Relations

(2014)

<u>Reflexive</u> $\forall a \in \mathbb{Z}$, $a^2 \in \mathbb{Z}$, so $5 \mid 5a^2$

then $5 \mid (7a^2 - 2a^2)$ so $7a^2 \equiv 2a^2 \mod 5$

Therefore aRa.

<u>Symmetry</u> $\forall a, b \in \mathbb{Z}$ s.t aRb:

$7a^2 \equiv 2b^2 \mod 5$

so $5 \mid 7a^2 - 2b^2$

so $\exists x \in \mathbb{Z}$ s.t. $5x = 7a^2 - 2b^2$.

Note: $2a^2 - 7b^2 = 2a^2(+5a^2 - 5a^2) - 2b^2 - 5b^2$

$= 7a^2 - 2b^2 + 5(-a^2 - b^2)$

Then $7b^2 - 2a^2 = 5x + 5(-a^2 - b^2) = 5(x - a^2 - b^2)$.

$7b^2 - 2a^2 = 5(a^2 + b^2 - x)$, and $a^2 + b^2 - x \in \mathbb{Z}$.

$5 \mid 7b^2 - 2a^2 \Rightarrow 7b^2 \equiv 2a^2 \mod 5$, so bRa.

Transitivity Let $a, b, c \in \mathbb{Z}$. s.t.

$a R b$ and $b R c$.

aRb : $7a^2 \equiv 2b^2 \mod 5$

bRc : $7b^2 \equiv 2c^2 \mod 5$

(goal: show $7a^2 \equiv 2c^2 \mod 5$)

$7a^2 - 2b^2 = 5x$, for some $x \in \mathbb{Z}$

$+$ $7b^2 - 2c^2 = 5y$, some $y \in \mathbb{Z}$

$7a^2 \underbrace{- 2b^2 + 7b^2} - 2c^2 = 5x + 5y$

so $7a^2 + 5b^2 - 2c^2 = 5x + 5y$

so $7a^2 - 2c^2 = 5(x + y - b^2)$

Then: $7a^2 \equiv 2c^2 \mod 5$

so aRc.

(2014)

$$[1] = \{ a \in \mathbb{Z} : aR1 \}$$
$$= \{ a \in \mathbb{Z} : 7a^2 \equiv 2 \cdot 1^2 \bmod 5 \}$$
$$= \{ a \in \mathbb{Z} : 7a^2 \equiv 2 \bmod 5 \}$$
$$= \{ a \in \mathbb{Z} : a \equiv 1 \bmod 5 \text{ or } a \equiv 4 \bmod 5 \}$$

Note:

$a \equiv 0 \bmod 5 \longrightarrow a^2 \equiv 0 \bmod 5 \longrightarrow 7a^2 \equiv 0 \bmod 5$

$\boxed{a \equiv 1 \bmod 5} \longrightarrow a^2 \equiv 1 \bmod 5 \longrightarrow \boxed{7a^2 \equiv 2 \bmod 5}$

$a \equiv 2 \bmod 5 \qquad a^2 \equiv 4 \bmod 5 \longrightarrow 7a^2 \equiv 3 \bmod 5$

$3 \qquad a^2 \equiv 4 \bmod 5 \qquad 3$

$\boxed{a \equiv 4 \bmod 5} \qquad a^2 \equiv 1 \bmod 5 \qquad \boxed{2}$

(2012) Let $b \in B$.

Then (by def of B) $b \equiv 1 \bmod 4$.

So: $4 \mid b-1$

So: $\exists x \in \mathbb{Z}$ s.t. $4x = b-1$

so $b = 4x+1$.

Then: $b^2 - 2b + 9 = (4x+1)^2 - 2(4x+1) + 9$

$= 16x^2 + \cancel{8x} + 1 - \cancel{8x} - 2 + 9$

$= 16x^2 + 8$

$= 8(2x^2+1)$

So $b^2 - 2b + 9 \equiv 0 \bmod 8$

so $b \in A$.

So: $B \subseteq A$.

2014 Short Ans

$\exists$ Inf. many countably infinite subsets of $\mathbb{N}$:

$\forall N \in \mathbb{N}$, define $A_N = \{ n \in \mathbb{N} : n \geq N \}$.

eg $A_1 = \mathbb{N}$

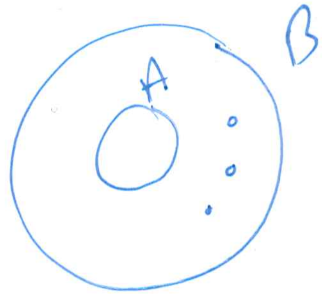
$A_2 = \mathbb{N} - \{1\}$

$A_3 = \mathbb{N} - \{1, 2\}$

$\vdots$

There are infinitely many A_N's.

Each is countably infinite.



2014 Short

$A = \{1\}$

$B = \{2\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}\}$

$\mathcal{P}(B) = \{\emptyset, \{2\}\}$

$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\}$

$\mathcal{P}(A \cup B) = \mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

The power set of a union is not the same as the union of power sets.

2013 Short

$\sim[\forall a \in A \ \exists b \in B \text{ s.t. } \forall c \in A, \ \underbrace{|a-b| \le 2 \Rightarrow b+c > 3}]$

$\exists a \in A \text{ s.t. } \forall b \in B \ \exists c \in A \text{ s.t. } |a-b| \le 2 \wedge b+c \le 3$

$\sim[P \Rightarrow Q]$

$\boxed{P \wedge \sim Q}$

$\boxed{P \Rightarrow \sim Q}$

$\sim P \vee \sim Q$

$P \Rightarrow Q$

If $\boxed{\text{not weekend}}^{P}$, $\boxed{\text{I get up at 7AM}}_{Q}$

Converse: $Q \Rightarrow P$

Contrapositive: $\sim Q \Rightarrow \sim P$

Converse: If I get up at 7am, then not weekend.

Contrapositive. If I do not get up at 7am, then it is the weekend.

IDEA

$$\mathbb{Z}: \quad \begin{aligned} 2n &\longrightarrow n \\ 2n+1 &\longrightarrow n \end{aligned}$$

Example of a function from theintegers to the integers that is surjective but not injective.

$f: \mathbb{Z} \to \mathbb{Z}$

$$f(x) = \begin{cases} x/2 & \underline{\text{if}} \quad x \text{ even} \\ (x-1)/2 & \underline{\text{if}} \quad x \text{ odd} \end{cases}$$

$0 \longrightarrow 0$	$-1 \longrightarrow -1$
$1 \longrightarrow 0$	$-2 \longrightarrow -1$
$2 \longrightarrow 1$	$-3 \longrightarrow -2$
$3 \longrightarrow 1$	$-4 \longrightarrow -2$
$4 \longrightarrow 2$	$\vdots$
$5 \longrightarrow 2$	
$6 \longrightarrow 3$	
$\vdots \qquad \vdots$	