then I am old as well.
(b) If Olaf is old, and I am not as old, then perhaps I am old as well (just slightly less so), or perhaps I am young. There is not enough information to tell.
(c) If Yuan is young, and I am older, then perhaps I am much older and I am old, or perhaps I am only a little older, and I am young. There is not enough information to tell.
(d) If Yuan is young, and I am *even younger*, then I must also be young.

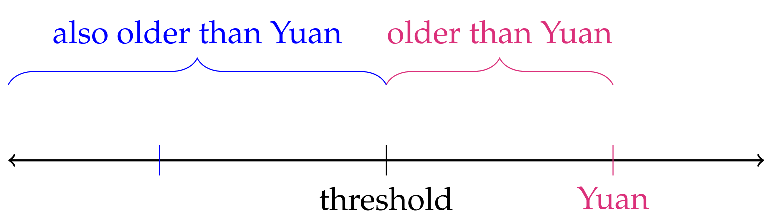
Another way to think about this is with a timeline of birthdates. People born before the threshold are old, and people born after it are young.



If I'm born before (older than) Olaf, I'm born before the threshold, so I'm old.
If I'm born after (younger than) Yuan I'm born after the threshold, so I'm young.



If I'm born after Olaf or before Yuan, I don't know which side of the threshold I'm on. I could be old or I could be young.



S-4: The comparison test is Theorem 5.3.8 in the text. However, rather than trying to memorize which way the inequalities go in all cases, we use the same reasoning as Question 3.

If a sequence has positive terms, it either converges, or it diverges to infinity, with the partial sums increasing and increasing without bound. If one sequence diverges, and the other sequence is larger, then the other sequence diverges–just like being older than an old person makes you old.

If $\sum a_n$ converges, and $\{a_n\}$ is the red (larger) series, then $\sum b_n$ converges: it's smaller than a sequence that doesn't add up to infinity, so it too does not add up to infinity.

If $\sum a_n$ diverges, and $\{a_n\}$ is the blue (smaller) series, then $\sum b_n$ diverges: it's larger than a sequence that adds up to infinity, so it too adds up to infinity.

In the other cases, we can't say anything. If $\{a_n\}$ is the red (larger) series, and $\sum a_n$ diverges, then perhaps $\{b_n\}$ behaves similarly to $\{a_n\}$ and $\sum b_n$ diverges, or perhaps $\{b_n\}$ is much, much smaller than $\{a_n\}$ and $\sum b_n$ converges.

Similarly, if $\{a_n\}$ is the blue (smaller) series, and $\sum a_n$ converges, then perhaps $\{b_n\}$ behaves similarly to $\{a_n\}$ and $\sum b_n$ converges, or perhaps $\{b_n\}$ is much, much bigger than $\{a_n\}$ and $\sum b_n$ diverges.

	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$ CONVERGES	inconclusive
and if $\{a_n\}$ is the blue series	inconclusive	then $\sum b_n$ DIVERGES

S-5: (a) Since $\sum \frac{1}{n}$ is divergent, we can only use it to prove series with *larger* terms are divergent. This is the case here, since $\frac{1}{n-1} > \frac{1}{n}$. So, the direct comparison test is valid.

For the limit comparison test, we calculate:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1}$$

Since the limit is a real number and not zero, also the limit comparison test is valid.

(b) Since the series $\sum \frac{1}{n^2}$ converges, we can only use the direct comparison test to show the convergence of a series if its terms have *smaller* absolute values. Indeed,

$$\left|\frac{\sin n}{n^2+1}\right| = \frac{|\sin n|}{n^2+1} < \frac{1}{n^2}$$

so the series are set for a direct comparison.

To check whether a limit comparison will work, we compute:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sin n}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} \sin n = \lim_{n\to\infty} (1) \sin n$$

The limit does not exist, so the limit comparison test is not a valid test to compare these two series.

(c) Since the series $\sum \frac{1}{n^3}$ converges, we can only use the direct comparison test to conclude something about a series with *smaller* terms. However,

$$\frac{n^3+5n+1}{n^6-2} > \frac{n^3}{n^6} = \frac{1}{n^3}.$$

Therefore the direct comparison test does not apply to this pair of series.

For the limit comparison test, we calculate:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\frac{n^3+5n+1}{n^6-2}}{\frac{1}{n^3}} = \lim_{n\to\infty}\frac{n^3+5n+1}{n^3-\frac{2}{n^3}}\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)\\ &= \lim_{n\to\infty}\frac{1+\frac{5}{n^2}+\frac{1}{n^3}}{1-\frac{2}{n^6}} = 1\end{aligned}$$

Since the limit is a nonzero real number, we can use the limit comparison test to compare this pair of series.

(d) Since the series $\sum \frac{1}{\sqrt[4]{n}}$ diverges, we can only use the direct comparison test to show that a series with *larger* terms diverges. However,

$$\frac{1}{\sqrt{n}} < \frac{1}{\sqrt[4]{n}}$$

so the direct comparison test isn't valid with this pair of series.

For the limit comparison test, we calculate:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt[4]{n}}} = \lim_{n\to\infty}\frac{1}{\sqrt[4]{n}} = 0$$

Since the limit is zero, the limit comparison test doesn't apply.

S-6: It diverges by the divergence test, because $\lim\limits_{n\to\infty} a_n \neq 0$.

S-7: The divergence test (Theorem 5.3.1 in the text) is inconclusive when $\lim\limits_{n\to\infty} a_n = 0$. We cannot use the divergence test to show that a series converges.

S-8: The integral test does not apply because $f(x)$ is not decreasing.

S-9: The inequality goes the wrong way, so the direct comparison test (with this comparison series) is inconclusive.

S-10: One possible answer: $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$. This series converges (it's a p–series with $p = 2 > 1$), but if we take the ratio of consecutive terms:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1$$

The limit of the ratio is 1, so the ratio test is inconclusive.

S-11: By the divergence test, for a series $\sum a_n$ to *converge*, we need $\lim\limits_{n\to\infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller. If $\lim\limits_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| < 1$ or (equivalently) $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1$, then $|a_{n+1}| > |a_n|$ for sufficiently large n, so the terms are actually *growing* in magnitude. That means the series diverges, by the divergence test.

S-12: The terms of the series only see a small portion of the domain of the integral. We can try to think of a function $f(x)$ that behaves "nicely" when x is a whole number (that is, it produces a sequence whose sum converges), but is more unruly when x is not a whole number.

For example, suppose $f(x) = \sin(\pi x)$. Then $f(x) = 0$ for every integer x, but this is not representative of the function as a whole. Indeed, our corresponding series has terms $\{a_n\} = \{0, 0, 0, \ldots\}$.

- $\displaystyle\int_1^\infty \sin(\pi x)\,\mathrm{d}x = \lim_{R\to\infty}\left[-\frac{1}{\pi}\cos(\pi x)\right]_1^R = \lim_{R\to\infty}\Big[-\cos(\pi R)\Big] - \frac{1}{\pi}$
  Since the limit does not exist, the integral diverges.
- $\displaystyle\sum_{n=1}^\infty \sin(\pi n) = \sum_{n=1}^\infty 0 = 0$. The series converges.

S-13: When n is very large, the term 2^n dominates the numerator, and the term 3^n dominates the denominator. So when n is very large $a_n \approx \frac{2^n}{3^n}$. Therefore we should take $b_n = \dfrac{2^n}{3^n}$. Note that, with this choice of b_n,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2^n+n}{3^n+1}\,\frac{3^n}{2^n} = \lim_{n\to\infty}\frac{1+n/2^n}{1+1/3^n} = 1$$

as desired.

S-14: (a) In general false. The harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges by the p–test with $p = 1$.

(b) Be careful. You were not told that the a_n's are positive. So this is false in general. If $a_n = (-1)^n\frac{1}{n}$, then $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ is again the harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$, which diverges.

(c) In general false. Take, for example, $a_n = 0$ and $b_n = 1$.

S-15: First, we'll check the divergence test. It doesn't always work, but if it does, it's likely the easiest path.

$$\lim_{n\to\infty}\frac{n^2}{3n^2+\sqrt{n}}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{1}{3+\frac{1}{n\sqrt{n}}} = \frac{1}{3} \neq 0$$

Since the limit of the terms being added is not zero, the series diverges by the divergence test.

S-16: This precise question was asked on a 2014 final exam. Note that the n^{th} term in the series is $a_n = \frac{5^k}{4^k+3^k}$ and does *not* depend on n! There are two possibilities. Either this was intentional (and the instructor was being particularly nasty) or it was a typo and the intention was to have $a_n = \frac{5^n}{4^n+3^n}$. In both cases, the limit

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{5^k}{4^k+3^k} = \frac{5^k}{4^k+3^k} \neq 0 \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{5^n}{4^n+3^n} = \lim_{n\to\infty} \frac{(5/4)^n}{1+(3/4)^n} = +\infty \neq 0
\end{aligned}$$

is nonzero, so the series diverges by the divergence test.

S-17: We usually check the divergence test first, to look for low-hanging fruit. The limit of the terms being added is zero:

$$\lim_{n\to\infty} \frac{1}{n+\frac{1}{2}} = 0$$

so the divergence test is inconclusive. That is, we need to look harder.

Next, we might consider a comparison test–these can also provide us (if we're lucky) with an easy path. The terms we're adding look somewhat like $\frac{1}{n}$, but our terms are *smaller* than these terms, which form the terms of the *divergent* harmonic series. So, a direct comparison seems unlikely. Now we search for more exotic tests.

Let $f(x) = \dfrac{1}{x+\frac{1}{2}}$. Note $f(x)$ is positive and decreases as x increases. So, by the integral test, which is Theorem 5.3.5 in the text, the given series converges if and only if the integral $\int_0^\infty \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x$ converges. Since

$$\begin{aligned}
\int_0^\infty \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x = \lim_{R\to\infty}\int_0^R \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x &= \lim_{R\to\infty}\left[\ln\left(x+\frac{1}{2}\right)\right]_{x=0}^{x=R} \\
&= \lim_{R\to\infty}\left[\ln\left(R+\frac{1}{2}\right)-\ln\frac{1}{2}\right]
\end{aligned}$$

diverges, the series diverges.

S-18: The terms of the series tend to 0, so we can't use the divergence test.

To generate a guess about its convergence, we do the following:

$$\sum \frac{1}{\sqrt{k}\sqrt{k+1}} = \sum \frac{1}{\sqrt{k^2+k}} \approx \sum \frac{1}{\sqrt{k^2}} = \sum \frac{1}{k}$$

We guess that our series behaves like the harmonic series, and the harmonic series diverges (which can be demonstrated by p-test or integral test). So, we guess that our series diverges. However, in order to directly compare our series to the harmonic series and show our series diverges, our terms would have to be *bigger* than the terms in the harmonic series, and this is not the case. So, we use limit comparison.

$$\frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2+k}}} = \frac{\sqrt{k^2+k}}{k} = \frac{\sqrt{k^2+k}}{\sqrt{k^2}} = \sqrt{\frac{k^2+k}{k^2}} = \sqrt{1+\frac{1}{k}}, \text{ so}$$
$$\lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2+k}}} = \lim_{k\to\infty} \sqrt{1+\frac{1}{k}} = 1$$

Since 1 is a real number greater than 0, by the Limit Comparison Test, $\sum \frac{1}{\sqrt{k}\sqrt{k+1}}$ diverges, like $\sum \frac{1}{k}$.

S-19: This is a geometric series with $r = 1.001$. Since $|r| > 1$, it is divergent.

S-20: This is a geometric series with $r = \frac{-1}{5}$. Since $|r| < 1$, it is convergent.

We want to use the formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, but our series does not start at 0, so we re-write it:

$$\sum_{n=3}^{\infty}\left(\frac{-1}{5}\right)^n = \sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n - \sum_{n=0}^{2}\left(\frac{-1}{5}\right)^n = \frac{1}{1-(-1/5)} - \left(1-\frac{1}{5}+\frac{1}{25}\right)$$
$$= \frac{1}{6/5} - 1 + \frac{1}{5} - \frac{1}{25} = \frac{5}{6} + \frac{-25+5-1}{25} = -\frac{1}{150}$$

S-21: For any integer n, $\sin(\pi n) = 0$, so $\sum \sin(\pi n) = \sum 0 = 0$. So, this series converges.

S-22: For any integer n, $\cos(\pi n) = \pm 1$, so $\lim\limits_{n\to\infty} \cos(\pi n) \neq 0$.
By the divergence test, this series diverges.

S-23: Factorials grow super fast. Like, wow, really fast. Even faster than exponentials. So the terms are going to zero, and the divergence test won't help us. Let's use ratio–it's a good go-to test with factorials.

$$\frac{a_{k+1}}{a_k} = \frac{\frac{e^{k+1}}{(k+1)!}}{\frac{e^k}{k!}} = \frac{e^{k+1}}{e^k}\cdot\frac{k!}{(k+1)!} = e\cdot\frac{k(k-1)\cdots(1)}{(k+1)(k)(k-1)\cdots(1)} = e\cdot\frac{1}{k+1} = \frac{e}{k+1}$$

Since e is a constant,

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{e}{k+1} = 0$$

Since $0 < 1$, by the ratio test, the series converges.

S-24: This is close to being in the form of a geometric series. First, we should have our powers be k, not $k+2$, but we notice $3^{k+2} = 3^k 3^2 = 9 \cdot 3^k$, so:

$$\sum_{k=0}^{\infty} \frac{2^k}{3^{k+2}} = \sum_{k=0}^{\infty} \frac{2^k}{9 \cdot 3^k} = \frac{1}{9} \sum_{k=0}^{\infty} \frac{2^k}{3^k} = \frac{1}{9} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

Now it looks like a geometric series with $r = \frac{2}{3}$

$$= \frac{1}{9}\left(\frac{1}{1-(2/3)}\right) = \frac{1}{3}$$

In conclusion: this (geometric) series is convergent, and its sum is $\dfrac{1}{3}$.

S-25: Usually with factorials, we want to use the divergence test or the ratio test. Since the terms are indeed tending towards zero, we are left with the ratio test.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1)!(n+1)!}{(2n+2)!}}{\frac{n!n!}{(2n)!}} = \frac{(n+1)!(n+1)!}{n!n!} \cdot \frac{(2n)!}{(2n+2)!} \\
&= \frac{(n+1)(n)(n-1)\cdots(1)}{n(n-1)\cdots(1)} \cdot \frac{(n+1)(n)(n-1)\cdots(1)}{n(n-1)\cdots(1)} \cdot \frac{(2n)(2n-1)(2n-2)\cdots(1)}{(2n+2)(2n+1)(2n)(2n-1)(2n-2)\cdots(1)} \\
&= (n+1)(n+1) \cdot \frac{1}{(2n+2)(2n+1)}, \text{ so}
\end{aligned}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{(n+1)(n+1)}{2(n+1)(2n+1)} = \lim_{n\to\infty} \frac{n+1}{4n+2} = \frac{1}{4}$$

Since the limit is a number less than 1, the series converges by the ratio test.

S-26: We want to make an estimation, when n gets big:

$$\frac{n^2+1}{2n^4+n} \approx \frac{n^2}{2n^4} = \frac{1}{2n^2}$$

Since $\sum \frac{1}{2n^2}$ is a convergent series (by p-test, or integral test), we guess that our series is convergent as well. If we wanted to use comparison test, we should have to show $\frac{n^2+1}{2n^4+n} < \frac{1}{2n^2}$, which seems unpleasant, so let's use limit comparison.

$$\lim_{n\to\infty} \frac{\frac{n^2+1}{2n^4+n}}{\frac{1}{2n^2}} = \lim_{n\to\infty} \frac{(n^2+1)2n^2}{2n^4+n} = \lim_{n\to\infty} \frac{2n^4+2n^2}{2n^4+n}\left(\frac{1/n^4}{1/n^4}\right) = \lim_{n\to\infty} \frac{2+\frac{2}{n^2}}{2+\frac{1}{n^3}} = 1$$

Since the limit is a positive finite number, by the Limit Comparison Test, $\sum \frac{n^1+1}{2n^4+n}$ does the same thing $\sum \frac{1}{2n^2}$ does: it converges.

S-27: First, we rule out some of the easier tests. The limit of the terms being added is zero, so the divergence test is inconclusive. The terms being added are smaller than the terms of the (divergent) harmonic series, $\sum \frac{1}{n}$, so we can't directly compare these two series, and there isn't another obvious series to compare ours to. However, the terms being added seem like a function we could integrate.

Let $f(x) = \dfrac{5}{x(\ln x)^{3/2}}$. Then $f(x)$ is positive and decreases as x increases. So the sum $\sum\limits_{3}^{\infty} f(n)$ and the integral $\displaystyle\int_3^\infty f(x)\,\mathrm{d}x$ either both converge or both diverge, by the integral test, which is Theorem 5.3.5 in the text. For the integral, we use the substitution $u = \ln x$, $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ to get

$$\int_3^\infty \frac{5\,\mathrm{d}x}{x(\ln x)^{3/2}} = \int_{\ln 3}^\infty \frac{5\,\mathrm{d}u}{u^{3/2}}$$

which converges by the p–test (which is Example 3.10.8 in the text) with $p = \frac{3}{2} > 1$.

S-28: Let $f(x) = \frac{1}{x(\ln x)^p}$. Then $f(x)$ is positive for $n \geqslant 3$, and $f(x)$ decreases as x increases. So, we can use the integral test, Theorem 5.3.5 in the text.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\mathrm{d}x = \lim_{R\to\infty}\int_2^R \frac{1}{(\ln x)^p}\frac{\mathrm{d}x}{x} = \lim_{R\to\infty}\int_{\ln 2}^{\ln R}\frac{1}{u^p}\mathrm{d}u \quad \text{with } u = \ln x,\, \mathrm{d}u = \frac{\mathrm{d}x}{x}$$

Using the results about p-series, Example 5.3.6 in the text, we know this integral converges if and only if $p > 1$, so the same is true for the series by the integral test.

S-29: As usual, let's see whether the "easy" tests work. The terms we're adding converge to zero:

$$\lim_{n\to\infty}\frac{e^{-\sqrt{n}}}{\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n}e^{\sqrt{n}}} = 0$$

so the divergence test is inconclusive. Our series isn't geometric, and it doesn't seem obvious how to compare it to a geometric series. However, the terms we're adding seem like they would make an integrable function.

Set $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$. For $x \geqslant 1$, this function is positive and decreasing (since it is the product of the two positive decreasing functions $e^{-\sqrt{x}}$ and $\frac{1}{\sqrt{x}}$). We use the integral test with this function. Using the substitution $u = \sqrt{x}$, so that $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$, we see that

$$\begin{aligned}
\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{R\to\infty} \int_1^R \left(\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x \right) \\
&= \lim_{R\to\infty} \left(\int_1^{\sqrt{R}} e^{-u} \cdot 2\,\mathrm{d}u \right) \\
&= \lim_{R\to\infty} \left(-2e^{-u} \Big|_1^{\sqrt{R}} \right) \\
&= \lim_{R\to\infty} \left(-2e^{-\sqrt{R}} + 2e^{-\sqrt{1}} \right) = 0 + 2e^{-1},
\end{aligned}$$

and so this improper integral converges. By the integral test, the given series also converges.

S-30: We first develop some intuition. For very large n, $3n^2$ dominates 7 so that

$$\frac{\sqrt{3n^2-7}}{n^3} \approx \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2}$$

The series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges by the p–test with $p = 2$, so we expect the given series to converge too.

To verify that our intuition is correct, it suffices to observe that

$$0 < a_n = \frac{\sqrt{3n^2-7}}{n^3} < \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2} = c_n$$

for all $n \geqslant 2$. As the series $\sum\limits_{n=2}^{\infty} c_n$ converges, the comparison test says that $\sum\limits_{n=2}^{\infty} a_n$ converges too.

S-31: We first develop some intuition. For very large k, k^4 dominates 1 so that the numerator $\sqrt[3]{k^4+1} \approx \sqrt[3]{k^4} = k^{4/3}$, and k^5 dominates 9 so that the denominator $\sqrt{k^5+9} \approx \sqrt{k^5} = k^{5/2}$ and the summand

$$\frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \approx \frac{k^{4/3}}{k^{5/2}} = \frac{1}{k^{7/6}}$$

The series $\sum\limits_{n=1}^{\infty} \dfrac{1}{k^{7/6}}$ converges by the p–test with $p = \frac{7}{6} > 1$, so we expect the given series to converge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_k = \frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \quad \text{and} \quad b_k = \frac{1}{k^{7/6}} = \frac{k^{4/3}}{k^{5/2}}$$

which is valid since

$$\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{\sqrt[3]{k^4+1}/k^{4/3}}{\sqrt{k^5+9}/k^{5/2}}=\lim_{k\to\infty}\frac{\sqrt[3]{1+1/k^4}}{\sqrt{1+9/k^5}}=1$$

exists. Since the series $\sum_{k=1}^{\infty} b_k$ is a convergent p–series (with ratio $p=\frac{7}{6}>1$), the given series converges.

Note: to apply the direct comparison test with our chosen comparison series, we would need to show that

$$\frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}}\leqslant\frac{1}{k^{7/6}}$$

for all k sufficiently large. However, this is not true: the opposite inequality holds when k is large.

S-32:

Solution 1: Let's see whether the divergence test works here.

$$\lim_{n\to\infty}\frac{n^4 2^{n/3}}{(2n+7)^4}\left(\frac{\frac{1}{n^4}}{\frac{1}{n^4}}\right)=\lim_{n\to\infty}\frac{2^{n/3}}{(2+7/n)^4}=\lim_{n\to\infty}\frac{2^{n/3}}{(2+0)^4}=\infty$$

The summands of our series do not converge to zero. By the divergence test, the series diverges.

Solution 2: Let's develop some intuition for a comparisn. For very large n, $2n$ dominates 7 so that

$$\frac{n^4 2^{n/3}}{(2n+7)^4}\approx\frac{n^4 2^{n/3}}{(2n)^4}=\frac{1}{16}2^{n/3}$$

The series $\sum_{n=1}^{\infty} 2^{n/3}$ is a geometric series with ratio $r=2^{1/3}>1$ and so diverges. (It also fails the divergence test.) We expect the given series to diverge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_n=\frac{n^4 2^{n/3}}{(2n+7)^4}\quad\text{and}\quad b_n=2^{n/3}$$

which is valid since

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^4}{(2n+7)^4}=\lim_{n\to\infty}\frac{1}{(2+7/n)^4}=\frac{1}{2^4}$$

exists and is nonzero. Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent geometric series (with ratio $r=2^{1/3}>1$), the given series diverges.

(It is possible to use the plain comparison test as well. One needs to show something like $a_n=\frac{n^4 2^{n/3}}{(2n+7)^4}\geqslant\frac{n^4 2^{n/3}}{(2n+7n)^4}=\frac{1}{9^4}b_n$.)

Solution 3: Alternately, one can apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^4 2^{(n+1)/3}/(2(n+1)+7)^4}{n^4 2^{n/3}/(2n+7)^4}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)^4(2n+7)^4}{n^4(2n+9)^4}\,\frac{2^{(n+1)/3}}{2^{n/3}} \\
&= \lim_{n\to\infty}\frac{(1+1/n)^4(2+7/n)^4}{(2+9/n)^4}\cdot 2^{1/3} = 1\cdot 2^{1/3} > 1.
\end{aligned}$$

Since the ratio of consecutive terms is greater than one, by the ratio test, the series diverges.

S-33: For large k, $k^4 \gg 2k^3 - 2$ and $k^5 \gg k^2 + k$ so

$$\frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k} \approx \frac{k^4}{k^5} = \frac{1}{k}.$$

This suggests that we apply the limit comparison test with $a_k = \frac{k^4-2k^3+2}{k^5+k^2+k}$ and $b_k = \frac{1}{k}$. Since

$$\begin{aligned}
\lim_{k\to\infty}\frac{a_k}{b_k} &= \lim_{k\to\infty}\frac{k^4-2k^3+2}{k^5+k^2+k}\cdot\frac{k}{1} = \lim_{k\to\infty}\frac{k^5-2k^4+k^2}{k^5+k^2+k} = \lim_{k\to\infty}\frac{1-2/k+1/k^3}{1+1/k^3+1/k^4} \\
&= 1
\end{aligned}$$

and since $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ diverges (by the p–test with $p = 1$), the given series diverges.

S-34: (a) For large n, $n^2 \gg n+1$ and so the numerator $n^2+n+1 \approx n^2$. For large n, $n^5 \gg n$ and so the denominator $n^5 - n \approx n^5$. So, for large n,

$$\frac{n^2+n+1}{n^5-n} \approx \frac{n^2}{n^5} = \frac{1}{n^3}.$$

This suggests that we apply the limit comparison test with $a_n = \frac{n^2+n+1}{n^5-n}$ and $b_n = \frac{1}{n^3}$. Since

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{(n^2+n+1)/(n^5-n)}{1/n^3} = \lim_{n\to\infty}\frac{n^5+n^4+n^3}{n^5-n} = \lim_{n\to\infty}\frac{1+1/n+1/n^2}{1-1/n^4} \\
&= 1
\end{aligned}$$

exists and is nonzero, and since $\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$ converges (by the p–test with $p = 3 > 1$), the given series converges.

(b) For large m, $3m \gg |\sin\sqrt{m}|$ and so

$$\frac{3m+\sin\sqrt{m}}{m^2} \approx \frac{3m}{m^2} = \frac{3}{m}.$$

This suggests that we apply the limit comparison test with $a_m = \frac{3m+\sin\sqrt{m}}{m^2}$ and $b_m = \frac{1}{m}$. (We could also use $b_m = \frac{3}{m}$.) Since

$$\begin{aligned}\lim_{m\to\infty}\frac{a_m}{b_m} &= \lim_{m\to\infty}\frac{(3m+\sin\sqrt{m})/m^2}{1/m} = \lim_{m\to\infty}\frac{3m+\sin\sqrt{m}}{m} = \lim_{m\to\infty} 3 + \frac{\sin\sqrt{m}}{m}\\ &= 3\end{aligned}$$

exists and is nonzero, and since $\sum\limits_{m=1}^{\infty}\frac{1}{m}$ diverges (by the p–test with $p=1$), the given series diverges.

S-35:

$$\begin{aligned}\sum_{n=5}^{\infty}\frac{1}{e^n} &= \sum_{n=5}^{\infty}\left(\frac{1}{e}\right)^n\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n - \sum_{n=0}^{4}\left(\frac{1}{e}\right)^n\\ &= \frac{1}{1-\frac{1}{e}} - \frac{1-\left(\frac{1}{e}\right)^5}{1-\frac{1}{e}}\\ &= \frac{\left(\frac{1}{e}\right)^5}{1-\frac{1}{e}} = \frac{1}{e^5\left(1-\frac{1}{e}\right)}\\ &= \frac{1}{e^5-e^4}\end{aligned}$$

S-36: This is a geometric series.

$$\sum_{n=2}^{\infty}\frac{6}{7^n} = \sum_{n=0}^{\infty}\frac{6}{7^{n+2}} = \sum_{n=0}^{\infty}\frac{6}{7^2}\cdot\frac{1}{7^n}$$

We use Theorem 5.2.5 in the text with $a = \frac{6}{7^2}$ and $r = \frac{1}{7}$.

$$= \frac{6}{7^2}\cdot\frac{1}{1-\frac{1}{7}} = \frac{6}{42} = \frac{1}{7}$$

S-37: (a)

Solution 1: The given series is

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots = \sum_{n=1}^{\infty} a_n \text{ with } a_n = \frac{1}{2n-1}$$

First we'll develop some intuition by observing that, for very large n, $a_n \approx \frac{1}{2n}$. We know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p–test with $p = 1$. So let's apply the limit comparison test with $b_n = \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$$

the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges. So the given series diverges.

Solution 2: The series

$$\begin{aligned} 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots &\geqslant \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots \\ &= \frac{1}{2}\Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\Big) \end{aligned}$$

The series in the brackets is the harmonic series which we know diverges, by the p–test with $p = 1$. So the series on the right hand side diverges. By the direct comparison test, the series on the left hand side diverges too.

(b) We'll use the ratio test with $a_n = \dfrac{(2n+1)}{2^{2n+1}}$. Since

$$\frac{a_{n+1}}{a_n} = \frac{(2n+3)}{2^{2n+3}}\frac{2^{2n+1}}{(2n+1)} = \frac{1}{4}\frac{(2n+3)}{(2n+1)} = \frac{1}{4}\frac{(2+3/n)}{(2+1/n)} \to \frac{1}{4} < 1 \text{ as } n \to \infty$$

the series converges.

S-38: (a) For very large k, $k \ll k^2$ so that

$$a_n = \frac{\sqrt[3]{k}}{k^2-k} \approx \frac{\sqrt[3]{k}}{k^2} = \frac{1}{k^{5/3}}.$$

We apply the limit comparison test with $b_k = \frac{1}{k^{5/3}}$. Since

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt[3]{k}/(k^2-k)}{1/k^{5/3}} = \lim_{k\to\infty} \frac{k^2}{k^2-k} = \lim_{k\to\infty} \frac{1}{1-1/k} = 1$$

exists and is nonzero, and $\sum_{k=1}^{\infty} \frac{1}{k^{5/3}}$ converges (by the p–test with $p = \frac{5}{3} > 1$), the given series converges by the limit comparison test.

(b) The k^{th} term in this series is $a_k = \frac{k^{10}10^k(k!)^2}{(2k)!}$. Factorials often work well with the ratio test, because they simplify so nicely in quotients.

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{(k+1)^{10}10^{k+1}((k+1)!)^2}{(2k+2)!} \cdot \frac{(2k)!}{k^{10}10^k(k!)^2} = 10\Big(\frac{k+1}{k}\Big)^{10}\frac{(k+1)^2}{(2k+2)(2k+1)} \\ &= 10\Big(1+\frac{1}{k}\Big)^{10}\frac{(1+1/k)^2}{(2+2/k)(2+1/k)} \end{aligned}$$

As k tends to ∞, this converges to $10 \times 1 \times \frac{1}{2\times 2} > 1$. So the series diverges by the ratio test.

(c) We'll use the integal test. The k^{th} term in the series is $a_k = \frac{1}{k(\ln k)(\ln\ln k)} = f(k)$ with $f(x) = \frac{1}{x(\ln x)(\ln\ln x)}$, which is continuous, positive and decreasing for $x \geqslant 3$.

$$\begin{aligned}\int_3^\infty f(x)\,\mathrm{d}x &= \int_3^\infty \frac{\mathrm{d}x}{x(\ln x)(\ln\ln x)} = \lim_{R\to\infty}\int_3^R \frac{\mathrm{d}x}{x(\ln x)(\ln\ln x)} \\ &= \lim_{R\to\infty}\int_{\ln 3}^{\ln R} \frac{\mathrm{d}y}{y\ln y} \qquad\qquad \text{with } y = \ln x,\ \mathrm{d}y = \frac{\mathrm{d}x}{x} \\ &= \lim_{R\to\infty}\int_{\ln\ln 3}^{\ln\ln R} \frac{\mathrm{d}t}{t} \qquad\qquad \text{with } t = \ln y,\ \mathrm{d}t = \frac{\mathrm{d}y}{y} \\ &= \lim_{R\to\infty}\Big[\ln t\Big]_{\ln\ln 3}^{\ln\ln R} = \infty\end{aligned}$$

Since the integral is divergent, the series is divergent as well by the integral test.

S-39: For large n, the numerator $n^3 - 4 \approx n^3$ and the denominator $2n^5 - 6n \approx 2n^5$, so the nth term is approximately $\frac{n^3}{2n^5} = \frac{1}{2n^2}$. So we apply the limit comparison test with $a_n = \frac{n^3-4}{2n^5-6n}$ and $b_n = \frac{1}{n^2}$. Since

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n^3-4)/(2n^5-6n)}{1/n^2} = \lim_{n\to\infty}\frac{1-\frac{4}{n^3}}{2-\frac{6}{n^4}} = \frac{1}{2}$$

exists and is nonzero, the given series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the series $\sum\limits_{n=1}^{\infty} b_n$ converges. Since the series $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series (with $p = 2$), both series converge.

S-40: (a) There are plenty of powers/factorials. So let's try the ratio test with $a_n = \frac{n^n}{9^n n!}$.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{9^{n+1}(n+1)!}\frac{9^n n!}{n^n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n\,9\,(n+1)} = \lim_{n\to\infty}\frac{(1+1/n)^n}{9} = \frac{e}{9}$$

Here we have used that $\lim\limits_{n\to\infty}(1+1/n)^n = e$. This can be taken as a definition of the number e, or you can find the limit using L'Hôpital's Rule. As $e < 9$, our series converges.

(b) We know that the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, by the p–test with $p = 2$, and also that $\ln n \geqslant 2$ for all $n \geqslant e^2$. So let's use the limit comparison test with $a_n = \frac{1}{n^{\ln n}}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{n^{\ln n}}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{1}{n^{\ln n - 2}} = 0$$

So our series converges, by the limit comparison test.

S-41: (a)

Solution 1:

- Our first task is to identify the potential sources of impropriety for this integral.
- The domain of integration extends to $+\infty$. On the domain of integration the denominator is never zero so the integrand is continuous. Thus the only problem is at $+\infty$.
- Our second task is to develop some intuition about the behavior of the integrand for very large x. When x is very large:
  - $|\sin x| \leqslant 1 \ll x$, so that the numerator $x + \sin x \approx x$, and
  - $1 \ll x^2$, so that denominator $1 + x^2 \approx x^2$, and
  - the integrand $\dfrac{x+\sin x}{1+x^2} \approx \dfrac{x}{x^2} = \dfrac{1}{x}$
- Now, since $\displaystyle\int_2^\infty \frac{dx}{x}$ diverges, we would expect $\displaystyle\int_2^\infty \frac{x+\sin x}{1+x^2}\,dx$ to diverge too.
- Our final task is to verify that our intuition is correct. To do so, we set

$$f(x) = \frac{x+\sin x}{1+x^2} \qquad\qquad g(x) = \frac{1}{x}$$

and compute

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x+\sin x}{1+x^2} \div \frac{1}{x} \\
&= \lim_{x\to\infty} \frac{(1+\sin x/x)x}{(1/x^2+1)x^2} \times x \\
&= \lim_{x\to\infty} \frac{1+\sin x/x}{1/x^2+1} \\
&= 1
\end{aligned}$$

- Since $\displaystyle\int_2^\infty g(x)\,dx = \int_2^\infty \frac{dx}{x}$ diverges, by Example 3.10.8 in the text[17], with $p = 1$, Theorem 3.10.22(b) in the text now tells us that $\displaystyle\int_2^\infty f(x)\,dx = \int_2^\infty \frac{x+\sin x}{1+x^2}\,dx$ diverges too.

Solution 2: Let's break up the integrand as $\dfrac{x+\sin x}{1+x^2} = \dfrac{x}{1+x^2} + \dfrac{\sin x}{1+x^2}$. First, we consider the integral $\displaystyle\int_2^\infty \frac{\sin x}{1+x^2}\,dx$.

- $\dfrac{|\sin x|}{1+x^2} \leqslant \dfrac{1}{1+x^2}$, so if we can show $\displaystyle\int \frac{1}{1+x^2}\,dx$ converges, we can conclude that $\displaystyle\int \frac{|\sin x|}{1+x^2}\,dx$ converges as well by the comparison test.

17 To change the lower limit of integration from 1 to 2, just apply Theorem 3.10.20 in the text.

- $\displaystyle\int_2^\infty \frac{1}{1+x^2}\,\mathrm{d}x \leqslant \int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$
- $\displaystyle\int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges (by the p–test with $p=2$)
- So the integral $\displaystyle\int_2^\infty \frac{\sin x}{1+x^2}\,\mathrm{d}x$ converges by the comparison test, and hence
- $\displaystyle\int_2^\infty \frac{\sin x}{1+x^2}\,\mathrm{d}x$ converges as well.

Therefore, $\displaystyle\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ converges if and only if $\displaystyle\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ converges. But

$$\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x = \lim_{r\to\infty}\int_2^r \frac{x}{1+x^2}\,\mathrm{d}x = \lim_{r\to\infty}\Big[\tfrac{1}{2}\ln(1+x^2)\Big]_2^r = \infty$$

diverges, so $\displaystyle\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ diverges.

(b) The problem is that $f(x) = \dfrac{x+\sin x}{1+x^2}$ is not a decreasing function. To see this, compute the derivative:

$$f'(x) = \frac{(1+\cos x)(1+x^2)-(x+\sin x)(2x)}{(1+x^2)^2} = \frac{(\cos x-1)x^2-2x\sin x+1+\cos x}{(1+x^2)^2}$$

If $x = 2m\pi$, the numerator is $0-0+1+1>0$.

Therefore, the integral test does not apply.

(c)

Solution 1: Set $a_n = \frac{n+\sin n}{1+n^2}$. We first try to develop some intuition about the behaviour of a_n for large n and then we confirm that our intuition was correct.

- *Step 1: Develop intuition.* When $n \gg 1$, the numerator $n+\sin n \approx n$, and the denominator $1+n^2 \approx n^2$ so that $a_n \approx \frac{n}{n^2} = \frac{1}{n}$ and it looks like our series should diverge by the p–test (Example 5.3.6 in the text) with $p=1$.
- *Step 2: Verify intuition.* To confirm our intuition we set $b_n = \frac{1}{n}$ and compute the limit

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n+\sin n}{1+n^2}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n[n+\sin n]}{1+n^2} = \lim_{n\to\infty}\frac{1+\frac{\sin n}{n}}{\frac{1}{n^2}+1} = 1$$

We already know that the series $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges by the p–test with $p=1$. So our series diverges by the limit comparison test, Theorem 5.3.11 in the text.

Solution 2: Since $\left|\frac{\sin n}{1+n^2}\right| \leqslant \frac{1}{n^2}$ and the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p–test with $p=2$, the series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{1+n^2}$ converges. Hence $\sum\limits_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ converges if and only if the series $\sum\limits_{n=1}^{\infty} \frac{n}{1+n^2}$ converges. Now $f(x) = \frac{x}{1+x^2}$ is a continuous, positive, decreasing function on $[1,\infty)$ since

$$f'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

is negative for all $x > 1$. We saw in part (a) that the integral $\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ diverges. So the integral $\int_1^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ diverges too and the sum $\sum\limits_{n=1}^{\infty} \frac{n}{1+n^2}$ diverges by the integral test. So $\sum\limits_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges.

S-42: Note that $\dfrac{e^{-\sqrt{x}}}{\sqrt{x}} = \dfrac{1}{\sqrt{x}e^{\sqrt{x}}}$ decreases as x increases. Hence, for every $n \geqslant 1$,

$$\frac{e^{-\sqrt{x}}}{\sqrt{x}} \geqslant \frac{e^{\sqrt{n}}}{\sqrt{n}} \qquad \text{for } x \text{ in the interval } [n-1, n]$$

$$\begin{aligned}
\text{So,} \qquad \int_{n-1}^{n} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x &\geqslant \int_{n-1}^{n} \frac{e^{-\sqrt{n}}}{\sqrt{n}}\,\mathrm{d}x \\
&= \left[\frac{e^{-\sqrt{n}}}{\sqrt{n}} x\right]_{x=n-1}^{x=n} \\
&= \frac{e^{-\sqrt{n}}}{\sqrt{n}}
\end{aligned}$$

Then, for every $N \geqslant 1$,

$$\begin{aligned}
E_N = \sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} &\leqslant \sum_{n=N+1}^{\infty} \int_{n-1}^{n} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x \\
&= \int_{N}^{N+1} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x + \int_{N+1}^{N+2} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x + \cdots \\
&= \int_{N}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x
\end{aligned}$$

Substituting $y = \sqrt{x}$, $dy = \frac{1}{2}\frac{\mathrm{d}x}{\sqrt{x}}$,

$$\int_{N}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = 2\int_{\sqrt{N}}^{\infty} e^{-y}\,dy = -2e^{-y}\Big|_{\sqrt{N}}^{\infty} = 2e^{-\sqrt{N}}$$

This shows that $\sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges and is between 0 and $2e^{-\sqrt{N}}$. Since $E_{14} = 2e^{-\sqrt{14}} = 0.047$, we may truncate the series at $n = 14$.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} &= \sum_{n=1}^{14} \frac{e^{-\sqrt{n}}}{\sqrt{n}} + E_{14} \\ &= \quad 0.3679 + 0.1719 + 0.1021 + 0.0677 + 0.0478 \\ &\quad + 0.0352 + 0.0268 + 0.0209 + 0.0166 + 0.0134 \\ &\quad + 0.0109 + 0.0090 + 0.0075 + 0.0063 + E_{14} \\ &= 0.9042 + E_{14}\end{aligned}$$

The sum is between 0.9035 and 0.9535. (This even allows for a roundoff error of 0.00005 in each term as we were calculating the partial sum.)

S-43: Let's get some intuition to guide us through a proof. Since $\sum\limits_{n=1}^{\infty} a_n$, converges a_n must converge to zero as $n \to \infty$. So, when n is quite large, $\frac{a_n}{1-a_n} \approx \frac{a_n}{1-0} = \frac{a_n}{1}$, and we know $\sum a_n$ converges. So, we want to separate the "large" indices from a finite number of smaller ones.

Since $\lim\limits_{n\to\infty} a_n = 0$, there must be[18] some integer N such that $\frac{1}{2} > a_n \geqslant 0$ for all $n > N$. Then, for $n > N$,

$$\frac{a_n}{1-a_n} \leqslant \frac{a_n}{1-1/2} = 2a_n$$

From the information in the problem statement, we know

$$\sum_{n=N+1}^{\infty} 2a_n = 2 \sum_{n=N+1}^{\infty} a_n \qquad \text{converges.}$$

So, by the direct comparison test,

$$\sum_{n=N+1}^{\infty} \frac{a_n}{1-a_n} \qquad \text{converges as well.}$$

Since the convergence of a series is not affected by its first N terms, as long as N is finite, we conclude

$$\sum_{n=1}^{\infty} \frac{a_n}{1-a_n} \qquad \text{converges.}$$

18 We could have chosen any positive number strictly less than 1, not only $\frac{1}{2}$.

S-44: By the divergence test, the fact that $\sum_{n=0}^{\infty}(1-a_n)$ converges guarantees that $\lim_{n\to\infty}(1-a_n)=0$, or equivalently, that $\lim_{n\to\infty}a_n=1$. So, by the divergence test, a second time, the fact that

$$\lim_{n\to\infty}2^n a_n=+\infty$$

guarantees that $\sum_{n=0}^{\infty}2^n a_n$ diverges too.

S-45: By the divergence test, the fact that $\sum_{n=1}^{\infty}\frac{na_n-2n+1}{n+1}$ converges guarantees that $\lim_{n\to\infty}\frac{na_n-2n+1}{n+1}=0$, or equivalently, that

$$0=\lim_{n\to\infty}\frac{n}{n+1}a_n-\lim_{n\to\infty}\frac{2n-1}{n+1}=\lim_{n\to\infty}a_n-2\iff\lim_{n\to\infty}a_n=2$$

The series of interest can be written $-\ln a_1+\sum_{n=1}^{\infty}\big[\ln(a_n)-\ln(a_{n+1})\big]$ which looks like a telescoping series. So we'll compute the partial sum

$$\begin{aligned}S_N&=-\ln a_1+\sum_{n=1}^{N}\big[\ln(a_n)-\ln(a_{n+1})\big]\\&=-\ln a_1+\big[\ln(a_1)-\ln(a_2)\big]+\big[\ln(a_2)-\ln(a_3)\big]+\cdots+\big[\ln(a_N)-\ln(a_{N+1})\big]\\&=-\ln(a_{N+1})\end{aligned}$$

and then take the limit $N\to\infty$

$$-\ln a_1+\sum_{n=1}^{\infty}\big[\ln(a_n)-\ln(a_{n+1})\big]=\lim_{N\to\infty}S_N=-\lim_{N\to\infty}\ln(a_{N+1})=-\ln 2=\ln\frac{1}{2}$$

S-46: We are told that $\sum_{n=1}^{\infty}a_n$ converges. Thus we must have that $\lim_{n\to\infty}a_n=0$. In particular, there is an index N such that $0\leqslant a_n\leqslant 1$ for all $n\geqslant N$. Then:

$$0\leqslant a_n^2\leqslant a_n\qquad\text{for } n>N$$

By the direct comparison test,

$$\sum_{n=N+1}^{\infty}a_n^2\qquad\text{converges.}$$

Since convergence doesn't depend on the first N terms of a series for any finite N,

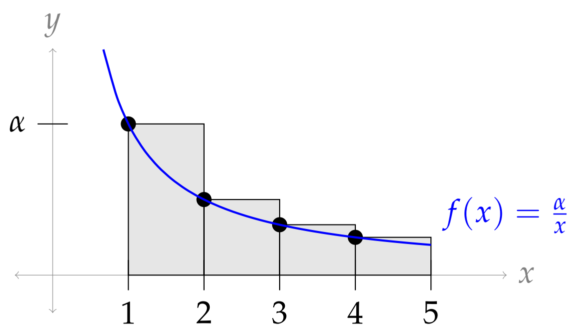
$$\sum_{n=1}^{\infty}a_n^2\qquad\text{converges as well.}$$

S-47: The most-commonly used word makes up α percent of all the words. So, we want to find α.

If we add together the frequencies of all the words, they should amount to 100%. That is,

$$\sum_{n=1}^{20,000} \frac{\alpha}{n} = 100$$

We can approximate the sum (with α left as a parameter) using the ideas behind the integral test. (See Example 5.3.4.)



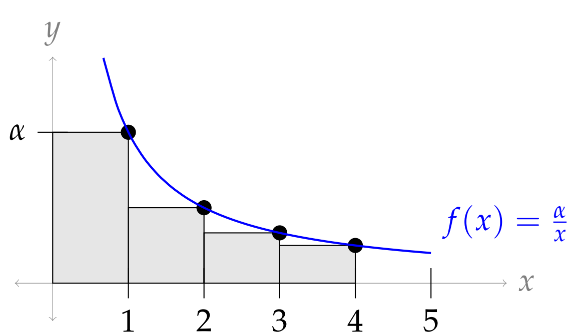
As we see in the diagram above, $\displaystyle\sum_{n=1}^{N} \frac{\alpha}{n}$ (which is the sum of the areas of the rectangles) is *greater* than $\displaystyle\int_1^{N+1} \frac{\alpha}{x}\mathrm{d}x$ (the area under the curve). That is,

$$\int_1^{N+1} \frac{\alpha}{x}\mathrm{d}x < \sum_{n=1}^{N} \frac{\alpha}{n}.$$

Using the fact that our language's 20,000 words make up 100% of the words used, we can find a lower bound for α.

$$\begin{aligned} 100 = \sum_{n=1}^{20,000} \frac{\alpha}{n} > \int_1^{20,001} \frac{\alpha}{x}\mathrm{d}x &= \Big[\alpha \ln(x)\Big]_1^{20,001} = \alpha \ln(20,001) \\ \alpha &< \frac{100}{\ln(20,001)} \end{aligned}$$

We can find an upper bound for α in a similar manner.



From the diagram, we see $\sum_{n=2}^{N} \frac{\alpha}{n}$ (which is the sum of the areas of the rectangles, excluding the first) is *less* than $\int_1^N \frac{\alpha}{x}\mathrm{d}x$. (The reason for excluding the first rectangle is to avoid comparing our series to an integral that diverges.) That is,

$$\sum_{n=2}^{N} \frac{\alpha}{n} < \int_1^N \frac{\alpha}{x}\mathrm{d}x\,.$$

Therefore,

$$\begin{aligned}
100 = \sum_{n=1}^{20{,}000} \frac{\alpha}{n} &= \alpha + \sum_{n=2}^{20{,}000} \frac{\alpha}{n} \\
&< \alpha + \int_1^{20{,}00} \frac{\alpha}{x}\mathrm{d}x = \alpha + \alpha\ln(20,000) = \alpha\left[1 + \ln(20,000)\right] \\
\alpha &> \frac{100}{1+\ln(20,000)}
\end{aligned}$$

Using a calculator, we see

$$9.17 < \alpha < 10.01$$

So, the most-commonly used word makes up about 9-10 percent of the total words.

S-48: Since no town has fewer than one person, the smallest town is the two-millionth town, with a single inhabitant. Therefore, there are 2 million towns, and the total population of the region is given by

$$\sum_{n=1}^{2\text{ mill}} \frac{2\text{ mill}}{n} = 2\times 10^6 \sum_{n=1}^{2\times 10^6} \frac{1}{n}$$

Generalizing our work in Question 47, we find the approximations:

$$\int_a^{b+1} \frac{1}{x}\mathrm{d}x < \sum_{n=a}^{b} \frac{1}{n} < \int_{a-1}^{b} \frac{1}{x}\mathrm{d}x$$

when $a \geqslant 2$.

We want our error to be less than one million, so we need to choose a value of a such that:

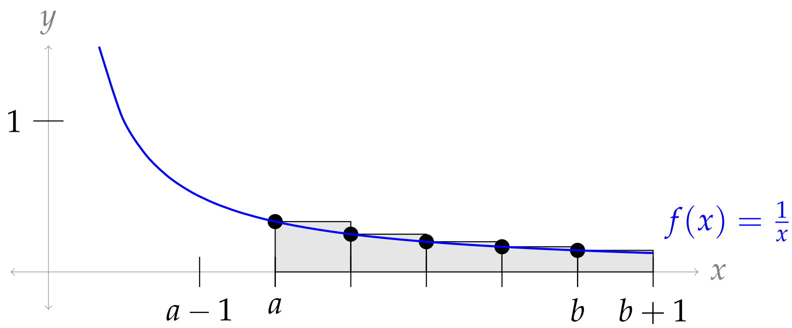


Figure 5.1: $\int\limits_{a}^{b+1} \frac{1}{x} \mathrm{d}x < \sum\limits_{n=a}^{b} \frac{1}{n}$

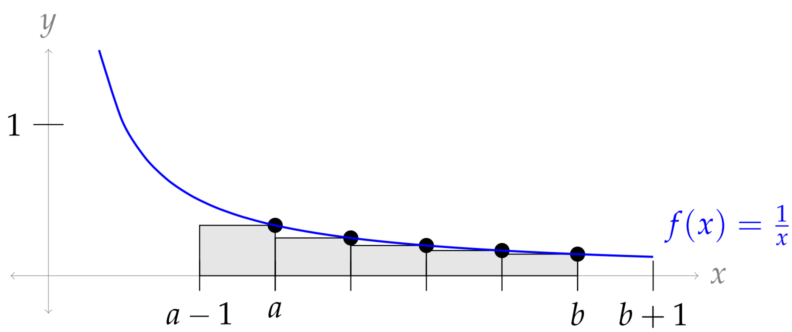


Figure 5.2: $\sum\limits_{n=a}^{b} \frac{1}{n} < \int\limits_{a-1}^{b} \frac{1}{x} \mathrm{d}x$

$$\underbrace{2 \times 10^6 \int_{a-1}^{2\times 10^6} \frac{1}{x} \mathrm{d}x}_{\text{upper bound}} - \underbrace{2 \times 10^6 \int_{a}^{2\times 10^6+1} \frac{1}{x} \mathrm{d}x}_{\text{lower bound}} < 10^6$$

$$\int_{a-1}^{2\times 10^6} \frac{1}{x} \mathrm{d}x - \int_{a}^{2\times 10^6+1} \frac{1}{x} \mathrm{d}x < \frac{1}{2}$$

$$\left[\ln\left(2 \times 10^6\right) - \ln(a-1)\right] - \left[\ln\left(2 \times 10^6 + 1\right) - \ln(a)\right] < \frac{1}{2}$$

$$\left[\ln\left(2 \times 10^6\right) - \ln\left(2 \times 10^6 + 1\right)\right] + \left[\ln(a) - \ln(a-1)\right] < \frac{1}{2}$$

$$\ln\left(\frac{2 \times 10^6}{2 \times 10^6 + 1}\right) + \ln\left(\frac{a}{a-1}\right) < \frac{1}{2}$$

The first term is extremely close to 0, so we ignore it.

$$\begin{aligned}
\ln\left(\frac{a}{a-1}\right) &< \frac{1}{2}\\
\frac{a}{a-1} &< e^{1/2}=\sqrt{e}\\
a &< a\sqrt{e}-\sqrt{e}\\
\sqrt{e} &< a(\sqrt{e}-1)\\
\frac{\sqrt{e}}{\sqrt{e}-1} &< a
\end{aligned}$$

Since $\dfrac{\sqrt{e}}{\sqrt{e}-1}\approx 2.5$, we use $a=3$. That is, we will approximate the value of $\displaystyle\sum_{n=3}^{2\times10^6}\frac{1}{n}$ using an integral. Then, we will use that approximation to estimate our total population.

$$\begin{array}{rcl}
\displaystyle\int_3^{2\times10^6+1}\frac{1}{x}\mathrm{d}x & <\displaystyle\sum_{n=3}^{2\times10^6}\frac{1}{n} < & \displaystyle\int_4^{2\times10^6}\frac{1}{x}\mathrm{d}x\\
\ln\left(2\times10^6+1\right)-\ln(3) & <\displaystyle\sum_{n=3}^{2\times10^6}\frac{1}{n} < & \ln\left(2\times10^6\right)-\ln(4)\\
1+\frac{1}{2}+\ln\left(2\times10^6+1\right)-\ln(3) & <\displaystyle\sum_{n=1}^{2\times10^6}\frac{1}{n} < & 1+\frac{1}{2}+\ln\left(2\times10^6\right)-\ln(4)\\
\frac{3}{2}+\ln\left(\frac{2\times10^6+1}{3}\right) & <\displaystyle\sum_{n=1}^{2\times10^6}\frac{1}{n} < & \frac{3}{2}+6\ln(10)-\ln(2)\\
2\times10^6\left(\frac{3}{2}+\ln\left(\frac{2}{3}\times10^6+\frac{1}{3}\right)\right) & <\displaystyle\sum_{n=1}^{2\times10^6}\frac{2\times10^6}{n} < & 2\times10^6\left(\frac{3}{2}+6\ln(10)-\ln(2)\right)\\
29,820,091 & <\text{population} < & 29,244,727
\end{array}$$

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S-1: False. For example if $b_n=\frac{1}{n}$, then $\sum\limits_{n=1}^{\infty}(-1)^{n+1}b_n=\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges by the alternating series test, but $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges by the p–test.

Remark: if we had added that $\{b_n\}$ is a sequence of *alternating terms*, then by Theorem 5.4.2 in the text, the statement would have been true. This is because $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ would either be equal to $\displaystyle\sum_{n=1}^{\infty}|b_n|$ or $\displaystyle-\sum_{n=1}^{\infty}|b_n|$.

S-2: Absolute convergence describes the situation where $\sum |a_n|$ converges (see Definition 5.4.1 in the text). By Theorem 5.4.2 in the text, this guarantees that also $\sum a_n$ converges.

Conditional convergence describes the situation where $\sum |a_n|$ diverges but $\sum a_n$ converges (see again Definition 5.4.1 in the text).

If $\sum a_n$ diverges, we just say it diverges. The reason is that if $\sum a_n$ diverges, we automatically know $\sum |a_n|$ diverges as well, so there's no need for a special name.

	$\sum a_n$ **converges**	$\sum a_n$ **diverges**
$\sum \lvert a_n\rvert$ **converges**	converges absolutely	not possible
$\sum \lvert a_n\rvert$ **diverges**	converges conditionally	diverges

S-3: The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{9n+5}$ converges by the alternating series test. On the other hand the series $\sum\limits_{n=1}^{\infty} \left|\frac{(-1)^n}{9n+5}\right| = \sum_{n=1}^{\infty} \frac{1}{9n+5}$ diverges by the limit comparison test with $b_n = \frac{1}{n}$. So the given series is conditionally convergent.

S-4: Note that $(-1)^{2n+1} = (-1) \cdot (-1)^{2n} = -1$. So we can simplify

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{1+n} = -\sum_{n=1}^{\infty} \frac{1}{1+n}$$

Since $\dfrac{1}{1+n} \geqslant \dfrac{1}{n+n} = \dfrac{1}{2n}$, $\sum\limits_{n=1}^{\infty} \dfrac{1}{1+n}$ diverges by the comparison test with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$. The extra overall factor of -1 in the original series does not change the conclusion of divergence.

S-5: Since

$$\lim_{n\to\infty} \frac{1+4^n}{3+2^{2n}} = \lim_{n\to\infty} \frac{1+4^n}{3+4^n} = 1$$

the alternating series test cannot be used. Indeed, $\lim\limits_{n\to\infty} (-1)^{n-1} \dfrac{1+4^n}{3+2^{2n}}$ does not exist (for very large n, $(-1)^{n-1}\frac{1+4^n}{3+2^{2n}}$ alternates between a number close to $+1$ and a number close to -1) so the divergence test says that the series diverges. (Note that "none of the above" cannot possibly be the correct answer — *every* series either converges absolutely, converges conditionally, or diverges.)

S-6: First, we'll develop some intuition. For very large n

$$\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right| \approx \left|\frac{\sqrt{n}\cos(n)}{n^2}\right| = \left|\frac{\cos(n)}{n^{3/2}}\right| \leqslant \frac{1}{n^{3/2}}$$

since $|\cos(n)| \leqslant 1$ for all n. By the p–test, the series $\sum\limits_{n=5}^{\infty} \frac{1}{n^p}$ converges for all $p > 1$. So we would expect the given series to converge absolutely.

Now, to confirm that our intuition is correct, we'll first try setting $a_n = \left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|$ and $b_n = \frac{1}{n^{3/2}}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\sqrt{n}\cdot\sqrt{n}^3|\cos n|}{n^2-1} \\
&= \lim_{n\to\infty} \frac{n^2|\cos n|}{n^2-1} = \lim_{n\to\infty} \left(\frac{n^2}{n^2-1}\right)|\cos n| \\
&= \lim_{n\to\infty} 1\cdot|\cos n|
\end{aligned}$$

Unfortunately, this limit doesn't exist, so we can't use the limit comparison theorem. We'll need a slightly different tactic.

Solution 1: Let's try the limit comparison test with a different p–series. The question is, which one. Let's start by leaving p as a variable, then see what happens.

We apply the limit comparison test with $a_n = \left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|$ and $b_n = \frac{1}{n^p}$. We'll choose a specific p shortly.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\left|\sqrt{n}\ \cos n/(n^2-1)\right|}{1/n^p} \\
&= \lim_{n\to\infty} \frac{n^{p+0.5}|\cos n|}{n^2(1-1/n^2)} = \lim_{n\to\infty} \frac{|\cos n|}{n^{(3/2-p)}(1-1/n^2)} \\
&= 0 \quad \text{if } p < \frac{3}{2}
\end{aligned}$$

the limit comparison test says that if $p < \frac{3}{2}$ and the series $\sum\limits_{n=5}^{\infty} b_n$ converges (which is the case if $p > 1$) then the series $\sum\limits_{n=5}^{\infty} \left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|$ also converges. So choosing any $1 < p < \frac{3}{2}$, for example $p = \frac{5}{4}$, we conclude that the given series converges absolutely.

Solution 2: Let's try to use the direct comparison test. When we were trying to develop intuition, we noticed the following:

$$\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right| \approx \left|\frac{\sqrt{n}\cos(n)}{n^2}\right| = \left|\frac{\cos(n)}{n^{3/2}}\right| \leqslant \frac{1}{n^{3/2}}$$

It's not the case that our terms are less than $\frac{1}{n^{3/2}}$, but perhaps they would be less than, say $\frac{2}{n^{3/2}}$. Let's trace our reasoning above backwards.

$$\frac{2}{n^{3/2}} \geqslant \left|\frac{2\cos n}{n^{3/2}}\right| = \left|\frac{\sqrt{n}\cos n}{(1/2)n^2}\right| \geqslant \left|\frac{\sqrt{n}\cos n}{n^2-1}\right|$$

where the final inequality holds for all $n \geqslant 2$.

Since $\sum\limits_{n=1}^{\infty} \frac{2}{n^{3/2}}$ converges by the p-test, our series converges as well by the direct comparison test.

S-7: We first develop some intuition about $\sum\limits_{n=1}^{\infty}\left|\dfrac{n^2-\sin n}{n^6+n^2}\right|$, where we take the absolute value of the summands to consider whether the series converges *absolutely*. For very large n, n^2 dominates $\sin n$ and n^6 dominates n^2 so that

$$\left|\frac{n^2-\sin n}{n^6+n^2}\right| \approx \frac{n^2}{n^6} = \frac{1}{n^4}$$

The series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^4}$ converges by the p–test with $p=4>1$. We expect the given series to converge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_n = \frac{n^2-\sin n}{n^6+n^2} \quad \text{and} \quad b_n = \frac{1}{n^4}$$

which is valid since

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left|\frac{(n^2-\sin n)}{n^6+n^2}\right|\cdot\frac{n^4}{1} = \lim_{n\to\infty}\frac{|n^6-n^4\sin n|}{n^6+n^2} = \lim_{n\to\infty}\frac{1-n^{-2}\sin n}{1+n^{-4}} = 1$$

exists and is nonzero. Since the series $\sum\limits_{n=1}^{\infty} b_n$ converges, the series $\sum\limits_{n=1}^{\infty}\dfrac{|n^2-\sin n|}{n^6+n^2}$ converges too. Therefore, the series $\sum\limits_{n=1}^{\infty}\dfrac{n^2-\sin n}{n^6+n^2}$ converges absolutely.

S-8: You might think that this series converges by the alternating series test. But you would be wrong. The problem is that $\{a_n\}$ does not converge to zero as $n\to\infty$, so that the series actually diverges by the divergence test. To verify that the n^{th} term does not converge to zero as $n\to\infty$ let's write $a_n = \frac{(2n)!}{(n^2+1)(n!)^2}$ (i.e. a_n is the n^{th} term without the sign) and check to see whether a_{n+1} is bigger than or smaller than a_n.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{((n+1)^2+1)((n+1)!)^2}\frac{(n^2+1)(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2}\frac{n^2+1}{(n+1)^2+1} \\ &= \frac{2(2n+1)}{(n+1)}\frac{1+1/n^2}{(1+1/n)^2+1/n^2} = 4\frac{1+1/2n}{1+1/n}\frac{1+1/n^2}{(1+1/n)^2+1/n^2}\end{aligned}$$

So

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4$$

and, in particular, for large n, $a_{n+1} > a_n$. Thus, for large n, a_n increases with n and so cannot converge to 0. So the series diverges by the divergence test.

S-9: This series converges by the alternating series test. We want to know whether it converges absolutely, so we consider the seris $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n(\ln n)^{101}}\right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{101}}$.

We've seen similar function before (e.g. Example 5.3.7 in the text, with $p = 101 > 1$) and it yields nicely to the integral test. Let $f(x) = \frac{1}{x(\ln x)^{101}}$. Note $f(x)$ is positive and decreasing for $n \geqslant 3$. Then by the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{101}}$ converges if and only if the integral $\int_2^{\infty} \frac{1}{x(\ln x)^{101}} \mathrm{d}x$ does. We evaluate the integral using the substitution $u = \ln x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^{101}} \mathrm{d}x &= \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^{101}} \mathrm{d}x \\
&= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^{101}}\,\mathrm{d}u \\
&= \lim_{b\to\infty} \left[\frac{-1}{100u^{100}}\right]_{\ln 2}^{\ln b} \\
&= \frac{1}{100(\ln 2)^{100}}
\end{aligned}$$

Since the integral converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{101}}$ converges, and therefore the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^{101}}$ converges absolutely.

S-10: The sequence has some positive terms and some negative terms, which limits the tests we can use. However, if we consider the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$, we can use the direct comparison test.

For every n, $|\sin n| < 1$, so $0 \leqslant \left|\dfrac{\sin n}{n^2}\right| < \dfrac{1}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by the direct comparison test, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ converges as well. Then $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely– in particular, it converges.

S-11: The terms of this series are sometimes negative (for odd values of n where $\sin n < \frac{1}{2}$) and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.

$$\begin{aligned}
-\frac{1}{4} \leqslant \frac{\sin n}{4} &\leqslant \frac{1}{4} \\
\Rightarrow \qquad \left(-\frac{1}{4}-\frac{1}{8}\right) \leqslant \left(\frac{\sin x}{4}-\frac{1}{8}\right) &< \left(\frac{1}{4}-\frac{1}{8}\right) \\
\Rightarrow \qquad -\frac{3}{8} \leqslant \left(\frac{\sin x}{4}-\frac{1}{8}\right) &< \frac{1}{8} \\
\Rightarrow \qquad 0 \leqslant \left|\frac{\sin x}{4}-\frac{1}{8}\right| &< \frac{3}{8} \\
\Rightarrow \qquad 0 \leqslant \left|\left(\frac{\sin x}{4}-\frac{1}{8}\right)^n\right| &< \left(\frac{3}{8}\right)^n
\end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{8}\right)^n$ converges (it's a geometric sum with $|r|<1$), by the direct comparison test, $\displaystyle\sum_{n=1}^{\infty}\left|\left(\frac{\sin x}{4}-\frac{1}{8}\right)^n\right|$ converges as well.

Then $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sin x}{4}-\frac{1}{8}\right)^n$ converges absolutely–and so it converges.

S-12: The terms of this series are sometimes negative and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.

$$\begin{aligned}
0 &\leqslant \sin^2 n \leqslant 1 \\
0 &\leqslant \cos^2 n \leqslant 1 \\
\text{So,} \qquad -1 &\leqslant \sin^2 n - \cos^2 n \leqslant 1 \\
-\frac{1}{2} = \left(-1+\frac{1}{2}\right) &\leqslant \left(\sin^2 n - \cos^2 n + \frac{1}{2}\right) \leqslant \left(1+\frac{1}{2}\right) = \frac{3}{2} \\
-\frac{1}{2}\cdot\frac{1}{2^n} &\leqslant \frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n} \leqslant \frac{3}{2}\cdot\frac{1}{2^n} \\
0 &\leqslant \left|\frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n}\right| \leqslant \frac{3}{2^{n+1}}
\end{aligned}$$

The series $\displaystyle\sum_{n=1}^{\infty}\frac{3}{2^{n+1}}$ converges, because it's a geometric series with $r=\frac{1}{2}$. By the direct comparison test, $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n}\right|$ converges as well. Then $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n}$

converges absolutely, so it converges.

S-13: (a)

Solution 1: We need to show that $\sum\limits_{n=1}^{\infty} 24n^2e^{-n^3}$ converges. If we replace n by x in the summand, we get $f(x) = 24x^2e^{-x^3}$, which we can integate. (Just substitute $u = x^3$.) So let's try the integral test. First, we have to check that $f(x)$ is positive and decreasing. It is certainly positive. To determine if it is decreasing, we compute

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 48xe^{-x^3} - 24 \times 3x^4e^{-x^3} = 24x(2 - 3x^3)e^{-x^3}$$

which is negative for $x \geqslant 1$. Therefore $f(x)$ is decreasing for $x \geqslant 1$, and the integral test applies. The substitution $u = x^3$, $\mathrm{d}u = 3x^2\ \mathrm{d}x$, yields

$$\int f(x)\,\mathrm{d}x = \int 24x^2e^{-x^3}\,\mathrm{d}x = \int 8e^{-u}\,\mathrm{d}u = -8e^{-u} + C = -8e^{-x^3} + C.$$

Therefore

$$\begin{aligned}\int_1^{\infty} f(x)\,\mathrm{d}x &= \lim_{R\to\infty}\int_1^R f(x)\,\mathrm{d}x = \lim_{R\to\infty}\Big[-8e^{-x^3}\Big]_1^R \\ &= \lim_{R\to\infty}\left(-8e^{-R^3} + 8e^{-1}\right) = 8e^{-1}\end{aligned}$$

Since the integral is convergent, the series $\sum\limits_{n=1}^{\infty} 24n^2e^{-n^3}$ converges and the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}24n^2e^{-n^3}$ converges absolutely.

Solution 2: Alternatively, we can use the ratio test with $a_n = 24n^2e^{-n^3}$. We calculate

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{24(n+1)^2e^{-(n+1)^3}}{24n^2e^{-n^3}}\right| \\ &= \lim_{n\to\infty}\left(\frac{(n+1)^2}{n^2}\frac{e^{n^3}}{e^{(n+1)^3}}\right) \\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 e^{-(3n^2+3n+1)} = 1\cdot 0 = 0 < 1,\end{aligned}$$

and therefore the series converges absolutely.

Solution 3: Alternatively, alternatively, we can use the limiting comparison test. First a little intuition building. Recall that we need to show that $\sum\limits_{n=1}^{\infty} 24n^2e^{-n^3}$ converges. The n^{th} term in this series is

$$a_n = 24n^2e^{-n^3} = \frac{24n^2}{e^{n^3}}$$

It is a ratio with both the numerator and denominator growing with n. A good rule of thumb is that exponentials grow a lot faster than powers. For example, if $n = 10$ the numerator is $2400 = 2.4 \times 10^3$ and the denominator is about 2×10^{434}. So we would guess that a_n tends to zero as $n \to \infty$. The question is "does a_n tend to zero fast enough with n that our series converges?". For example, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by the p–test with $p = 2$). So if a_n tends to zero faster than $\frac{1}{n^2}$ does, our series will converge. So let's try the limiting convergence test with $a_n = 24n^2e^{-n^3} = \frac{24n^2}{e^{n^3}}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{24n^2e^{-n^3}}{1/n^2} = \lim_{n\to\infty} \frac{24n^4}{e^{n^3}}$$

By l'Hôpital's rule, twice,

$$\begin{aligned}
\lim_{x\to\infty} \frac{24x^4}{e^{x^3}} &= \lim_{x\to\infty} \frac{4 \times 24x^3}{3x^2e^{x^3}} && \text{by l'Hôpital}\\
&= \lim_{x\to\infty} \frac{32x}{e^{x^3}} && \text{just cleaning up}\\
&= \lim_{x\to\infty} \frac{32}{3x^2e^{x^3}} && \text{by l'Hôpital, again}\\
&= 0
\end{aligned}$$

That's it. The limit comparison test now tells us that $\sum_{n=1}^{\infty} a_n$ converges.

(b) In part (a) we saw that $24n^2e^{-n^3}$ is positive and decreasing. The limit of this sequence equals 0 (as can be shown with l'Hôpital's Rule, just as we did at the end of the third solution of part (a)). Therefore, we can use the alternating series test, so that the error made in approximating the infinite sum $S = \sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^{n-1}24n^2e^{-n^3}$ by the sum of its first N terms, $S_N = \sum\limits_{n=1}^{N} a_n$, lies between 0 and the first omitted term, a_{N+1}. If we use 5 terms, the error satisfies

$$|S - S_5| \leqslant |a_6| = 24 \times 36e^{-6^3} \approx 1.3 \times 10^{-91}$$

S-14: The error in our approximation using through term N is at most $\frac{1}{(2(N+1))!}$. We want $\frac{1}{(2(N+1))!} < \frac{1}{1000}$. By checking small values of N, we see that $8! = 40320 > 1000$, so if $N = 3$, then $\frac{1}{2(N+1)!} = \frac{1}{40320} < \frac{1}{1000}$. So, for our approximation, it suffices to consider the first four terms of our series.

$$\begin{aligned}
\cos(2) \approx \sum_{N=0}^{3} \frac{(-1)^n}{(2n)!} &= \frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}\\
&= 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}\\
&= \frac{720 - 360 + 30 - 1}{720} = \frac{379}{720}
\end{aligned}$$

When we use a calculator, we see

$$\frac{389}{720} = 0.540\overline{27}$$
$$\cos(1) \approx 0.540302$$
$$\cos(1) - \frac{389}{720} \approx 0.000024528 \approx \frac{1}{40770}$$

So, our error is reasonably close to our bound of $\frac{1}{40320}$, and far smaller than $\frac{1}{1000}$.

S-15: The terms of this series are sometimes negative and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.

If n is prime, then

$$\left|\frac{a_n}{e^n}\right| = \left|-\frac{e^{n/2}}{e^n}\right| = \frac{1}{e^{n/2}} = \left(\frac{1}{\sqrt{e}}\right)^n$$

If n is not prime, then

$$\left|\frac{a_n}{e^n}\right| = \left|-\frac{n^2}{e^n}\right| = \frac{n^2}{e^n}$$

For n sufficiently large, $n^2 < e^{n/2}$, so for n sufficiently large,

$$\frac{n^2}{e^n} \leqslant \left(\frac{1}{\sqrt{e}}\right)^n.$$

Since $e > 1$, then $\sqrt{e} > 1$, so the geometric series $\sum \left(\frac{1}{\sqrt{e}}\right)^n$ has $|r| = r = \frac{1}{\sqrt{e}} < 1$, so it converges. By the direct comparison test, $\sum\limits_{n=1}^{\infty} \left|\frac{a_n}{e^n}\right|$ converges as well. Then $\sum\limits_{n=1}^{\infty} \frac{a_n}{e^n}$ converges absolutely, so it converges.

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S-1:

$$f(1) = \sum_{n=0}^{\infty} \left(\frac{3-1}{4}\right)^n$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

This is a geometric series with $r = \frac{1}{2}$.

$$= \frac{1}{1-\frac{1}{2}} = 2$$

S-2: Following Theorem 5.5.12, we differentiate our function term-by-term.

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{(x-5)^n}{n!+2} \\ f'(x) &= \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \left\{ \frac{(x-5)^n}{n!+2} \right\} \\ &= \sum_{n=1}^{\infty} \frac{n(x-5)^{n-1}}{n!+2} \end{aligned}$$

Keep in mind that x is our variable, and *for each term*, n is constant.

S-3: If $x = c$, then

$$\begin{aligned} f(x) &= A_a(c-c)^a + A_{a+1}(c-c)^{a+1} + A_{a+2}(c-c)^{a+2} + \cdots \\ &= A_a \cdot 0 + A_{a+1} \cdot 0 + A_{a+2} \cdot 0 + \cdots \\ &= 0 \end{aligned}$$

So, $f(x)$ converges (to the constant 0) when $x = c$. (Had we allowed $a = 0$, it would be possible for $f(x)$ to converge to a nonzero number A_0, because we use the convention $0^0 = 1$.)

Depending on the sequence $\{A_n\}$, it's possible that $f(x)$ diverges for all $x \neq c$. For example, suppose $A_n = n!$, so $f(x) = \sum_{n=0}^{\infty} n!(x-c)^n$. If $x \neq c$, then the limit $\lim_{n\to\infty} \left| \frac{(n+1)!(x-c)^{n+1}}{n!(x-c)^n} \right| = \lim_{n\to\infty} (n+1)|x-c|$ is infinity, since $x - c \neq 0$. So, the series diverges.

We've now shown that the series definitely converges at $x = c$, but at any other point, it may fail to converge.

S-4: According to Theorem 5.5.9 in the text, because $f(x)$ diverges somewhere, and because it converges at a point other than its centre, $f(x)$ has a positive radius of convergence R. That is, $f(x)$ converges whenever $|x-5| < R$, and it diverges whenever $|x-5| > R$.

If $R > 6$, then $|11-5| < R$, so $f(x)$ converges at $x = 11$; since we are told $f(x)$ *diverges* at $x = 11$, we see $R \leqslant 6$.

If $R < 6$, then $|-1-5| > R$, so $f(x)$ diverges at $x = -1$; since we are told $f(x)$ *converges* at $x = -1$, we see $R \geqslant 6$.

Therefore, $R = 6$.

S-5: (a) We apply the ratio test for the series whose k^{th} term is $a_k = (-1)^k 2^{k+1} x^k$. Then

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{k+2}x^{k+1}}{(-1)^k 2^{k+1}x^k}\right| \\ &= \lim_{k\to\infty}|2x| = |2x|\end{aligned}$$

Therefore, by the ratio test, the series converges for all x obeying $|2x| < 1$, i.e. $|x| < \frac{1}{2}$, and diverges for all x obeying $|2x| > 1$, i.e. $|x| > \frac{1}{2}$. So the radius of convergence is $R = \frac{1}{2}$.

Alternatively, one can set $A_k = (-1)^k 2^{k+1}$ and compute

$$A = \lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{k+2}}{(-1)^k 2^{k+1}}\right| = \lim_{k\to\infty} 2 = 2$$

so that $R = \frac{1}{A} = \frac{1}{2}$, again.

(b) The series is

$$\sum_{k=0}^{\infty}(-1)^k 2^{k+1}x^k = 2\sum_{k=0}^{\infty}(-2x)^k = 2\sum_{k=0}^{\infty} r^k\Big|_{r=-2x} = 2\times\frac{1}{1-r} = \frac{2}{1+2x}$$

for all $|r| = |2x| < 1$, i.e. all $|x| < \frac{1}{2}$.

S-6: We apply the ratio test for the series whose k^{th} term is $a_k = \frac{x^k}{10^{k+1}(k+1)!}$. Then

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{x^{k+1}}{10^{k+2}(k+2)!}\cdot\frac{10^{k+1}(k+1)!}{x^k}\right| \\ &= \lim_{k\to\infty}\left|\frac{10^{k+1}}{10^{k+2}}\right|\cdot\left|\frac{(k+1)!}{(k+2)!}\right|\cdot\left|\frac{x^{k+1}}{x^k}\right| \\ &= \lim_{k\to\infty}\frac{1}{10(k+2)}|x| = 0 < 1\end{aligned}$$

for all x. Therefore, by the ratio test, the series converges for all x and the radius of convergence is $R = \infty$.

Alternatively, one can set $A_k = \dfrac{1}{10^{k+1}(k+1)!}$ and compute $A = \lim\limits_{k\to\infty}\left|\dfrac{A_{k+1}}{A_k}\right| = 0$, so that R is again $+\infty$.

S-7: We apply the ratio test with $a_n = \frac{(x-2)^n}{n^2+1}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(x-2)^n}\right| \\
&= \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1}|x-2| \\
&= \lim_{n\to\infty}\frac{1+1/n^2}{(1+1/n)^2+1/n^2}|x-2| \\
&= |x-2|
\end{aligned}$$

So, the series converges if $|x-2|<1$ and diverges if $|x-2|>1$. That is, the radius of convergence is 1.

S-8: We apply the ratio test for the series whose n^{th} term is $a_n = \dfrac{(-1)^n(x+2)^n}{\sqrt{n}}$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{\sqrt{n+1}}\ \frac{\sqrt{n}}{(x+2)^n}\right| \\
&= \lim_{n\to\infty}|x+2|\ \frac{\sqrt{n}}{\sqrt{n+1}} \\
&= \lim_{n\to\infty}|x+2|\ \frac{1}{\sqrt{1+1/n}} \\
&= |x+2|
\end{aligned}$$

So the series must converge when $|x+2|<1$ and must diverge when $|x+2|>1$. So, its radius of convergence is 1.

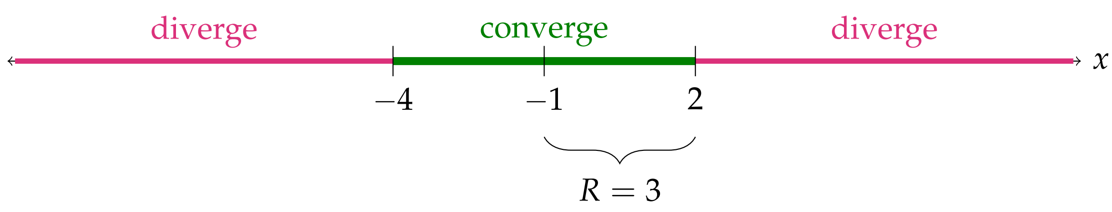
S-9: We apply the ratio test for the series whose n^{th} term is $a_n = \frac{(-1)^n}{n+1}\left(\frac{x+1}{3}\right)^n$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n+2}\left(\frac{x+1}{3}\right)^{n+1}}{\frac{(-1)^n}{n+1}\left(\frac{x+1}{3}\right)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\right|\cdot\left|\frac{n+1}{n+2}\right|\cdot\left|\frac{(x+1)^{n+1}}{(x+1)^n}\right|\cdot\left|\frac{3^n}{3^{n+1}}\right| \\
&= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)\cdot\left|\frac{x+1}{3}\right| = \frac{|x+1|}{3}
\end{aligned}$$

Therefore, by the ratio test, the series converges when $\frac{|x+1|}{3}<1$ and diverges when $\frac{|x+1|}{3}>1$. In particular, it converges when

$$|x+1|<3 \iff -3<x+1<3 \iff -4<x<2$$

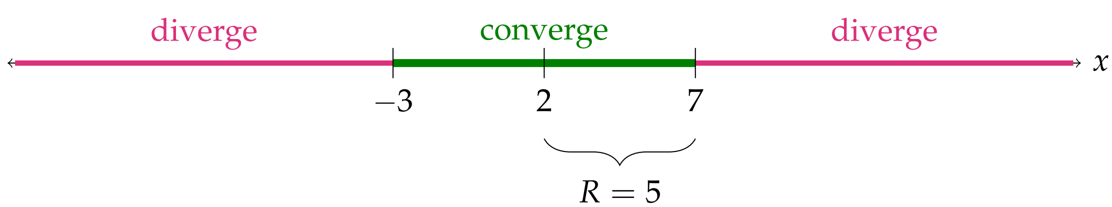
and the radius of convergence is $R=3$. (Alternatively, one can set $A_n=\frac{(-1)^n}{(n+1)3^n}$ and compute $A=\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|=\frac{1}{3}$, so that $R=\frac{1}{A}=3$.)



S-10: We first apply the ratio test with $a_n = \frac{(x-2)^n}{n^{4/5}(5^n-4)}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^{4/5}(5^{n+1}-4)}\cdot\frac{n^{4/5}(5^n-4)}{(x-2)^n}\right| \\
&= \lim_{n\to\infty}\frac{n^{4/5}(5^n-4)}{(n+1)^{4/5}(5^{n+1}-4)}|x-2| \\
&= \lim_{n\to\infty}\frac{(1-4/5^n)}{(1+1/n)^{4/5}(5-4/5^n)}|x-2| \\
&= \frac{|x-2|}{5}
\end{aligned}$$

Therefore the series converges if $|x-2| < 5$ and diverges if $|x-2| > 5$. So, its radius of convergence is 5.



S-11: We apply the ratio test with $a_n = \frac{(x+2)^n}{n^2}$. Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+2)^{n+1}}{(n+1)^2}}{\frac{(x+2)^n}{n^2}}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}|x+2| = \lim_{n\to\infty}\frac{1}{(1+1/n)^2}|x+2| = |x+2|$$

we have convergence for

$$|x+2| < 1 \iff -1 < x+2 < 1 \iff -3 < x < -1$$

and divergence for $|x+2| > 1$. So, the largest possible open interval of convergence is $(-3,-1)$.

S-12: We apply the ratio test with $a_n = \frac{4^n}{n}(x-1)^n$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{4^{n+1}(x-1)^{n+1}/(n+1)}{4^n(x-1)^n/n}\right| \\ &= \lim_{n\to\infty} 4|x-1|\frac{n}{n+1} \\ &= 4|x-1|\lim_{n\to\infty}\frac{n}{n+1} = 4|x-1|\cdot 1.\end{aligned}$$

the series converges if

$$4|x-1| < 1 \iff -1 < 4(x-1) < 1 \iff -\frac{1}{4} < x-1 < \frac{1}{4} \iff \frac{3}{4} < x < \frac{5}{4}$$

and diverges if $4|x-1| > 1$.

So the interval of convergence will be from $\frac{3}{4}$ to $\frac{5}{4}$; we don't need to decide whether the endpoints are included or not.

S-13: We apply the ratio test with $a_n = (-1)^n \frac{(x-1)^n}{2^n(n+2)}$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{2^{n+1}(n+3)}\ \frac{2^n(n+2)}{(x-1)^n}\right| \\ &= \lim_{n\to\infty}\frac{|x-1|}{2}\frac{n+2}{n+3} \\ &= \frac{|x-1|}{2}\lim_{n\to\infty}\frac{1+2/n}{1+3/n} = \frac{|x-1|}{2}\end{aligned}$$

the series converges if

$$\frac{|x-1|}{2} < 1 \iff |x-1| < 2 \iff -2 < (x-1) < 2 \iff -1 < x < 3$$

and diverges if $|x-1| > 2$. So the series has radius of convergence 2. That means the interval of convergence will be from -1 to 3. We don't need to decide which endpoints are included.

S-14: We apply the ratio test with $a_n = (-1)^n n^2 (x-a)^{2n}$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2(x-a)^{2(n+1)}}{(-1)^n n^2 (x-a)^{2n}}\right| \\ &= \lim_{n\to\infty}|x-a|^2\frac{(n+1)^2}{n^2} \\ &= |x-a|^2\lim_{n\to\infty}\left(1+{}^1\!/\!{}_n\right)^2 = |x-a|^2\cdot 1.\end{aligned}$$

the series converges if

$$|x-a|^2 < 1 \iff |x-a| < 1 \iff -1 < x-a < 1 \iff a-1 < x < a+1$$

and diverges if $|x-a| > 1$.

So the interval of convergence is from $a-1$ to $a+1$. We don't know which endpoints are included.

S-15: (a) We apply the ratio test for the series whose k^{th} term is $A_k = \frac{(x+1)^k}{k^2 9^k}$. Then

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| &= \lim_{k\to\infty}\left|\frac{(x+1)^{k+1}}{(k+1)^2 9^{k+1}}\;\frac{k^2 9^k}{(x+1)^k}\right| \\ &= \lim_{k\to\infty}|x+1|\;\frac{1}{9}\;\frac{k^2}{(k+1)^2} \\ &= \lim_{k\to\infty}|x+1|\;\frac{1}{9}\;\frac{1}{(1+1/k)^2} \\ &= \frac{|x+1|}{9}\end{aligned}$$

So the series must converge when $|x+1| < 9$ and must diverge when $|x+1| > 9$. So its interval of convergence is from -10 to 8. We don't know whether -10 and 8 are included.

(b) The partial sum

$$\sum_{k=1}^{N}\left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}}\right) = \left(\frac{a_1}{a_2}-\frac{a_2}{a_3}\right) + \left(\frac{a_2}{a_3}-\frac{a_3}{a_4}\right) + \cdots + \left(\frac{a_N}{a_{N+1}}-\frac{a_{N+1}}{a_{N+2}}\right) = \frac{a_1}{a_2} - \frac{a_{N+1}}{a_{N+2}}$$

We are told that $\sum_{k=1}^{\infty}\left(\frac{a_k}{a_{k+1}}-\frac{a_{k+1}}{a_{k+2}}\right)=\frac{a_1}{a_2}$. This means that the above partial sum converges to $\frac{a_1}{a_2}$ as $N\to\infty$, or equivalently, that

$$\lim_{N\to\infty}\frac{a_{N+1}}{a_{N+2}}=0$$

and hence that

$$\lim_{k\to\infty}\frac{|a_{k+1}(x-1)^{k+1}|}{|a_k(x-1)^k|}=|x-1|\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|}$$

is infinite for any $x\neq 1$. So, by the ratio test, this series converges only for $x=1$.

S-16: Using the geometric series $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$,

$$\frac{x^3}{1-x}=x^3\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}x^{n+3}=\sum_{n=3}^{\infty}x^n$$

S-17: We can find $f(x)$ by differentiating its integral, or antidifferentiating its derivative. In the latter case, we'll have to solve for the arbitrary constant of integration; in the former case, we do not. (Remember that many different functions have the same derivative, but a single function has only one derivative.) To avoid the necessity of finding the arbitrary constant, we can ignore the given equation for $f'(x)$, which makes the problem much simpler. This is the method used in Solution 1.

Solution 1 Using the Fundamental Theorem of Calculus Part 1:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_5^x f(t)\mathrm{d}t\right\}&=f(x)\\
\text{So,}\qquad f(x)&=\frac{\mathrm{d}}{\mathrm{d}x}\left\{3x+\sum_{n=0}^{\infty}\frac{(x-1)^{n+1}}{n(n+1)^2}\right\}\\
&=3+\sum_{n=1}^{\infty}\frac{(n+1)(x-1)^n}{n(n+1)^2}\\
&=3+\sum_{n=1}^{\infty}\frac{(x-1)^n}{n(n+1)}
\end{aligned}$$

Solution 2 Suppose we had used $f'(x)$ instead. We would antidifferentiate to find:

$$\begin{aligned} f(x) &= \int \left(\sum_{n=0}^{\infty} \frac{(x-1)^n}{n+2} \right) \mathrm{d}x \\ &= \left(\sum_{n=0}^{\infty} \frac{(x-1)^{n+1}}{(n+1)(n+2)} \right) + C \\ &= \left(\sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)} \right) + C \end{aligned}$$

Notice $f(1) = 0 + C$. So, to find C, we must find $f(1)$. We can't get that information from $f'(x)$, so our only option is to consider the given formula for $\int_5^x f(t)\mathrm{d}t$. Using the Fundamental Theorem of Calculus Part 1:

$$\begin{aligned} f(1) &= \frac{\mathrm{d}}{\mathrm{d}x} \left\{ \int_5^x f(t)\mathrm{d}t \right\} \Big|_{x=1} \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \left\{ 3x + \sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n(n+1)^2} \right\} \Bigg|_{x=1} \\ &= \left[3 + \sum_{n=1}^{\infty} \frac{(n+1)(x-1)^n}{n(n+1)^2} \right]_{x=1} \\ &= \left[3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)} \right]_{x=1} \\ &= 3 + \sum_{n=1}^{\infty} \frac{0^n}{n(n+1)} \\ &= 3 \end{aligned}$$

So, $f(x) = 3 + \displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)}$.

Note that in Solution 2, we did the same calculation as Solution 1, and more.

S-18: We apply the ratio test for the series whose n^{th} term is either $a_n = \frac{x^n}{3^{2n} \ln n}$ or $a_n = \left|\frac{x^n}{3^{2n} \ln n}\right|$. For both series

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{3^{2(n+1)} \ln(n+1)} \frac{3^{2n} \ln n}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x \ln n}{3^2 \ln(n+1)} \right| = \lim_{n\to\infty} \left| \frac{x \ln n}{3^2 [\ln(n) + \ln(1+1/n)]} \right| \\ &= \lim_{n\to\infty} \left| \frac{x}{3^2 [1 + \ln(1+1/n)/\ln(n)]} \right| \\ &= \frac{|x|}{9} \end{aligned}$$

Therefore, by the ratio test, our series converges absolutely when $|x| < 9$ and diverges when $|x| > 9$.

For $x = -9$, $\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ which converges by the alternating series test.

For $x = +9$, $\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ which is the same series as $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right|$. We shall shortly show that $n \geqslant \ln n$, and hence $\frac{1}{\ln n} \geqslant \frac{1}{n}$ for all $n \geqslant 1$. This implies that the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by comparison with the divergent series $\sum_{n=2}^{\infty} \frac{1}{n^p}\bigg|_{p=1}$. This yelds both divergence for $x = 9$ and also the failure of absolute convergence for $x = -9$.

Finally, we show that $n - \ln n > 0$, for all $n \geqslant 1$. Set $f(x) = x - \ln x$. Then $f(1) = 1 > 0$ and

$$f'(x) = 1 - \frac{1}{x} \geqslant 0 \qquad \text{for all } x \geqslant 1$$

So $f(x)$ is (strictly) positive when $x = 1$ and is increasing for all $x \geqslant 1$. So $f(x)$ is (strictly) positive for all $x \geqslant 1$.

S-19: (a) Differentiating both sides of

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

gives

$$\sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

Now multiplying both sides by x gives

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

as desired.

(b) Differentiating both sides of the conclusion of part (a) gives

$$\sum_{n=0}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2 - 2x(x-1)}{(1-x)^4} = \frac{(1-x)(1-x+2x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

Now multiplying both sides by x gives

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$

We know that differentiation preserves the radius of convergence of power series. So this series has radius of convergence 1 (the radius of convergence of the original geometric series). So the series converges for $-1 < x < 1$.

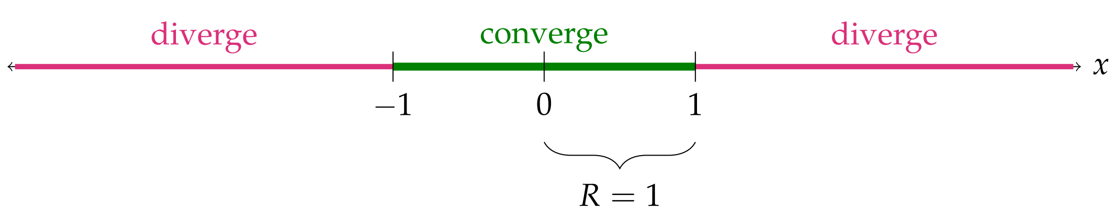
S-20: By the divergence test, the fact that $\sum_{n=0}^{\infty}(1-b_n)$ converges guarantees that $\lim_{n\to\infty}(1-b_n)=0$, or equivalently, that $\lim_{n\to\infty} b_n = 1$. So, by equation (5.5.2) in the text, the radius of convergence is

$$R = \left[\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|\right]^{-1} = \left[\frac{1}{1}\right]^{-1} = 1 \tag{5.7}$$

S-21: (a) We know that the radius of convergence R obeys

$$\frac{1}{R} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n}{n+1}\,\frac{(n+1)a_{n+1}}{na_n} = 1\frac{C}{C} = 1$$

because we are told that $\lim_{n\to\infty} na_n = C$. So $R=1$.



(b) Just knowing that the radius of convergence is 1, we know that the series converges for $|x|<1$ and diverges for $|x|>1$.

S-22: First, we differentiate.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n (x-c)^n \\
f'(x) &= \sum_{n=0}^{\infty} nA_n (x-c)^{n-1} \\
&= \sum_{n=1}^{\infty} nA_n (x-c)^{n-1} \\
f'(c) &= \sum_{n=1}^{\infty} nA_n \cdot 0^{n-1} \\
&= A_1 \cdot 1 + 2A_2 \cdot 0 + 3A_3 \cdot 0 + \cdots \\
&= A_1
\end{aligned}$$

So, if $A_1 = 0$, then $f'(c) = 0$. That is, $f(x)$ has a critical point at $x = c$.

To determine the behaviour of this critical point, we use the second derivative test.

$$\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} nA_n (x-c)^{n-1} \\
f''(x) &= \sum_{n=1}^{\infty} n(n-1)A_n (x-c)^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)A_n (x-c)^{n-2} \\
f''(c) &= \sum_{n=2}^{\infty} n(n-1)A_n \cdot 0^{n-2} \\
&= 2(1)A_2 \cdot 0^0 + 3(2)A_3 \cdot 0^1 + 4(3)A_4 \cdot 0^2 + \cdots \\
&= 2A_2
\end{aligned}$$

Following the second derivative test, $x = c$ is the location of a local maximum if $A_2 < 0$, and it is the location of a local minimum if $A_2 > 0$. (If $A_2 = 0$, the critical point may or may not be a local extremum.)

S-23: We recognize $\displaystyle\sum_{n=3}^{\infty} \frac{n}{5^{n-1}}$ as $f(x) = \displaystyle\sum_{n=3}^{\infty} n \cdot x^{n-1}$, evaluated at $x = \dfrac{1}{5}$. We should figure out what $f(x)$ is in equation form (as opposed to power series form). Notice that this looks similar to the derivative of the geometric series $\sum x^n$.

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \qquad \text{when } |x|<1\\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty} x^n\right\}\\
\frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1}\\
&= 1x^0 + 2x^1 + \sum_{n=3}^{\infty} nx^{n-1}\\
&= 1 + 2x + \sum_{n=3}^{\infty} nx^{n-1}\\
\text{So,} \qquad \frac{1}{(1-x)^2} - 1 - 2x &= \sum_{n=3}^{\infty} nx^{n-1}\\
\text{Setting } x = \frac{1}{5}: \qquad \frac{1}{(1-1/5)^2} - 1 - \frac{2}{5} &= \sum_{n=3}^{\infty} n\left(\frac{1}{5}\right)^{n-1}\\
\left(\frac{5}{4}\right)^2 - 1 - \frac{2}{5} &= \sum_{n=3}^{\infty} \frac{n}{5^{n-1}}
\end{aligned}$$

So, our series evaluates to $\frac{25}{16} - 1 - \frac{2}{5} = \frac{13}{80}$.

Solutions to Exercises 5.6 — Jump to TABLE OF CONTENTS

S-1: All functions A, B, and C intersect the function $y = f(x)$ when $x = 2$. B is a constant function, so this is the constant approximation. A is the tangent line, so A is the linear approximation. C is a tangent parabola, so C is the quadratic approximation.

S-2: Following how a Taylor series is constructed, the Taylor series and the function agree at the point chosen as the centre. So, $T(5) = \arctan^3\left(e^5 + 7\right)$.

If we were evaluating a Taylor series at a point *other than its centre*, we would generally need to check that (a) the series converges, and (b) it converges to the same value as the function we used to create it.

S-3: These are listed in Theorem 5.6.5 in the text. However, it's possible to figure out many of them without a lot of memorization. For example, $e^0 = \cos(0) = \frac{1}{1-0} = 1$, while $\sin(0) = \ln(1+0) = \arctan(0) = 0$. So by plugging in $x = 0$ to the series listed, we can divide them into these two categories.

The derivative of sine is cosine, so we can also look for one series that is the derivative of another. The derivative of e^x is e^x, so we can look for a series that is its own derivative.

Furthermore, sine and arctangent are odd functions and only II and IV are odd. Cosine is an even function and only III is even.

Alternately, we can find the first few terms of each series using the definition of a Taylor series, and match them up.

All together, the functions correspond to the following series:

A - V B - I C - IV D - VI E - II F - III

S-4:

(a) Using the definition of a Taylor series, we know

$$\sum_{n=0}^{\infty} \frac{n^2}{(n!+1)}(x-3)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$$

So, the coefficient of $(x-3)^{20}$ is $\frac{f^{(20)}(3)}{20!}$ (using the definition). Using the given series, the coefficient of $(x-3)^{20}$ is $\frac{20^2}{20!+1}$. So,

$$\begin{aligned} \frac{f^{(20)}(3)}{20!} &= \frac{20^2}{20!+1} \\ \Rightarrow \qquad f^{(20)}(3) &= 20^2\left(\frac{20!}{20!+1}\right) \end{aligned}$$

(which is extremely close to 20^2).

(b) Using the definition of a Taylor series, we know

$$\sum_{n=0}^{\infty} \frac{n^2}{(n!+1)}(x-3)^{2n} = \sum_{k=0}^{\infty} \frac{g^{(k)}(3)}{k!}(x-3)^k$$

So, the coefficient of $(x-3)^{20}$ is $\frac{g^{(20)}(3)}{20!}$ (using the definition). Looking at the given series, the coefficient of $(x-3)^{20}$ occurs when $n=10$, so it is $\frac{10^2}{10!+1}$. So,

$$\begin{aligned} \frac{g^{(20)}(3)}{20!} &= \frac{10^2}{10!+1} \\ \Rightarrow \qquad g^{(20)}(3) &= 10^2\left(\frac{20!}{10!+1}\right) \end{aligned}$$

(c) With the previous two examples in mind, we find the Maclaurin series for $h(x)$. (Using the series representation will be much easier than differentiating $h(x)$ directly twenty times.) Recall from the text that we know the Maclaurin series for $\arctan x$.

$$\begin{aligned}
\arctan(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\
\arctan(5x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(5x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{4n+2} \\
\frac{\arctan(5x^2)}{x^4} &= \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{4n-2} \\
\sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k &= \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}}{2n+1} x^{4n-2}
\end{aligned}$$

Using the definition of a Maclaurin series, the coefficient of x^{22} is $\dfrac{h^{(22)}(0)}{22!}$. This occurs in the given series when $n = 6$, so

$$\begin{aligned}
\frac{h^{(22)}(0)}{22!} &= (-1)^6 \frac{5^{2\times 6+1}}{2 \times 6 + 1} = \frac{5^{13}}{13} \\
\Rightarrow \qquad h^{(22)}(0) &= \frac{22! \cdot 5^{13}}{13}
\end{aligned}$$

Similarly, the coefficient of x^{20} in the Maclaurin series is $\dfrac{h^{(20)(0)}}{20!}$. Since no term x^{20} occurs in our series, that coefficient is 0, so $h^{(20)}(0) = 0$.

S-5: The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n$$

That is, we need a general description of $f^{(n)}(1)$. To find this, we take a few derivatives, and look for a pattern.

$$\begin{aligned}
f(x) &= \ln(x) & f(1) &= 0 \\
f'(x) &= x^{-1} & f'(1) &= 1 \\
f''(x) &= (-1)x^{-2} & f''(1) &= -1 \\
f^{(3)}(x) &= (-2)(-1)x^{-3} & f^{(3)}(1) &= 2! \\
f^{(4)}(x) &= (-3)(-2)(-1)x^{-4} & f^{(4)}(1) &= -3! \\
f^{(5)}(x) &= (-4)(-3)(-2)(-1)x^{-5} & f^{(5)}(1) &= 4! \\
f^{(6)}(x) &= (-5)(-4)(-3)(-2)(-1)x^{-6} & f^{(6)}(1) &= -5! \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^{n-1}(n-1)!\, x^{-n} & f^{(n)}(1) &= (-1)^{n-1}(n-1)!
\end{aligned}$$

Using the convention $0! = 1$, our pattern for $f^{(n)}(1)$ begins when $n = 1$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

S-6: To find the Taylor series for sine, centred at $a = \pi$, we'll need to know the various derivatives of sine at π.

$$\begin{aligned}
f(x) &= \sin x & f(\pi) &= 0 \\
f'(x) &= \cos x & f'(\pi) &= -1 \\
f''(x) &= -\sin x & f''(\pi) &= 0 \\
f'''(x) &= -\cos x & f'''(\pi) &= 1 \\
f^{(4)}(x) &= \sin x = f(x) & f^{(4)}(\pi) &= 0
\end{aligned}$$

Even derivatives are 0; odd derivatives alternate between -1 and $+1$. (If you're following along with the derivation of the Maclaurin series for sine in the text, note $f^{(n)}(\pi) = -f^{(n)}(0)$.)

In our Taylor series, every even-indexed term will be zero, and we will be left with only odd-indexed terms. If we let n be our index, then the term $2n+1$ will capture all the odd numbers. Since the signs alternate, $f^{(2n+1)}(\pi) = (-1)^{n+1}$. So, our Taylor series is:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{f^{(k)}(\pi)}{k!}(x-\pi)^k &= \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(\pi)}{(2n+1)!}(x-\pi)^{2n+1} \qquad \text{(since the even terms are all zero)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}
\end{aligned}$$

S-7: The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(10)}{n!}(x-10)^n$$

That is, we need a general description of $g^{(n)}(10)$. To find this, we take a few derivatives, and look for a pattern.

$$
\begin{aligned}
g(x) &= x^{-1} & g(10) &= \frac{1}{10} \\
g'(x) &= (-1)x^{-2} & g'(10) &= \frac{-1}{10^2} \\
g''(x) &= (-2)(-1)x^{-3} & g''(10) &= \frac{(-1)^2 2!}{10^3} \\
g^{(3)}(x) &= (-3)(-2)(-1)x^{-4} & g^{(3)}(10) &= \frac{(-1)^3 3!}{10^4} \\
g^{(4)}(x) &= (-4)(-3)(-2)(-1)x^{-5} & g^{(4)}(10) &= \frac{(-1)^4 4!}{10^5} \\
g^{(5)}(x) &= (-5)(-4)(-3)(-2)(-1)x^{-6} & g^{(5)}(10) &= \frac{(-1)^5 5!}{10^6} \\
&\vdots & &\vdots \\
g^{(n)}(x) &= (-1)^n n! x^{-(n+1)} & g^{(n)}(10) &= \frac{(-1)^n n!}{10^{n+1}}
\end{aligned}
$$

Using the convention $0! = 1$, our pattern for $g^{(n)}(10)$ begins when $n = 0$.

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!}(x-10)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n! 10^{n+1}}(x-10)^n \\
&= -\sum_{n=0}^{\infty} \frac{(x-10)^n}{(-10)^{n+1}} \\
&= \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{10-x}{10} \right)^n
\end{aligned}
$$

For fixed x, we recognize this as a geometric series with $r = \frac{10-x}{10}$. So it converges precisely when $|r| < 1$, i.e.

$$
\begin{aligned}
\left| \frac{10-x}{10} \right| &< 1 \\
|10-x| &< 10 \\
-10 < x - 10 &< 10 \\
0 < x &< 20
\end{aligned}
$$

So, its interval of convergence is $(0, 20)$.

S-8: The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!}(x-a)^n$$

That is, we need a general description of $h^{(n)}(a)$. To find this, we take a few derivatives, and look for a pattern.

$$\begin{aligned} h(x) &= e^{3x} & \qquad h(a) &= e^{3a} \\ h'(x) &= 3e^{3x} & h'(a) &= 3e^{3a} \\ h''(x) &= 3^2e^{3x} & h''(a) &= 3^2e^{3a} \\ h'''(x) &= 3^3e^{3x} & h'''(a) &= 3^3e^{3a} \\ &\vdots & &\vdots \\ h^{(n)}(x) &= 3^ne^{3x} & h^{(n)}(a) &= 3^ne^{3a} \end{aligned}$$

The pattern for $h^{(n)}(a)$ holds for all (whole numbers) $n \geqslant 0$. So, our Taylor series for $h(x)$ is

$$\sum_{n=0}^{\infty} \frac{3^ne^{3a}}{n!}(x-a)^n$$

To find its radius of convergence, we use the ratio test.

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{3^{n+1}e^{3a}(x-a)^{n+1}}{(n+1)!} \cdot \frac{n!}{3^ne^{3a}(x-a)^n}\right| \\ &= \left|\frac{3^{n+1}}{3^n} \cdot \frac{e^{3a}}{e^{3a}} \cdot \frac{n!}{(n+1)!} \cdot \frac{(x-a)^{n+1}}{(x-a)^n}\right| \\ &= 3 \cdot \frac{1}{n+1} \cdot |x-a| \\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{3}{n+1} \cdot |x-a|\right] = 0 \end{aligned}$$

Our series converges for every value of x, so its radius of convergence is ∞.

S-9: Substituting $y = 2x$ into $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is valid for all $-1 < y < 1$) gives

$$f(x) = \frac{1}{2x-1} = -\frac{1}{1-2x} = -\sum_{n=0}^{\infty}(2x)^n = -\sum_{n=0}^{\infty} 2^nx^n \qquad \text{for all } -\tfrac{1}{2} < x < \tfrac{1}{2}$$

S-10: Substituting first $y = -x$ and then $y = 2x$ into $\dfrac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is valid for all $-1 < y < 1$) gives

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \text{for all } -1 < x < 1$$
$$\frac{1}{1-(2x)} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n \qquad \text{for all } -\tfrac{1}{2} < x < \tfrac{1}{2}$$

Hence, for all $-\frac{1}{2} < x < \frac{1}{2}$,

$$\begin{aligned} f(x) &= \frac{3}{x+1} - \frac{1}{2x-1} = \frac{3}{1-(-x)} + \frac{1}{1-2x} = 3\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} 2^n x^n \\ &= \sum_{n=0}^{\infty} \big(3(-1)^n + 2^n\big) x^n \end{aligned}$$

So $b_n = 3(-1)^n + 2^n$.

S-11: We found the Taylor series for e^{3x} from scratch in Question 8. If we hadn't just done that, we could easily find it by modifying the series for e^x.

Substituting $y = 3x$ into the exponential series

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

gives

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n\, x^n}{n!}$$

so that c_5, the coefficient of x^5, which appears only in the $n = 5$ term, is $c_5 = \dfrac{3^5}{5!}$

S-12: Since

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \ln(1+2t) = \frac{2}{1+2t} = 2\sum_{n=0}^{\infty} (-2t)^n \qquad \text{if } |2t| < 1 \text{ i.e. } |t| < \frac{1}{2}$$

and $f(0) = 0$, we have

$$f(x) = \int_0^x f'(t)\,\mathrm{d}t = 2\sum_{n=0}^{\infty} \int_0^x (-1)^n 2^n t^n\,\mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n 2^{n+1} \frac{x^{n+1}}{n+1} \qquad \text{for all } |x| < \frac{1}{2}$$

S-13: We just need to substitute $y = x^3$ into the known Maclaurin series for $\sin y$, to get the Maclaurin series for $\sin(x^3)$, and then multiply the result by x^2.

$$\begin{aligned}
\sin y &= y - \frac{y^3}{3!} + \cdots \\
\sin(x^3) &= x^3 - \frac{x^9}{3!} + \cdots \\
x^2 \sin(x^3) &= x^5 - \frac{x^{11}}{3!} + \cdots
\end{aligned}$$

so $a = 1$ and $b = -\frac{1}{3!} = -\frac{1}{6}$.

S-14: Recall that

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \cdots$$

Setting $y = -x^2$, we have

$$\begin{aligned}
e^{-x^2} &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots \\
e^{-x^2} - 1 &= -x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots \\
\frac{e^{-x^2} - 1}{x} &= -x + \frac{x^3}{2} - \frac{x^5}{6} + \cdots \\
\int \frac{e^{-x^2} - 1}{x} \, \mathrm{d}x &= C - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{36} + \cdots
\end{aligned}$$

S-15: Recall that

$$\arctan(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{2n+1}$$

Setting $y = 2x$, we have

$$\begin{aligned}
\int x^4 \arctan(2x) \, \mathrm{d}x &= \int \left(x^4 \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} \right) \mathrm{d}x \\
&= \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+5}}{2n+1} \right) \mathrm{d}x \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+6}}{(2n+1)(2n+6)} + C \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+6}}{(2n+1)(n+3)} + C
\end{aligned}$$

S-16: Substituting $y = -3x^3$ into $\dfrac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ gives

$$\frac{\mathrm{d}f}{\mathrm{d}x} = x \cdot \frac{1}{1+3x^3} = x \sum_{n=0}^{\infty} \left(-3x^3\right)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+1}$$

Now integrating,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{3n+2}}{3n+2} + C$$

To have $f(0) = 1$, we need $C = 1$. So, finally

$$f(x) = 1 + \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{3n+2} x^{3n+2}$$

S-17: We're given a big hint: that our series resembles the Taylor series for arctangent.

The terms of arctangent are $(-1)^n \dfrac{x^{2n+1}}{2n+1}$. Our terms resemble those terms, with x^{2n+1} replaced by $\dfrac{1}{3^n}$.

Since $3^n = \left(\sqrt{3}\right)^{2n} = \frac{1}{\sqrt{3}}\left(\sqrt{3}\right)^{2n+1}$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\left(\sqrt{3}\right)^{2n+1}} = \sqrt{3} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\bigg|_{x=\frac{1}{\sqrt{3}}} = \sqrt{3} \arctan \frac{1}{\sqrt{3}}$$
$$= \sqrt{3}\,\frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}$$

S-18: Recall that $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$. So

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right]_{x=-1} = \left[e^x\right]_{x=-1} = e^{-1}$$

S-19: Recall that $e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$. So

$$\sum_{k=0}^{\infty} \frac{1}{e^k k!} = \left[\sum_{k=0}^{\infty} \frac{x^k}{k!}\right]_{x=1/e} = \left[e^x\right]_{x=1/e} = e^{1/e}$$

S-20: Recall that $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$. So

$$\sum_{k=0}^{\infty} \frac{1}{\pi^k k!} = \Big[\sum_{k=0}^{\infty} \frac{x^k}{k!}\Big]_{x=1/\pi} = \Big[e^x\Big]_{x=1/\pi} = e^{1/\pi}$$

This series differs from the given one only in that it starts with $k = 0$ while the given series starts with $k = 1$. So

$$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = \sum_{k=0}^{\infty} \frac{1}{\pi^k k!} - \underbrace{1}_{k=0} = e^{1/\pi} - 1$$

S-21: Recall, from Theorem 5.6.5 in the text, that, for all $-1 < x \leqslant 1$,

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

(To get from the first sum to the second sum we substituted $n = k+1$. If you don't see why the two sums are equal, write out the first few terms of each.) So

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n} = \Big[\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}\Big]_{x=1/2} = \Big[\ln(1+x)\Big]_{x=1/2} = \ln(3/2)$$

S-22: Write

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n &= \sum_{n=1}^{\infty} \frac{n}{n!} e^n + \sum_{n=1}^{\infty} \frac{2}{n!} e^n \\
&= \sum_{n=1}^{\infty} \frac{e^n}{(n-1)!} + 2\sum_{n=1}^{\infty} \frac{e^n}{n!} \\
&= e\sum_{n=1}^{\infty} \frac{e^{n-1}}{(n-1)!} + 2\sum_{n=1}^{\infty} \frac{e^n}{n!} \\
&= e\sum_{n=0}^{\infty} \frac{e^n}{n!} + 2\sum_{n=1}^{\infty} \frac{e^n}{n!}
\end{aligned}$$

Recall that $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$. So

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n &= e\Big[\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big]_{x=e} + 2\Big[\sum_{n=1}^{\infty} \frac{x^n}{n!}\Big]_{x=e} = e\Big[e^x\Big]_{x=e} + 2\Big[e^x - 1\Big]_{x=e} = e^{e+1} + 2(e^e - 1) \\
&= (e+2)e^e - 2
\end{aligned}$$

S-23: Let's use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}}\right| \\ &= \lim_{n\to\infty} 2\frac{n}{n+1} = 2 > 1\end{aligned}$$

So, the series diverges.

Remark: it's tempting to note that $\ln(1+y) = \sum_{n=0}^{\infty}(-1)^n\frac{y^{n+1}}{n+1} = -\sum_{n=1}^{\infty}\frac{(-y)^n}{n}$, and try to substitute in $y=-2$. But, the Maclaurin series for $\ln(1+y)$ has radius of convergence $R=1$, so it doesn't converge at $y=-2$. Furthermore, $\ln(1+(-2)) = \ln(-1)$, but this is undefined.

S-24: Our series looks something like the Taylor series for sine, $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1}\left(1+2^{2n+1}\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[\left(\frac{\pi}{4}\right)^{2n+1}+\left(\frac{\pi}{2}\right)^{2n+1}\right] \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1} \\ &= \sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}\right) \\ &= \frac{1}{\sqrt{2}}+1 = \frac{1+\sqrt{2}}{\sqrt{2}}\end{aligned}$$

S-25: (a)

Solution 1: The naive strategy is to set $a_n = \frac{x^{2n}}{(2n)!}$ and apply the ratio test.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}}\right| &= \left|\frac{x^{2n+2}}{x^{2n}}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right| \\ &= \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} \\ &= 0\end{aligned}$$

This is smaller than 1 no matter what x is. So the series converges for all x.

Solution 2: Alternatively, the sneaky way is to observe that both $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and

$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$ are known to converge for all x. So

$$\frac{1}{2}\big(e^x + e^{-x}\big) = \sum_{n \text{ even}} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

also converges for all x.

(b) Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then:

$$\begin{aligned}
e &= \sum_{n=0}^{\infty} \frac{1}{n!} \\
e^{-1} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \\
e + e^{-1} &= \sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} = 2 \sum_{n \text{ even}} \frac{1}{n!} = 2 \sum_{n=0}^{\infty} \frac{1}{(2n)!}
\end{aligned}$$

Hence $\sum_{n=0}^{\infty} \frac{1}{(2n)!} = \frac{1}{2}\left(e + \frac{1}{e}\right)$.

S-26: The Taylor Series for e^x is not alternating, so we'll use Theorem 5.6.1-b in the text to bound the error in a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c strictly between a and x. In our case, $a = 0$ and $x = 1$. So, we want to find a value of N such that

$$\left| \frac{f^{(N+1)}(c)}{(N+1)!} (1-0)^{N+1} \right| = \frac{e^c}{(N+1)!} < 5 \times 10^{-11}$$

for *all* c in $(0,1)$.

If c is between 0 and 1, then e^c is between 1 and e. However, since the purpose of this problem is to approximate e precisely, it doesn't make much sense to use e in our bound. Since e is less than 3, then $e^c < 3$ for all c in $(0,1)$. Now we can search for an appropriate value of N.

N	$\frac{3}{(N+1)!}$
10	$\frac{3}{11!} = \frac{1}{9^{10}} \approx 8 \times 10^{-8}$
11	$\frac{3}{12!} \approx 6 \times 10^{-9}$
12	$\frac{3}{13!} \approx 5 \times 10^{-10}$
13	$\frac{3}{14!} \approx 3 \times 10^{-11}$

So, it suffices to use the partial sum S_{13}.

S-27: The Taylor Series for $\ln(1-x)$ is not alternating, so we'll use Theorem 5.6.1-b in the text to bound the error in a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c strictly between a and x. In our case, $a = 0$ and $x = \frac{1}{10}$. So, we want to find a value of N such that

$$\left|\frac{f^{(N+1)}(c)}{(N+1)!}\left(\frac{1}{10}\right)^{N+1}\right| < 5 \times 10^{-11}$$

for *all* c in $(0, \frac{1}{10})$.

To find this N, we to know $f^{(N+1)}(x)$. Just like when we create a Taylor polynomial from scratch, we'll differentiate $f(x)$ several times, and look for a pattern.

$$f(x) = \ln(1-x) \qquad f^{(6)}(x) = \frac{-2(3)(4)(5)}{(1-x)^6}$$

$$f'(x) = \frac{-1}{1-x} \qquad f^{(7)}(x) = \frac{-2(3)(4)(5)(6)}{(1-x)^7}$$

$$f''(x) = \frac{-1}{(1-x)^2} \qquad \vdots$$

$$f'''(x) = \frac{-2}{(1-x)^3} \qquad f^{(N+1)}(x) = \frac{-N!}{(1-x)^{N+1}}$$

$$f^{(4)}(x) = \frac{-2(3)}{(1-x)^4}$$

$$f^{(5)}(x) = \frac{-2(3)(4)}{(1-x)^5}$$

Now we want a reasonable bound on $f^{(N+1)}(c)$, when c is in $(0, \frac{1}{10})$. Note that in this range, $1-c > 0$.

$$\begin{aligned}
& 0 < c < \frac{1}{10} \\
\Rightarrow \quad & \frac{9}{10} < 1-c < 1 \\
\Rightarrow \quad & \left(\frac{9}{10}\right)^{N+1} < (1-c)^{N+1} < 1 \\
\Rightarrow \quad & 1 < \frac{1}{(1-c)^{N+1}} < \left(\frac{10}{9}\right)^{N+1} \\
\Rightarrow \quad & N! < \frac{N!}{(1-c)^{N+1}} < N!\left(\frac{10}{9}\right)^{N+1}
\end{aligned}$$

This bound provides us with a "worst-case scenario" error. We don't know exactly what c is, but we don't need to–the bound above holds for *all* c between 0 and $\frac{1}{10}$.

Now we're ready to choose an N that results in a sufficiently small error bound.

$$\left|\frac{f^{(N+1)}(c)}{(N+1)!}\left(\frac{1}{10}\right)^{N+1}\right| < \frac{N!\left(\frac{10}{9}\right)^{N+1}}{(N+1)!}\left(\frac{1}{10}\right)^{N+1} = \frac{1}{9^{N+1}\cdot(N+1)}$$

So, we want: $\quad \dfrac{1}{9^{N+1}\cdot(N+1)} < 5\times 10^{-11}$

To find an appropriate N, we test several values.

N	$\dfrac{1}{9^{N+1}\cdot(N+1)}$
8	$\dfrac{1}{9\cdot 9^9} = \dfrac{1}{9^{10}} \approx 3\times 10^{-10}$
9	$\dfrac{1}{10\cdot 9^{10}} \approx 3\times 10^{-11}$

So, it suffices to use the partial sum S_9.

S-28: We'll use Theorem 5.6.1-b in the text to bound the error of a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c strictly between a and x. In our case, $a=0$ and x is in $(-2,1)$. So, we want to find a value of N such that

$$\left|\frac{f^{(N+1)}(c)}{(N+1)!}(x)^{N+1}\right| < 5\times 10^{-11}$$

for *all* x in $(-2,1)$, and *all* c in $(-2,1)$.

To find this N, we to know $f^{(N+1)}(x)$. Just like when we create a Taylor polynomial from scratch, we'll differentiate $f(x)$ several times, and look for a pattern.

$$\begin{aligned}
f(x) = \sinh(x) &= \frac{e^x - e^{-x}}{2}\\
f'(x) &= \frac{e^x + e^{-x}}{2}\\
f''(x) &= \frac{e^x - e^{-x}}{2}\\
f'''(x) &= \frac{e^x + e^{-x}}{2}
\end{aligned}$$

That is, even derivatives of $f(x)$ are $f(x)$, and odd derivatives of $f(x)$ are $\frac{e^x+e^{-x}}{2}$ (which, incidentally, is the function called $\cosh x$).

Now we want a reasonable bound on $f^{(N+1)}(c)$, when c is in $(-2,1)$. Since powers of e are always positive, we begin by noting that $0 < \frac{e^x-e^{-x}}{2} < \frac{e^x+e^{-x}}{2}$. So, all derivatives of $f(x)$ are bounded above by $\frac{e^x+e^{-x}}{2}$.

$$\begin{aligned}
& -2 < c < 1 \\
\Rightarrow \quad & e^{-2} < e^c < e \quad \text{and } e^{-1} < e^{-c} < e^2 \\
\Rightarrow \quad & f^{(N+1)}(c) < \frac{e^c+e^{-c}}{2} \qquad < \frac{e^2+e^2}{2} = e^2 < 9
\end{aligned}$$

This bound provides us with a "worst-case scenario" error. We don't know exactly what c is, but we don't need to–the bound above holds for *all* c between -2 and 1.

We also don't know exactly what x will be, only that it's between -2 and 1. So, we note $|x|^{N+1} < 2^{N+1}$.

Now we're ready to choose an N that results in a sufficiently small error bound.

$$\left|\frac{f^{(N+1)}(c)}{(N+1)!}(x)^{N+1}\right| < \frac{9 \cdot 2^{N+1}}{(N+1)!}$$

$$\text{So, we want:} \qquad \frac{9 \cdot 2^{N+1}}{(N+1)!} < 5 \times 10^{-11}$$

To find an appropriate N, we test several values.

N	$\frac{9 \cdot 2^{N+1}}{(N+1)!}$
10	$\frac{9 \cdot 2^{11}}{(11)!} \approx 5 \times 10^{-4}$
15	$\frac{9 \cdot 2^{16}}{(16)!} \approx 3 \times 10^{-8}$
17	$\frac{9 \cdot 2^{18}}{(18)!} \approx 4 \times 10^{-10}$
18	$\frac{9 \cdot 2^{19}}{(19)!} \approx 4 \times 10^{-11} < 5 \times 10^{-11}$

So, it suffices to use the partial sum S_{18}.

S-29: We'll use Theorem 5.6.1-b in the text to bound the error in a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c strictly between a and x. In our case, $a = \frac{1}{2}$, $x = -\frac{1}{3}$, and we are given the nth derivative of $f(x)$:

$$\begin{aligned} E_6 &= \frac{f^{(7)}(c)}{7!}\left(-\frac{1}{3}-\frac{1}{2}\right)^7 \\ &= \frac{1}{7!}\cdot\frac{6!}{2}\left[(1-c)^{-7}+(-1)^6(1+c)^{-7}\right]\left(-\frac{5}{6}\right)^7 \\ &= \frac{-5^7}{14\cdot 6^7}\cdot\left[(1-c)^{-7}+(1+c)^{-7}\right] \qquad = \end{aligned}$$

for some c in $(-\frac{1}{3}, \frac{1}{2})$.

We want to provide actual numeric bounds for this expression. That is, we want to find the absolute max and min of

$$E(c) = \frac{-5^7}{14\cdot 6^7}\cdot\left[(1-c)^{-7}+(1+c)^{-7}\right]$$

over the interval $\left(-\frac{1}{3}, \frac{1}{2}\right)$. Absolute extrema occur at endpoints and critical points. So, we'll start by differentiating $E(c)$, and finding its critical points (if any) in the interval $\left(-\frac{1}{3}, \frac{1}{2}\right)$.

$$\begin{aligned} E(c) &= \frac{-5^7}{14\cdot 6^7}\cdot\left[(1-c)^{-7}+(1+c)^{-7}\right] \\ E'(c) &= \frac{-5^7}{14\cdot 6^7}\cdot\left[7(1-c)^{-8}-7(1+c)^{-8}\right] = 0 \\ (1-c)^{-8} &= (1+c)^{-8} \\ 1-c &= 1+c \\ c &= 0 \end{aligned}$$

Since $E'(c)$ is defined over our entire interval, its only critical point is $c = 0$.

- $E(0) = \frac{-5^7}{14\cdot 6^7}[2]$
- $E\left(-\frac{1}{3}\right) = \frac{-5^7}{14\cdot 6^7}\left[\left(\frac{4}{3}\right)^{-7}+\left(\frac{2}{3}\right)^{-7}\right] = \frac{-5^7}{14\cdot 6^7}\left[\left(\frac{3}{4}\right)^7+\left(\frac{3}{2}\right)^7\right]$
- $E\left(\frac{1}{2}\right) = \frac{-5^7}{14\cdot 6^7}\left[\left(\frac{1}{2}\right)^{-7}+\left(\frac{3}{2}\right)^{-7}\right] = \frac{-5^7}{14\cdot 6^7}\left[2^7+\left(\frac{2}{3}\right)^7\right]$

We want to decide which of these numbers is biggest, and which smallest. Note that 2^7 is much, much bigger than $(3/2)^7$, and both $(3/4)^7$ and $(2/3)^7$ are less than one. Furthermore, $(3/2)^7$ is much larger than 2. So: $\left[2^7+(2/3)^7\right] > \left[(3/2)^7+(3/4)^7\right] > 2$. Therefore,

$$\frac{-5^7}{14\cdot 6^7}\left[2^7+\left(\frac{2}{3}\right)^7\right] < \frac{-5^7}{14\cdot 6^7}\left[\left(\frac{3}{4}\right)^7+\left(\frac{3}{2}\right)^7\right] < \frac{-5^7}{14\cdot 6^7}[2]$$

We conclude that the error E_6 is in the interval

$$\left(\frac{-5^7}{14 \cdot 6^7} \left[2^7 + \left(\frac{2}{3} \right)^7 \right] \quad , \quad \frac{-5^7}{14 \cdot 6^7} [2] \right)$$

or, equivalently,

$$\left(\frac{-5^7}{14 \cdot 3^7} \left[1 + \frac{1}{3^7} \right] \quad , \quad \frac{-5^7}{7 \cdot 6^7} \right)$$

which is approximately $(-0.199, -0.040)$.

S-30: Using the Maclaurin series expansions of $\cos x$ and e^x,

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
1 + x - e^x &= -\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \\
\frac{1 - \cos x}{1 + x - e^x} &= \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots}{-\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots} = \frac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots}
\end{aligned}$$

we have

$$\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \to 0} \frac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots} = \frac{\frac{1}{2!}}{-\frac{1}{2!}} = -1$$

S-31: Using the Maclaurin series expansion of $\sin x$,

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\sin x - x + \frac{x^3}{6} &= \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\frac{\sin x - x + \frac{x^3}{6}}{x^5} &= \frac{1}{5!} - \frac{x^2}{7!} + \cdots
\end{aligned}$$

we have

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x \to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \cdots \right) = \frac{1}{5!} = \frac{1}{120}$$

Remark: to solve this using l'Hôpital's rule we would differentiate five times, making series a practical alternative.

S-32: Our limit has the indeterminate form 1^∞; as with l'Hôpital's rule, we can change it to a friendlier form using the natural logarithm.

$$f(x) = \left(1+x+x^2\right)^{2/x}$$
$$\ln(f(x)) = \ln\left[\left(1+x+x^2\right)^{2/x}\right] = \frac{2}{x}\ln\left(1+x+x^2\right)$$

Recall $\ln(1+y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}y^n}{n}$, and set $y = x+x^2$. The series converges when $|y| < 1$, and since we only consider values of x that are very close to 0, we can assume $|x+x^2| < 1$.

$$\begin{aligned}
\ln(f(x)) &= \frac{2}{x}\ln\left(1+(x+x^2)\right) = \frac{2}{x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+x^2)^n}{n} \\
&= \frac{2}{x}\left[(x+x^2) - \frac{(x+x^2)^2}{2} + \frac{(x+x^2)^3}{3} - \cdots\right] \\
&= 2+2x - \frac{(x+x^2)^2}{2x} + \frac{(x+x^2)^3}{3x} - \cdots \\
&= 2+2x - \frac{(x^2+x)(1+x)}{2} + \frac{(x^2+x)^2(1+x)}{3} - \cdots \\
\lim_{x\to 0}\ln(f(x)) &= \lim_{x\to 0}\left[2+2x - \frac{(x^2+x)(1+x)}{2} + \frac{(x^2+x)^2(1+x)}{3} - \cdots\right] \\
&= 2+0+0\cdots = 2 \\
\lim_{x\to 0} f(x) &= \lim_{x\to 0} e^{\ln f(x)} = e^2
\end{aligned}$$

S-33: We have an indeterminate form 1^∞. We can use a natural logarithm to change this to a friendlier form. Furthermore, to avoid negative powers, we substitute $y = \frac{1}{2x}$. As x grows larger and larger, y gets closer and closer to zero, while staying positive.

$$\begin{aligned}
\ln\left[\left(1+\frac{1}{2x}\right)^x\right] &= x\ln\left(1+\frac{1}{2x}\right) = \frac{1}{2y}\ln(1+y) \\
&= \frac{1}{2y}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}y^n \\
&= \frac{1}{2y}\left[y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\right] \\
&= \left[\frac{1}{2} - \frac{y}{4} + \frac{y^2}{6} - \frac{y^3}{8} + \cdots\right] \\
\lim_{x\to\infty}\ln\left[\left(1+\frac{1}{2x}\right)^x\right] &= \lim_{y\to 0^+}\left[\frac{1}{2} - \frac{y}{4} + \frac{y^2}{6} - \frac{y^3}{8} + \cdots\right] = \frac{1}{2} \\
\lim_{x\to\infty}\left[\left(1+\frac{1}{2x}\right)^x\right] &= e^{1/2} = \sqrt{e}
\end{aligned}$$

S-34: The factor $(n+1)(n+2)$ reminds us of a derivative. Start with the geometric series.

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty} x^n\right\} \\
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{(1-x)^2}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=1}^{\infty} nx^{n-1}\right\} \\
\frac{2}{(1-x)^3} &= \sum_{n=1}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)x^n
\end{aligned}$$

Let $x = \frac{1}{7}$. Then $|x| < 1$, so our series converges.

$$\begin{aligned}
\frac{2}{(1-1/7)^3} &= \sum_{n=0}^{\infty} (n+2)(n+1)\left(\frac{1}{7}\right)^n \\
\frac{2}{(6/7)^3} &= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{7^n}
\end{aligned}$$

S-35: Recall the Taylor series for arctangent is:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

There are similarities between this and our given series: skipping powers of x, and a denominator that's not factorial. We'll try to manipulate it to look like our series. First, we antidifferentiate, to get a factor of $(2n+2)$ on the bottom.

$$\int \arctan x \, \mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} + C$$

We can find the antiderivative of arctangent using integration by parts. Let $u = \arctan x$ and $\mathrm{d}v = \mathrm{d}x$; then $\mathrm{d}u = \frac{1}{1+x^2}\mathrm{d}x$ and $v = x$.

$$\int \arctan x \, \mathrm{d}x = x \arctan x - \int \frac{x}{1+x^2} \, \mathrm{d}x + C$$

Now, we use the substitution $w = 1 + x^2$, $\mathrm{d}w = 2x\mathrm{d}x$.

$$= x \arctan x - \frac{1}{2}\ln(1+x^2) + C$$

So, $$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} = x \arctan x - \frac{1}{2}\ln(1+x^2) + C$$

To find C, we evaluate both sides of the equation at $x = 0$.

$$0 = 0 \arctan 0 - \frac{1}{2}\ln(1) + C = C$$

Therefore, $$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} = x \arctan x - \frac{1}{2}\ln(1+x^2)$$

Multiplying both sides by x^2,

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+4}}{(2n+1)(2n+2)} = x^3 \arctan x - \frac{x^2}{2}\ln(1+x^2)$$

S-36:

(a) We'll start, as we usually do, by finding a pattern for $f^{(n)}(0)$.

$$\begin{aligned}
f(x) &= (1-x)^{-1/2} \\
f'(x) &= \frac{1}{2}(1-x)^{-3/2} \\
f''(x) &= \frac{1\cdot 3}{2^2}(1-x)^{-5/2} \\
f'''(x) &= \frac{1\cdot 3\cdot 5}{2^3}(1-x)^{-7/2} \\
f^{(4)}(x) &= \frac{1\cdot 3\cdot 5\cdot 7}{2^4}(1-x)^{-9/2} \\
&\vdots \\
f^{(n)}(x) &= \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2^n}(1-x)^{-(2n+1)/2} \\
f^{(n)}(0) &= \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2^n}
\end{aligned}$$

We could leave it like this, but we simplify, to make our work cleaner later on.

$$\begin{aligned}
&= \frac{1}{2^n}\cdot\frac{(2n)!}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)} \\
&= \frac{1}{2^n}\cdot\frac{(2n)!}{2^n\cdot n!} \\
&= \frac{(2n)!}{2^{2n}\, n!}
\end{aligned}$$

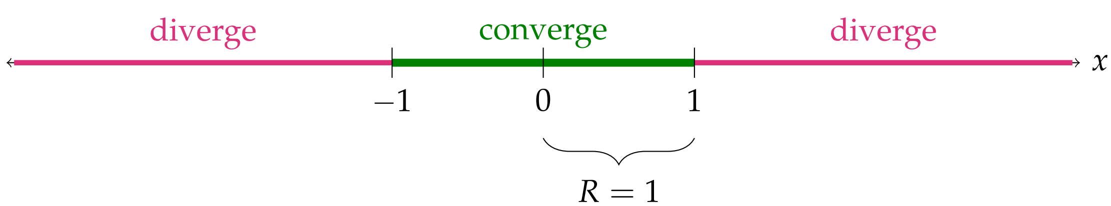
This pattern holds for $n \geqslant 0$. Now, we can write our Maclaurin series for $f(x)$.

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} x^n$$

To find the radius of convergence, we use the ratio test.

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(2n+2)!}{2^{2n+2}((n+1)!)^2} \cdot \frac{2^{2n}\,(n!)^2}{(2n)!} \cdot |x| \\
&= \frac{(2n+2)!}{(2n)!} \left(\frac{n!}{(n+1)!}\right)^2 \cdot \frac{2^{2n}}{2^{2n+2}} |x| \\
&= (2n+2)(2n+1) \left(\frac{1}{n+1}\right)^2 \cdot \frac{1}{4} |x| \\
&= \frac{4n^2+4n+2}{4n^2+8n+4} |x| \\
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{4n^2+4n+2}{4n^2+8n+4} |x|\right] = |x|
\end{aligned}$$

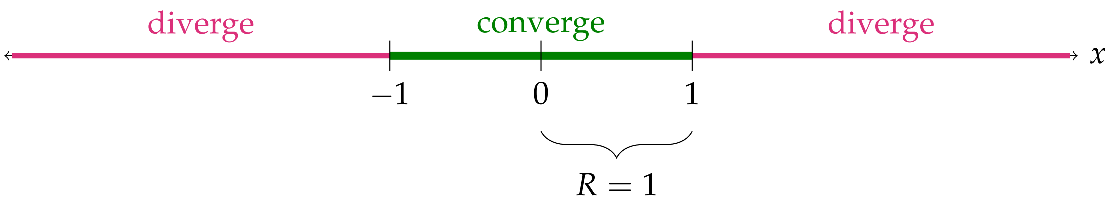
So, the radius of convergence is $R = 1$.



(b) We note the derivative of the arcsine function is $\dfrac{1}{\sqrt{1-x^2}} = f(x^2)$. With this insight, we can manipulate our Taylor series for $f(x)$ into a Taylor series for arcsine.

$$\begin{aligned}
\frac{1}{\sqrt{1-x}} &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} x^n \\
\frac{1}{\sqrt{1-x^2}} &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} x^{2n} \\
\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \left(\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} x^{2n}\right) \mathrm{d}x \\
\arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2(2n+1)} x^{2n+1} + C \\
\arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2(2n+1)} x^{2n+1}
\end{aligned}$$

where we found the value of C by setting $x = 0$. Its radius of convergence is also 1, by Theorem 5.5.12.



S-37: We use that

$$\ln(1+y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y^n}{n} \qquad \text{for all } -1 < y \leqslant 1$$

with $y = \dfrac{x-2}{2}$ to give

$$\begin{aligned} \ln(x) &= \ln(2+x-2) = \ln\left[2\left(1+\frac{x-2}{2}\right)\right] \\ &= \ln 2 + \ln\left(1+\frac{x-2}{2}\right) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}(x-2)^n \end{aligned}$$

It converges when $-1 < y \leqslant 1$, or equivalently, $0 < x \leqslant 4$.

S-38: Using the geometric series expansion with $r = -t^4$,

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \implies \frac{1}{1+t^4} = \sum_{n=0}^{\infty} \left(-t^4\right)^n = \sum_{n=0}^{\infty} (-1)^n t^{4n}$$

Substituting this into our integral,

$$\begin{aligned} I(x) &= \int_0^x \frac{1}{1+t^4}\,\mathrm{d}t \\ &= \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{4n}\right) \mathrm{d}t \\ &= \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+1}}{4n+1}\right]_{t=0}^{t=x} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1} \end{aligned}$$

S-39: Using the Taylor series expansion of e^x with $x = -t$,

$$e^{-t} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \implies e^{-t} - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{t^n}{n!} \implies \frac{e^{-t} - 1}{t} = \sum_{n=1}^{\infty} (-1)^n \frac{t^{n-1}}{n!}$$

Substituting this into our integral,

$$I(x) = \int_0^x \frac{e^{-t} - 1}{t}\, \mathrm{d}t = \sum_{n=1}^{\infty} (-1)^n \int_0^x \frac{t^{n-1}}{n!}\, \mathrm{d}t = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \cdot n!}$$

S-40: (a) Using the Taylor series expansion of $\sin x$ with $x = t$,

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \implies \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$$

So

$$\Sigma(x) = \int_0^x \frac{\sin t}{t}\, \mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \int_0^x \frac{t^{2n}}{(2n+1)!}\, \mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$$

(b) The critical points of $\Sigma(x)$ are the solutions of $\Sigma'(x) = 0$ or where $\Sigma'(x)$ do not exist. By the Fundamental Theorem of Calculus $\Sigma'(x) = \frac{\sin x}{x}$. $\Sigma'(x) = \frac{\sin x}{x}$ is not defined at $x = 0$ and $\Sigma'(x) = 0$ at $x = \pm\pi, \pm 2\pi, \cdots$. So the critical points of $\Sigma(x)$ are $x = 0, \pm\pi, \pm 2\pi, \cdots$. The absolute maximum occurs at $x = \pi$.

(c) Substituting in $x = \pi$,

$$\begin{aligned}
\Sigma(\pi) &= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)(2n+1)!} \\
&= \pi - \frac{\pi^3}{3 \cdot 3!} + \frac{\pi^5}{5 \cdot 5!} - \frac{\pi^7}{7 \cdot 7!} + \cdots \\
&= 3.1416 - 1.7226 + 0.5100 - 0.0856 + 0.0091 - 0.0007 + \cdots
\end{aligned}$$

The series for $\Sigma(\pi)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is no larger than the first omitted term. So

$$\Sigma(\pi) = 3.1416 - 1.7226 + 0.5100 - 0.0856 + 0.0091 = 1.8525$$

with an error of magnitude at most $0.0007 + 0.0005$ (the 0.0005 is the maximum possible accumulated roundoff error in all five retained terms).

S-41: Using the Taylor series expansion of $\cos t$,

$$
\begin{aligned}
\cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} \\
\frac{\cos t - 1}{t^2} &= -\frac{1}{2!} + \frac{t^2}{4!} - \frac{t^4}{6!} + \cdots \quad = \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n-2}}{(2n)!} \\
I(x) = \int_0^x \frac{\cos t - 1}{t^2}\,\mathrm{d}t &= -\frac{x}{2!} + \frac{x^3}{4!3} - \frac{x^5}{6!5} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!(2n-1)}
\end{aligned}
$$

S-42: Using the Taylor series expansions of $\sin x$ and $\cos x$ with $x = t$,

$$
\begin{aligned}
\sin t &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} && = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \\
t \sin t &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+2}}{(2n+1)!} && = \quad t^2 - \frac{t^4}{3!} + \frac{t^6}{5!} - \frac{t^8}{7!} + \cdots \\
&= -\sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n-1)!} \\
\cos t &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} && = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots \\
\cos t - 1 &= \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} && = -\frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots \\
\cos t + t \sin t - 1 &= \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} - \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n-1)!} \\
&= \sum_{n=1}^{\infty} (-1)^n t^{2n} \left(\frac{1}{(2n)!} - \frac{1}{(2n-1)!} \right) && = \left(1 - \frac{1}{2!}\right) t^2 - \left(\frac{1}{3!} - \frac{1}{4!}\right) t^4 + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^n t^{2n} \left(\frac{1}{(2n)!} - \frac{2n}{(2n)!} \right) \\
&= \sum_{n=1}^{\infty} (-1)^n t^{2n} \left(\frac{1-2n}{(2n)!} \right) && = \left(\frac{2}{2!} - \frac{1}{2!}\right) t^2 - \left(\frac{4}{4!} - \frac{1}{4!}\right) t^4 + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} t^{2n} \left(\frac{2n-1}{(2n)!} \right) && = \frac{1}{2!} t^2 - \frac{3}{4!} t^4 + \frac{5}{6!} t^6 - \frac{7}{8!} t^8 + \cdots \\
\frac{\cos t + t \sin t - 1}{t^2} &= \sum_{n=1}^{\infty} (-1)^{n+1} t^{2n-2} \left(\frac{2n-1}{(2n)!} \right) && = \frac{1}{2!} t - \frac{3}{4!} t^2 + \frac{5}{6!} t^4 - \frac{7}{8!} t^6 + \cdots
\end{aligned}
$$

Now, we're ready to integrate.

$$\begin{aligned}
I(x) = \int_0^x \left(\frac{\cos t + t\sin t - 1}{t^2}\right) &= \int_0^x \left(\sum_{n=1}^{\infty}(-1)^{n+1}t^{2n-2}\left(\frac{2n-1}{(2n)!}\right)\right)\mathrm{d}t \\
&= \left[\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^{2n-1}}{(2n)!}\right]_0^x \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n)!}
\end{aligned}$$

S-43: (a) Substituting $x = -t$ into the known power series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, we see that:

$$\begin{aligned}
e^{-t} &= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots \\
1 - e^{-t} &= t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \cdots \\
\frac{1-e^{-t}}{t} &= 1 - \frac{t}{2!} + \frac{t^2}{3!} - \frac{t^3}{4!} + \cdots \\
\int \frac{1-e^{-t}}{t}\,\mathrm{d}t &= C + x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \frac{x^4}{4\cdot 4!} + \cdots
\end{aligned}$$

Finally, $f(0) = 0$ (since $f(0)$ is an integral from 0 to 0) and so $C = 0$. Therefore

$$f(x) = \int_0^x \frac{1-e^{-t}}{t}\,\mathrm{d}t = x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \frac{x^4}{4\cdot 4!} + \cdots.$$

We can also do this calculation entirely in summation notation: $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, and so

$$\begin{aligned}
e^{-t} &= \sum_{n=0}^{\infty}\frac{(-t)^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n t^n}{n!} \\
1 - e^{-t} &= -\sum_{n=1}^{\infty}\frac{(-1)^n t^n}{n!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1} t^n}{n!} \\
\frac{1-e^{-t}}{t} &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1} t^{n-1}}{n!} \\
f(x) = \int_0^x \frac{1-e^{-t}}{t}\,\mathrm{d}t &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1} x^n}{n\cdot n!}
\end{aligned}$$

(b) We set $a_n = A_n x^n = \frac{(-1)^{n-1}}{n \cdot n!} x^n$ and apply the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-1)^n x^{n+1}/((n+1)\cdot(n+1)!)}{(-1)^{n-1}x^n/(n\cdot n!)}\right| \\
&= \lim_{n\to\infty} \left(\frac{|x|^{n+1}}{|x|^n} \frac{n\cdot n!}{(n+1)\cdot(n+1)!}\right) \\
&= \lim_{n\to\infty} \left(|x| \frac{n}{(n+1)^2}\right) \qquad \text{since } (n+1)! = (n+1)\, n! \\
&= 0
\end{aligned}$$

This is smaller than 1 no matter what x is. So the series converges for all x.

S-44:

$$\begin{aligned}
& e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \geqslant 1 + x \qquad \text{for all } x \geqslant 0 \\
\implies & e^x - 1 \geqslant x \\
\implies & \frac{x^3}{e^x - 1} \leqslant \frac{x^3}{x} = x^2 \\
\implies & \int_0^1 \frac{x^3}{e^x - 1}\, \mathrm{d}x \leqslant \int_0^1 x^2\, \mathrm{d}x = \frac{1}{3}
\end{aligned}$$

S-45: (a) We know that $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Replacing x by $-x$, we also have $e^{-x} = \sum\limits_{n=0}^{\infty} \frac{(-x)^n}{n!}$ for all x and hence

$$\cosh(x) = \frac{1}{2}\left[e^x + e^{-x}\right] = \frac{1}{2}\Big[\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\Big] = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

for all x. In particular, the interval of convergence is all real numbers.

(b) Using the power series expansion of part (a),

$$\cosh(2) = 1 + \frac{2^2}{2!} + \frac{2^4}{4!} + \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} = \frac{11}{3} + \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$$

So it suffices to show that $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} \leqslant 0.1$. Let's write $b_n = \frac{2^{2n}}{(2n)!}$. The first term in $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$ is

$$b_3 = \frac{2^6}{6!} = \frac{2^6}{6 \times 5 \times 4 \times 3 \times 2} = \frac{4}{45}$$

The ratio between successive terms in $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$ is

$$\frac{b_{n+1}}{b_n} = \frac{2^{2n+2}/2^{2n}}{(2n+2)!/(2n)!} = \frac{4}{(2n+2)(2n+1)} \leqslant \frac{4}{8 \times 7} = \frac{1}{14} \qquad \text{for all } n \geqslant 3$$

Hence

$$\sum_{n=3}^{\infty}\frac{2^{2n}}{(2n)!}\leqslant \overbrace{\frac{4}{45}}^{b_3}+\overbrace{\frac{4}{45}\times\frac{1}{14}}^{b_4\leqslant}+\overbrace{\frac{4}{45}\times\frac{1}{14^2}}^{b_5\leqslant}+\overbrace{\frac{4}{45}\times\frac{1}{14^3}}^{b_6\leqslant}+\cdots$$
$$=\frac{4}{45}\,\frac{1}{1-\frac{1}{14}}=\frac{4}{45}\,\frac{14}{13}=\frac{56}{585}<\frac{1}{10}$$

(c) Comparing

$$\cosh(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{\left(t^2\right)^n}{(2n)!}\qquad\text{and}\qquad e^{\frac{1}{2}t^2}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}t^2\right)^n}{n!}=\sum_{n=0}^{\infty}\frac{\left(t^2\right)^n}{2^n n!}$$

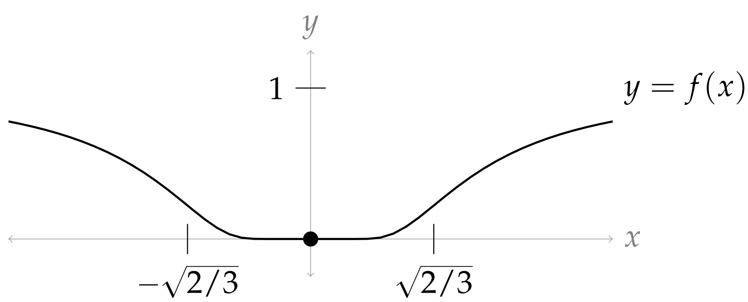
we see that it suffices to show that $(2n)!\geqslant 2^n n!$. Now. for all $n\geqslant 1$,

$$(2n)!=\overbrace{1\times 2\times\cdots\times n}^{n\text{ factors}}\,\overbrace{(n+1)\times(n+2)\times\cdots\times 2n}^{n\text{ factors}}$$
$$\geqslant\overbrace{1\times 2\times\cdots\times n}^{n\text{ factors}}\,\overbrace{2\times 2\times\cdots\times 2}^{n\text{ factors}}$$
$$=2^n\,n!$$

S-46:

(a) To sketch $y=f(x)$, we note the following:

- $f(x)$ is never negative.
- $\lim\limits_{x\to\pm\infty} f(x)=e^0=1$, so the curve has horizontal asymptotes in both directions at $y=1$.
- $\lim\limits_{x\to\pm 0} f(x)=\lim\limits_{x\to\pm 0}\frac{1}{e^{1/x^2}}=\lim\limits_{u\to+\infty}\frac{1}{e^u}=0=f(0)$, so the curve is continuous at $x=0$.
- For $x\neq 0$, $f'(x)=\frac{2}{x^3}e^{-1/x^2}$, so our curve is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$
- For $x\neq 0$, $f''(x)=2x^{-6}(2-3x^2)e^{-1/x^2}$, so our curve is concave up on $(-\sqrt{2/3},\sqrt{2/3})$, and concave down elsewhere.



(b) Since $f^{(n)}(0) = 0$ for all whole n (that is, the graph is really quite flat at the origin), and since $f(0) = 0$, the Maclaurin series for $f(x)$ is $\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$.

(c) The Maclaurin series converges for all real values of x (to the constant 0).

(d) Since $e^y > 0$ for any real y, we see $f(x) = 0$ only when $x = 0$. So, $f(x)$ is only equal to its Maclaurin series at the single point $x = 0$.

Remark: the function $f(x)$ is an example of a function whose Maclaurin series converges, but not to $f(x)$! To describe this behaviour, we say $f(x)$ is *non-analytic*.

S-47:

Solution 1: Since $f(x)$ is odd, $f(-x) = -f(x)$ for all x in its domain. We plug this into our power series, then consider the even-indexed terms and the odd-indexed terms separately.

$$\begin{aligned}
f(-x) &= -f(x) \\
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(-x)^n &= -\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!}(-x)^{2n+1} + \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!}(-x)^{2n} &= -\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{n!} x^{2n+1} - \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{n!} x^{2n} \\
-\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} + \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= -\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{n!} x^{2n+1} - \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{n!} x^{2n} \\
\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= -\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{n!} x^{2n} \\
2\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= 0 \\
\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= 0
\end{aligned}$$

Solution 2: Alternately, we could note the following:

- Since all derivative of $f(x)$ exist, all its derivatives are continuous.
- The derivative of an odd function is even, and the derivative of an even function is odd.
- So, the even-indexed derivatives of $f(x)$ are continuous, odd functions.
- Every continuous, odd function passes through the origin. That is, $f^{(2n)}(0) = 0$.
- So, every term in the series is 0.